If $d$ is Super-Metric, Then $d/(1 + d)$ is Super-Metric

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Abstract

If a function $d$ is metric, a well-known result is that $d/(1 + d)$ is also metric. We consider $m$-ary analogs of the binary notion of semimetric, called hemi-metrics and super-metrics. The metrics are totally symmetric maps from $X^{m+1}$ into $\mathbb{R}_{\geq 0}$. It is shown that, if $d$ is super-metric, then $d/(1 + d)$ is also super-metric.

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1 Hemi-metrics and super-metrics

A metric is a function that defines a distance between two elements of a set. We consider generalizations of the notion of metric in the direction of distances between three or more elements.

Deza and Rosenberg [4] introduced the following notion. Let $m$ be a positive integer and $X$ a set with at least $m+2$ elements. A function $d : X^{m+1} \to \mathbb{R}$ is called $m$-hemi-metric if (see, also [1,2,5]):

1. $d$ is non-negative, i.e., $d(x_1, \ldots, x_{m+1}) \geq 0$ for all $x_1, \ldots, x_{m+1} \in X$.

2. $d$ is totally symmetric, i.e., satisfies $d(x_1, \ldots, x_{m+1}) = d(x_{\pi(1)}, \ldots, x_{\pi(m+1)})$ for all $x_1, \ldots, x_{m+1} \in X$ and for any permutation $\pi$ of $\{1, \ldots, m+1\}$. 
3. \(d\) is zero conditioned, i.e. \(d(x_1, \ldots, x_{m+1}) = 0\) if and only if \(x_1, \ldots, x_{m+1}\) are not pairwise distinct.

4. For all \(x_1, \ldots, x_{m+2} \in X\), \(d\) satisfies the \(m\)-simplex inequality:

\[
d(x_1, \ldots, x_{m+1}) \leq \sum_{i=1}^{m+1} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}).
\]

(1)

The notion of \(m\)-hemi-metric is an \(m\)-ary analog of the binary notion of semi-metric. An important special case of the \(m\)-hemi-metric is the following notion obtained for \(m = 2\). A function \(d : X^3 \to \mathbb{R}\) is called a 2-metric if \(d\) is non-negative, totally symmetric, zero conditioned, and satisfies the tetrahedron inequality:

\[
d(x_1, x_2, x_3) \leq d(x_1, x_2, x_4) + d(x_1, x_3, x_4) + d(x_2, x_3, x_4).
\]

(2)

Interpreting \(d(x_1, x_2, x_3)\) as the area of the triangle with vertices \(x_1, x_2\) and \(x_3\), the tetrahedron inequality specifies that the area of each triangle face of the tetrahedron formed by \(x_1, x_2, x_3\) and \(x_4\) does not exceed the sum of the areas of the remaining faces. Alternative axiom systems are considered in [6-11].

Deza and Dutour [3] introduced the following notion. Let \(s\) be a positive real number. A function \(d : X^{m+1} \to \mathbb{R}\) is called \((m, s)\)-super-metric if \(d\) is non-negative, totally symmetric, zero conditioned, and satisfies the \((m, s)\)-simplex inequality

\[
sd(x_1, \ldots, x_{m+1}) \leq \sum_{i=1}^{m+1} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}).
\]

(3)

An \((m, s)\)-super-metric is an \(m\)-hemi-metric if \(s \geq 1\). Furthermore, a \(m\)-hemi-metric is a \((m, 1)\)-super-metric and a semi-metric is a \((1, 1)\)-super-metric.

For the ordinary metric, a well-known result is that, if \(d\) is metric, then \(d/(1 + d)\) and \(\min\{1, d\}\) are also metric. In Section 2 we present an analogous result for the function \(d/(1 + d)\) for hemi-metrics and super-metrics. In Section 3 we present an analogous result for the function \(\min\{1, d\}\) for hemi-metrics and the \((2, 2)\)-super-metric.

## 2 Function \(d/(1 + d)\)

Lemma 2.1 considers the notion of \(m\)-hemi-metric. Lemma 2.3 considers the notion of \((m, s)\)-super-metric for \(s \geq 1\). Lemma 2.2 is used in the proof of Lemmas 2.3 and 3.2.

**Lemma 2.1.** Let \(d\) be \(m\)-hemi-metric. Then \(d/(1 + d)\) is \(m\)-hemi-metric.
If $d$ is super-metric, then $d/(1 + d)$ is super-metric

Proof. Non-negativity of $d/(1 + d)$ follows from the non-negativity of $d$. Furthermore, total symmetry and axiom 3 follow from the identity

$$
\frac{d(x_1, \ldots, x_{m+1})}{1 + d(x_1, \ldots, x_{m+1})} = 1 - \frac{1}{1 + d(x_1, \ldots, x_{m+1})},
$$

and the fact that $d$ is totally symmetric and zero conditioned. Thus, we must show that $d/(1 + d)$ satisfies (1).

Because $d/(1 + d)$ is strictly increasing in $d$, and since $d$ satisfies (1), we have

$$
\frac{d(x_1, \ldots, x_{m+1})}{1 + d(x_1, \ldots, x_{m+1})} \leq \sum_{i=1}^{m+1} \frac{d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})}{1 + \sum_{i=1}^{m+1} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})}.
$$

Furthermore, for all $i \in \{1, \ldots, m+1\}$ we have the inequality

$$
\frac{d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})}{1 + \sum_{j=1}^{m+1} d(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{m+2})} \leq \frac{d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})}{1 + d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})}.
$$

Summing (6) over all $i \in \{1, \ldots, m+1\}$, and combining the resulting inequality with inequality (5), completes the proof.

Lemma 2.2. Suppose $s > 1$ and let $d$ be $(m, s)$-super-metric. Then $d$ satisfies the inequality

$$
(s - 1)d(x_1, \ldots, x_{m+1}) \leq \sum_{i=2}^{m+1} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}).
$$

Proof. Interchanging the roles of $x_1$ and $x_{m+2}$ in (3), and dividing the result by $s$, we obtain

$$
d(x_2, \ldots, x_{m+2}) \leq \frac{1}{s} \sum_{i=2}^{m+2} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}). \tag{8}
$$

Adding inequalities (3) and (8) yields

$$
(s - \frac{1}{s}) d(x_1, \ldots, x_{m+1}) \leq \left(1 + \frac{1}{s}\right) \sum_{i=2}^{m+1} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}), \tag{9}
$$

which is equivalent to (7).
Lemma 2.3. Suppose $s \geq 1$ and let $d$ be $(m, s)$-super-metric. Then $d/(1+d)$ is $(m, s)$-super-metric.

Proof. The case $s = 1$ is proved in Lemma 2.1. Therefore, suppose $s > 1$. The proof of non-negativity, total symmetry and axiom 3 is analogous to the proof of Lemma 2.1. We must show that $d$ satisfies (3).

Because $d/(1+d)$ is strictly increasing in $d$, and since $d$ satisfies (3), we have

$$
\frac{d(x_1, \ldots, x_{m+1})}{1 + d(x_1, \ldots, x_{m+1})} \leq \frac{1}{s} \sum_{i=1}^{m+1} d(x_1, \ldots, x_{i-1}, x_i+1, \ldots, x_{m+2}).
$$

(10)

After multiplying both sides of (10) by $s$, we may write the result as

$$
\frac{sd(x_1, \ldots, x_{m+1})}{1 + d(x_1, \ldots, x_{m+1})} \leq \sum_{i=1}^{m+1} \frac{d(x_1, \ldots, x_{i-1}, x_i+1, \ldots, x_{m+2})}{1 + \frac{1}{s} \sum_{j=1}^{m+1} d(x_1, \ldots, x_{j-1}, x_j+1, \ldots, x_{m+2})}.
$$

(11)

Due to Lemma 2.2, combined with the total symmetry of $d$, we have for all $i \in \{1, \ldots, m+1\}$,

$$(s-1)d(x_1, \ldots, x_{i-1}, x_i+1, \ldots, x_{m+2}) \leq \sum_{j=1}^{m+1} d(x_1, \ldots, x_{j-1}, x_j+1, \ldots, x_{m+2}) - d(x_1, \ldots, x_{i-1}, x_i+1, \ldots, x_{m+2}).
$$

(12)

Adding $d(x_1, \ldots, x_{i-1}, x_i+1, \ldots, x_{m+2})$ to both sides of (12), and dividing the result by $s$, we have for all $i \in \{1, \ldots, m+1\}$,

$$
d(x_1, \ldots, x_{i-1}, x_i+1, \ldots, x_{m+2}) \leq \frac{1}{s} \sum_{j=1}^{m+1} d(x_1, \ldots, x_{j-1}, x_j+1, \ldots, x_{m+2}).
$$

(13)

Furthermore, using (13), we have, for all $i \in \{1, \ldots, m+1\}$, the inequality

$$
\frac{d(x_1, \ldots, x_{i-1}, x_i+1, \ldots, x_{m+2})}{1 + \frac{1}{s} \sum_{j=1}^{m+1} d(x_1, \ldots, x_{j-1}, x_j+1, \ldots, x_{m+2})} \leq \frac{d(x_1, \ldots, x_{i-1}, x_i+1, \ldots, x_{m+2})}{1 + d(x_1, \ldots, x_{i-1}, x_i+1, \ldots, x_{m+2})}.
$$

(14)

Summing (14) over all $i \in \{1, \ldots, m+1\}$, and combining the result with (11), completes the proof. \qed
If \( d \) is super-metric, then \( d/(1 + d) \) is super-metric

### 3 Function \( \min \{1, d\} \)

Lemma 3.1 considers the notion of \( m \)-hemi-metric. Lemma 3.2 considers the notion of \((2,2)\)-super-metric.

**Lemma 3.1.** Let \( d \) be \( m \)-hemi-metric. Then \( \min \{1, d\} \) is \( m \)-hemi-metric.

**Proof.** Non-negativity, symmetry and axiom 3 of \( \min \{1, d\} \) follow from the analogous properties of \( d \). Thus, we must show that \( \min \{1, d\} \) satisfies (1).

We go through the various cases.

Suppose there is an \( j \in \{1, \ldots, m+1\} \) such that

\[
d(x_1, \ldots, x_{m+1}) \leq d(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{m+2}).
\]

(15)

In this case we have

\[
\min \{1, d(x_1, \ldots, x_{m+1})\} \leq \min \{1, d(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{m+2})\}
\]

\[
\leq \sum_{i=1}^{m+1} \min \{1, d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})\}.
\]

(16)

Thus, we may assume that, for all \( i \in \{1, \ldots, m+1\} \), we have

\[
d(x_1, \ldots, x_{m+1}) \geq d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}).
\]

(17)

Suppose \( d(x_1, \ldots, x_{m+1}) \leq 1 \). In this case we have, for all \( i \in \{1, \ldots, m+2\} \),

\[
\min \{1, d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})\} = d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}),
\]

and it follows that \( \min \{1, d\} \) satisfies (1) because \( d \) satisfies (1).

Next, suppose \( d(x_1, \ldots, x_{m+1}) > 1 \). Furthermore, suppose there is an \( j \in \{1, \ldots, m+1\} \) such that \( d(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{m+2}) \geq 1 \). In this case we have

\[
\min \{1, d(x_1, \ldots, x_{m+1})\} = 1 = \min \{1, d(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{m+2})\}
\]

\[
\leq \sum_{i=1}^{m+1} \min \{1, d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})\}.
\]

(18)

Therefore, suppose that \( d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2}) \leq 1 \) for all \( i \in \{1, \ldots, m+1\} \). In this final case we have, since \( d \) satisfies (1),

\[
\min \{1, d(x_1, \ldots, x_{m+1})\} = 1 < d(x_1, \ldots, x_{m+1})
\]

\[
\leq \sum_{i=1}^{m+1} d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})
\]

\[
= \sum_{i=1}^{m+1} \min \{1, d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m+2})\}.
\]

(19)

This completes the proof. \( \Box \)
Lemma 3.2. Let $d$ be $(2,2)$-super-metric. Then $\min\{1, d\}$ is $(2,2)$-super-metric.

Proof. Non-negativity, symmetry and axiom 3 of $\min\{1, d\}$ follow from the analogous properties of $d$. Thus, we must show that $\min\{1, d\}$ satisfies

$$2d(x_1, x_2, x_3) \leq d(x_1, x_2, x_4) + d(x_1, x_3, x_4) + d(x_2, x_3, x_4),$$

which is a strong version of tetrahedron inequality (2) [6,8,9,11]. We go through the various cases.

First, suppose $d(x_1, x_2, x_3) \leq 1$. In addition, suppose at least two of the three quantities on the right-hand side of (20) $\geq 1$. In this case we have

$$2 \min\{1, d(x_1, x_2, x_3)\} = 2d(x_1, x_2, x_3) \leq 2 = 1 + 1$$

$$\leq \min\{1, d(x_1, x_2, x_4)\} + \min\{1, d(x_1, x_3, x_4)\} + \min\{1, d(x_2, x_3, x_4)\}.$$ 

Furthermore, without loss of generality, suppose that $d(x_1, x_2, x_4) > 1$ and $d(x_1, x_3, x_4), d(x_2, x_3, x_4) \leq 1$. In this case we have

$$\min\{1, d(x_1, x_2, x_3)\} = d(x_1, x_2, x_3) \leq 1 = \min\{1, d(x_1, x_2, x_4)\}.$$ 

(21)

We also have, using Lemma 2.2,

$$\min\{1, d(x_1, x_2, x_3)\} = d(x_1, x_2, x_3) \leq d(x_1, x_3, x_4) + d(x_2, x_3, x_4)$$

$$= \min\{1, d(x_1, x_3, x_4)\} + \min\{1, d(x_2, x_3, x_4)\}.$$ 

(22)

Combining (21) and (22) gives the desired inequality.

Moreover, suppose all three quantities on the right-hand side of (20) $\leq 1$. In this case we have, since $d$ satisfies (20),

$$2 \min\{1, d(x_1, x_2, x_3)\} = 2d(x_1, x_2, x_3)$$

$$\leq d(x_1, x_2, x_4) + d(x_1, x_3, x_4) + d(x_2, x_3, x_4)$$

$$= \min\{1, d(x_1, x_2, x_4)\} + \min\{1, d(x_1, x_3, x_4)\} + \min\{1, d(x_2, x_3, x_4)\}.$$ 

Second, suppose $d(x_1, x_2, x_3) > 1$. In addition, suppose at least two of the three quantities on the right-hand side of (20) $\geq 1$. In this case we have

$$2 \min\{1, d(x_1, x_2, x_3)\} = 2 = 1 + 1$$

$$\leq \min\{1, d(x_1, x_2, x_4)\} + \min\{1, d(x_1, x_3, x_4)\} + \min\{1, d(x_2, x_3, x_4)\}.$$ 

Furthermore, without loss of generality, suppose that $d(x_1, x_2, x_4) \geq 1$ and $d(x_1, x_3, x_4), d(x_2, x_3, x_4) \leq 1$. In this case we have

$$2 \min\{1, d(x_1, x_2, x_3)\} = 2 < d(x_1, x_2, x_3) + \min\{1, d(x_1, x_2, x_4)\}.$$ 

(23)
If $d$ is super-metric, then $d/(1 + d)$ is super-metric

We also have, using Lemma 2.2,
\[
d(x_1, x_2, x_3) \leq d(x_1, x_3, x_4) + d(x_2, x_3, x_4) \\
= \min \{1, d(x_1, x_3, x_4)\} + \min \{1, d(x_2, x_3, x_4)\}.
\] (24)

Combining (23) and (24) gives the desired inequality.

Finally, suppose all three quantities on the right-hand side of (20) $\leq 1$. In this case we have, since $d$ satisfies (20),
\[
2 \min \{1, d(x_1, x_2, x_3)\} = 2 < 2d(x_1, x_2, x_3) \\
\leq d(x_1, x_2, x_4) + d(x_1, x_3, x_4) + d(x_2, x_3, x_4) \\
= \min \{1, d(x_1, x_2, x_4)\} + \min \{1, d(x_1, x_3, x_4)\} + \min \{1, d(x_2, x_3, x_4)\}.
\]

This completes the proof.

References


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