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Fully Annihilator Small Stable Modules

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Abstract

Let R be an associative ring with non-zero identity and M be a left R-module. A submodule N of M is called annihilator small (briefly a-small), if for every submodule L of M with N+L=M, then l_R(L)=l_R(M). The properties of a-small submodules have been studied and characterizations of a-small cyclic submodules have been investigated. The sum of a-small submodules is studied. Moreover, we shall introduce fully annihilator small stable module (briefly FASS module) where M is called a FASS module if every annihilator small submodule of M is stable. Characterizations of FASS modules are proven.

Keywords: Annihilator small submodules, Fully stable modules, Annihilator small regular modules

1. Introduction

Throughout this work R will denote an associative ring with non-zero identity, M a left R-module. A submodule N of M is called *small*, if for every submodule K of M with N+K=M, then K=M [5]. Recently, many authors have been interested in studying different kinds of a-small submodules as in [3] and [4], where the authors in [3] introduced the concept of R-annihilator small submodules, that is; a submodule N of an R-module M is called R-annihilator small, if whenever N+K=M, where K a submodule of M; then $l_R(K)$=0. This has motivated us in turn to introduce the concept of annihilator small submodules, in way that a submodule N of M is called annihilator small (briefly a-small) in case l_R(K)=$l_R(M)$, where K is a submodule of M; whenever N+K=M. It is clear that every small submodule is a-small, but the converse is not true generally as examples can show next, while the two definitions become equal if M is faithful, recalling that M is called *faithful* in case $l_R(M) = 0$. Remember that *singular submodule* of an R-module M denoted by Z(M)={m∈M | $l_R(m)$ is essential in R} [5], We shall study the properties of a- small submodules, and define a subset of M that consists of all annihilator small elements ($denoted\ by\ AS_M$), as well as; we shall denote the sum of all annihilator small submodules of M by $J_a(M)$, and study its properties and the relation between it and the Jacobson radical. Finally, we shall introduce the concept of fully annihilator small stable modules as a generalization of fully stable modules [1]. Recall that a submodule N of an R-module M is called stable in case for every R-homomorphism $\alpha: N \longrightarrow M$ we have $\alpha(N) \subseteq N$ and M is called fully stable if every submodule of M is stable. Characterizations and properties of this concept is studied involving the satisfaction of Baer's criterion on a-small cyclic submodules and its effect on M being a FASS module. Recall that, a submodule N of M is said to satisfy Baer's criterion if for each $\beta: N \longrightarrow M$ there exists an element $r \in R$ such that $\beta(n) = rn$ for each $n \in N$ [1].In this paper, we are also interested to study the relation between M being a FASS module and $End_R(M)$ being commutative.

2. Annihilator small submodules

Definition 2.1: A submodule N of an R-module M is called *annihilator small* (briefly *a-small*) in M, and denoted by N a≪ M; if whenever N+K=M for each submodule K of M, then $l_R(K)=l_R(M)$. Where l_R denotes the left annihilators in R. A left ideal I of R is annihilator small if for each left ideal J of R with I+J=R, implies that l_R(J)=0.

Examples and remarks 2.2:

1. It is clear that every small submodule is annihilator small, but the converse is not true generally. For example, in the $\mathbb{Z}$-module $\mathbb{Z}$, (0) is the only small submodule while for every n>1, there exists m such that n$\mathbb{Z}$+m$\mathbb{Z}$=$\mathbb{Z}$ and $l_R(m\mathbb{Z})$=0= $l_R(\mathbb{Z})$.
2. If M is a faithful R-module then the concepts of annihilator small submodules and R-annihilator small submodules are equivalent.
3. There are annihilator small submodules that are direct summands as in the $\mathbb{Z}_2$-module M=$\mathbb{Z}_2 \oplus \mathbb{Z}_2$, where it is clear that A=$\mathbb{Z}_2 \oplus (0)$ is a direct summand of M, $M = A \oplus \mathbb{Z}_2 = A \oplus < (\bar{1}, \bar{1}) >$ and $l_{\mathbb{Z}_2}$(M)=0=$l_{\mathbb{Z}_2}(< (\bar{1}, \bar{1}) >)$.

Recall that, M is called *prime* if l_R(N)=l_R(M) for every non-zero submodule N of M[5]. M is called *quasi-Dedekind* if Hom(M/N, M)=0 for every proper submodule N of M[6], it is mentioned in [6] that every quasi-Dedekind module is prime.

The proof of the following proposition is obvious.

Proposition 2.3: Let M be a prime R-module. Then every proper submodule of M is annihilator small. In particular, every proper submodule of a quasi-Dedekind R-module is annihilator small.

It is mentioned in [6, p.25] that $\mathbb{Q}$ as $\mathbb{Z}$-module is quasi-Dedekind, and hence by the use of proposition (2.3) we get that every proper submodule of $\mathbb{Q}$ is annihilator small, but only finitely generated submodules of $\mathbb{Q}$ are small.

Proposition 2.4: Let M be an R-module with submodules A$\subseteq N$. If N a$\ll M$ then A a$\ll M$.

Proof: Let X be a submodule of M such that A+X=M, since A$\subseteq N$ hence N+X=M. By N being a-small in M then $l_R(X) = l_R(M)$ and hence A a$\ll$ M. ■

Proposition 2.5: Let M be an R-module with submodules A$\subseteq N$, if A a$\ll N$ and $l_R(N) = l_R(M)$ then A a$\ll M$.
Proof: Let X be any submodule of M such that A+X=M, now N∩M=N∩ (A+X) implies that N=A+(N∩X) by the modular law. Since A a$\ll N$, thus $l_R(N \cap X) = l_R(N)$. But $l_R(X) \subseteq l_R(N \cap X) = l_R(N) = l_R(M)$ implies that $l_R(X) \subseteq l_R(M)$ and then $l_R(X) = l_R(M)$, hence X a$\ll M$. ■

Proposition 2.6: Let M and N be R-modules and $\alpha: M \longrightarrow N$ an R-monomorphism if W a$\ll$ M then $\alpha(W)$ a$\ll$ $\alpha(M)$.

Proof: Let U be a submodule of N such that $\alpha(W)$+U=$\alpha(M)$, now U$\subseteq N$ implies $\alpha^{-1}(U) \subseteq \alpha^{-1}(N) = M$ and $\alpha(\alpha^{-1}(U))$=U∩ $Im(\alpha) = U \cap \alpha(M) = U$. Now, $\alpha^{-1}\big(\alpha(W)\big) + \alpha^{-1}(U) = \alpha^{-1}(\alpha(M))$ and then W+$\alpha^{-1}(U) = M$ this implies that $l_R(\alpha^{-1}(U))$=$l_R(M)$ since W a$\ll$ M. Let X=$\alpha^{-1}(U)$ then $l_R(X)$=$l_R(M)$. Let $r\in l_R(U)$=$l_R(\alpha(X))$, thus $r\alpha(X)$=0 $\Longrightarrow \alpha(rX)$=0 $\Longrightarrow rX = 0 \Longrightarrow r \in l_R(X) \Longrightarrow l_R(U) \subseteq l_R(X) = l_R(M) \Longrightarrow l_R(U) = l_R(M) \subseteq l_R\big(\alpha(M)\big) \Longrightarrow l_R(U) = l_R\big(\alpha(M)\big)$. Hence, $\alpha(W)$ a$\ll \alpha(M)$. ■

Corollary 2.7: Let M and N be R-modules and $\alpha: M \to N$ an R-monomorphism such that $l_R\big(\alpha(M)\big) = l_R(N)$, if W a$\ll$ M then $\alpha(W)$ a$\ll N$.

In the same manner of the definition of Jacobson radical related to small submodules, we will state a definition related to annihilator small submodules in the following. But first we need this definition.

Definition 2.8: Let M be an R-module and $a\in M$. We say that an element *a* in M is annihilator small if Ra is annihilator small submodule of M. let $AS_M = \{a \in M | Ra$ a$\ll M\}$.

Note that AS_M is not a submodule of M. In fact, it is not closed under addition, for example in the $\mathbb{Z} - module\ \mathbb{Z}$ we have that 3,-2 ∈ $AS_{\mathbb{Z}}$ but 3-2=1$\notin AS_{\mathbb{Z}}$.

We can see by the use of proposition (2.4) that if M is an R-module and $a\in AS_M$, *then* $Ra \subseteq AS_M$. Moreover, if A a$\ll$ M then A$\subseteq AS_M$.

Definition 2.9: Let M be an R-module. Denote $J_a(M)$ for the sum of all annihilator small submodules of M.

It is clear that $AS_M \subsetneq J_a(M)$ for every R-module M. The $\mathbb{Z}-module\ \mathbb{Z}$ is an example of this inclusion being proper, where $n\mathbb{Z}$ is a-small for each $n\neq 1,-1$ in $\mathbb{Z}$, hence $J_a(\mathbb{Z}) = \sum_{n\mathbb{Z}\ a\ll\mathbb{Z}} n\mathbb{Z} = \mathbb{Z}$, but $AS_{\mathbb{Z}} = \{n \in \mathbb{Z} | n\mathbb{Z}\ a \ll\ \mathbb{Z}\} = \{n\mathbb{Z} | n \neq 1,-1\}$.

Recall that, if T is an arbitrary proper submodule of a right R-module M and N a submodule of M, then N is called *T-essential* provided that N ⊈ T and for each submodule K of M, N∩K⊆T implies that K⊆T [8].

We introduce the following singularity of modules.

Definition 2.10: Let M be an R-module and J be an arbitrary left ideal of R. define the subset Z(J,M)of M by Z(J,M)={x∈M| l_R(x) is J-essential in R}, it is easy to show that Z(J,M)={x∈M| Ix=0 for some J-essential left ideal I of R}. It is clear that Z(0,M)=Z(M) for any R-module M.

Proposition 2.11: Let M be an R-module and J an arbitrary proper left ideal of R. Then Z(J,M) is a submodule of M, and it is called the singular submodule of M relative to J.

Proof: It is clear that Z(J,M) is non-empty. Let *x,y* ∈ Z(J,M), then there exist two J-essential left ideals A and B of R with A*x=0* and B*y=0*. Now, A∩B is J-essential and (A∩B)(*x-y*)=0 [7] and thus *x-y* ∈ Z(J,M). For each *r*∈R, since $l_R(x) \subseteq l_R(rx)$ and $l_R(x)$ is J-essential in R hence *rx*∈ Z(J,M). ■

Lemma 2.12: Let M be a non-zero R-module and N a submodule of M. If $l_R(N)$ is $l_R(M)$-essential in R, then $r_M(l_R(N))$ is a-small in M; in particular, N is a-small in M.

Proof: Let X be a submodule of M with X+$r_M(l_R(N))$=M. Then $l_R(X) \cap l_R\left(r_M\left(l_R(N)\right)\right) = l_R(X) \cap l_R(N) = l_R(M)$, since $l_R(N)$ is $l_R(M)$-essential in R then $l_R(X) \subseteq l_R(M)$ and hence $r_M(l_R(N))$ is a- small in M. The last assertion follows from proposition (2.4). ■

Corollary 2.13: Let M be a non-zero R-module. If m ∈ Z($l_R(M)$,M), then Rm is a-small in M.

Proof: Let m∈ $Z(l_R(M), M)$. Then $l_R(m)\ is\ l_R(M)$-essential in R, and by lemma (2.12) we have Rm is a-small in M. ■

Note that the converse of lemma(2.12) is true if $r_M(A \cap B) = r_M(A) + r_M(B)$ for each left ideals A and B of R. For this, let T be a left ideal of R with $l_R(N) \cap T \subseteq l_R(M)$. Then

$$M \subseteq r_M\left(l_R(M)\right) \subseteq r_M(l_R(N) \cap T) = r_M(l_R(N)) + r_M(T).$$

Since $r_M(l_R(N))$ is a-small in M, then $T \subseteq l_R\left(r_M(T)\right) \subseteq l_R(M)$. This shows that $l_R(N)\ is\ l_R(M)$-essential in R.

Proposition 2.14: Let M be a non-zero finitely generated R-module and K a submodule of M. If K is a-small in M, then so is K+J(M)+Z(J,M) where J=$l_R(M)$.

Proof: Let X be a submodule of M such that K+J(M)+Z(J,M)+X=M. Since M is finitely generated, then $\{m_i\}_{i=1}^{n}$ is a set of generators of M and M= $\sum_{i=1}^{n} Rm_i$, and J(M) is small in M; that is, K+Z(J,M)+X=M. Now, for each m_i ∈ M we have $m_i = k_i + z_i + x_i$ where k_i ∈ K, z_i ∈ Z(J,M) and x_i ∈ X for each *i*=1,…,n. Thus M= K+$\sum_{i=1}^{n} Rz_i$+X and since K is a-small in M by our assumption.

Thus $l_R(M)$=$l_R(\sum_{i=1}^{n} Rz_i$+X)=$l_R(\sum_{i=1}^{n} Rz_i)$∩ $l_R(X)$=($\cap_{i=1}^{n} l_R(Rz_i)$) ∩ ($l_R(X)$). But z_i ∈ Z(J,M), thus $l_R(z_i)$ is $l_R(M)$-essential in R for each *i*=1,…,n, and hence $\cap_{i=1}^{n} l_R(Rz_i)$ is $l_R(M)$-essential in R [2]. Thus $l_R(X) \subseteq l_R(M)$, and hence K+J(M)+Z(J,M) is a-small submodule of M. ■

Corollary 2.15: Let M be a finitely generated R-module. Then J(M)+Z(J,M) is a-small in M where J=$l_R(M)$.

The proof of the following proposition is as that in lemma (2.12).

Proposition 2.16: let M be an R-module such that Z(J,M) is finitely generated. If K is an a-small submodule of M, then so is K+Z(J,M).

In the following we give a characterization of cyclic annihilator small submodules.

Theorem 2.17: Let M be an R-module and m∈M. Then the following statements are equivalent:

1. Rm a≪ M.
2. $\cap_{i\in I} l_R(m_i - r_i m) = l_R(M)\ for\ each\ r_i \in R.$
3. There exists j∈I such that $rm_j \notin Rrm$ for all $r \notin l_R(M)$.

Proof: (1)⟹ (2) For each i∈I, $m_i = m_i - r_i m + r_i m$ and hence M=$\sum_{i\in I} R(m_i - r_i m) + Rm$. By (1) we have $l_R(M) = l_R(\sum_{i\in I} R(m_i - r_i m))$ =$\cap_{i\in I} l_R(m_i - r_i m)$.
(2)⟹ (1) Let X be a submodule of M with X+Rm=M. Then for each i∈I $m_i = x_i + r_i m,\ r_i \in R\ and\ x_i \in X$. Let t∈ $l_R(X)$, then $tm_i = tr_i m + tx_i = l_R$(M).
(2)⟹ (3) Let $r \notin l_R(M)$ and assume that $rm_i \in Rrm$ for all i∈ I. Then $rm_i = r_i rm = rr_i m$ for all *i*∈*I*, so by (1) $r \in \cap_{i\in I} l_R(m_i - r_i m) = l_R(M)$ which is a contradiction.
(3)⟹ (2) Let $r \in \cap_{i\in I} l_R(m_i - r_i m)$ and hence $r \in l_R(m_i - r_i m)$ for all $i \in I$. Thus $rm_{i=} rr_i m = r_i rm$ for all $i \in I$, so $rm_i \in Rrm$. By (2) $r \in l_R(M)$ and hence $\cap_{i\in I} l_R(m_i - r_i m) \subseteq l_R(M)$ and $\cap_{i\in I} l_R(m_i - r_i m) = l_R(M)$ for all $r_i \in R$. ■

Theorem 2.18: Let R be a commutative ring, M=$\sum_{i\in I} Rm_i$ and K a submodule of M. Then the following statements are equivalent:

1. K a≪ M.
2. $\cap_{i\in I} l_R R(m_i - k_i) = l_R(M)$ for all $k_i \in K$.

Proof: (1)$\Longrightarrow$ (2) For each $i \in I$, let $k_i \in K$. Then $m_i = m_i - k_i + k_i$ for each $i \in I$. Then $M = \sum_{i\in I} R(m_i - k_i) + K$, by (1) we obtain $l_R(M) = l_R(\sum_{i\in I} R(m_i - k_i)) = \cap_{i\in I} l_R(R(m_i - k_i))$.

(2)$\Longrightarrow$ (1) Let A be a submodule of M with M=A+K. Then for each $i \in I$ $m_i = a_i + k_i$ where $a_i \in A$ and $k_i \in K$. Hence $a_i = m_i - k_i$ for each $i \in I$ and $M = \sum_{i\in I} R(m_i - k_i) + K$. Now, let $t \in l_R(A)$ then $ta_i = t(m_i - k_i)$ for each $i \in I$ and hence $t \in l_R\big(R(m_i - k_i)\big) = l_R(M)$ by (2), so $l_R(A) \subseteq l_R(M)$. Thus K a$\ll$ M. ■

Next, properties and characterization of $J_a(M)$ are given.

Proposition 2.19: Let M be an R-module such that $AS_M \neq \phi$, then we have the following:

1. $J_a(M)$ is a submodule of M and contains every annihilator small submodule of M.
2. $J_a(M) = \{a_1 + a_2 + \cdots + a_n; a_i \in AS_M \text{ for each } i, n \geq 1\}$.
3. $J_a(M)$ is generated by AS_M.
4. If M is finitely generated, then J(M) $\subseteq J_a(M)$.

Proof:

1. Let $\{N_\lambda | \lambda \in \Lambda\}$ be the set of all annihilator small submodules of M, thus $J_a(M) = \sum_{\lambda\in\Lambda} N_\lambda$. Let $x,y \in J_a(M)$, this means that $x = \sum_{\lambda\in\Lambda} x_\lambda$ and $y = \sum_{\lambda\in\Lambda} y_\lambda$ where $x_\lambda, y_\lambda \in N_\lambda$ for each $\lambda \in \Lambda$ and $x_\lambda, y_\lambda \neq 0$ for at most a finite number of $\lambda \in \Lambda$. Then $x+y = \sum_{\lambda\in\Lambda}(x_\lambda + y_\lambda)$ such that $x_\lambda + y_\lambda \in N_\lambda$ for each $\lambda \in \Lambda$, $x+y \in J_a(M)$. Now, let $r \in R$ and $x \in J_a(M)$ it is an easy matter to see that $rx \in J_a(M)$. Hence, $J_a(M)$ is a submodule of M. it is clear from the definition of $J_a(M)$that it contains every a-small submodule of M.
2. Follows from (1) and $AS_M \subseteq J_a(M)$.
3. Since $AS_M \subseteq J_a(M)$, then $< AS_M > \subseteq J_a(M)$. Clearly, $J_a(M) \subseteq < AS_M >$. Hence, $J_a(M)$is generated by AS_M.
4. Since M is finitely generated then J(M) $\ll$ M, hence J(M) a$\ll$ M and by (1) J(M) $\subseteq J_a(M)$. ■

Proposition 2.20: Let M be an R-module such that $AS_M \neq \phi$. Then the following statements are equivalent:

1. AS_M is closed under addition; that is, a finite sum of a-small elements is a-small.
2. $J_a(M) = AS_M$.

Proof:

(1)$\Longrightarrow$(2) Let $a_1 + a_2 + \cdots + a_n \in J_a(M)$, $a_i \in A_i$ *i=1,…,n*, A_i is a-small in M for each *i=1,…,n.* then Ra_i a$\ll$ M by proposition (2.4). Hence $a_i \in AS_M$ for each *i=1,…,n*, by the assumption in (1) we get that $a_1 + \cdots + a_n \in AS_M$. thus $J_a(M) \subseteq AS_M$ and hence $J_a(M) = AS_M$.

(2)$\Longrightarrow$ (1) Let $x,y \in AS_M$, since $AS_M \subseteq J_a(M)$ then $x, y \in J_a(M)$ and by using proposition (2.19) we have $x + y \in J_a(M)$. Hence, $x + y \in AS_M$ (by our assumption); that is, AS_M is closed under addition. We can prove that a finite sum of annihilator small elements is annihilator small by the use of induction. ■

Proposition 2.21: Let M be an R-module such that $AS_M \neq \phi$. If considering the following statements:

1. $J_a(M)$ is an annihilator small submodule of M.
2. If K and L are annihilator small submodules of M, then K+L is an annihilator small submodule of M.
3. AS_M is closed under addition; that is, sum of annihilator small elements of M is annihilator small.
4. $J_a(M) = AS_M$.

Then $(1) \Longrightarrow (2) \Longrightarrow (3) \Longleftrightarrow (4)$. If M is finitely generated, then $(1) \Longleftrightarrow (2)$.

Proof:

(1)$\Longrightarrow$ (2) Let K,L be a-small in M, then K+L$\subseteq J_a(M)$ which is a-small by assumption. Thus by using proposition (2.4) we get K+L a$\ll$ M.

(2)$\Longrightarrow$ (3) Let $x, y \in AS_M$, then *Rx*, *Ry* are a-small in M, and hence by (2) *Rx*+*Ry* is annihilator small in M. But *R(x+y)*⊆*Rx*+*Ry* and by using proposition (2.4) we get *R(x+y)* a$\ll$ M. Hence, $x+y \in AS_M$.

(3)$\Longleftrightarrow$ (4) By proposition (2.20).

Now, let M be finitely generated to prove (2)$\Longrightarrow$ (1). Consider $\{m_1, m_2, \dots, m_n\}$ to be the set of generators of M. Let X be a submodule of M such that $J_a(M)$+X=M, then $m_i = a_i + x_i$ such that $a_i \in J_a(M)$ and $x_i \in$ X for each *i*=1,…,*n*. Thus $\sum_{i=1}^{n} Rm_i = \sum_{i=1}^{n} Ra_i + \sum_{i=1}^{n} Rx_i$ and hence M=$\sum_{i=1}^{n} Ra_i + X$. Now, since $a_i \in J_a(M)$ and since $(2) \Longrightarrow (3) \Longleftrightarrow (4)$ we get $J_a(M) = AS_M$; that is, $a_i \in AS_M$ and hence Ra_i a$\ll$ M thus $l_R(X) = l_R(M)$ implies that $J_a(M)$ a$\ll$ M. ■

Proposition 2.22: Let M be a finitely generated R-module and $J_a(M)$ a$\ll$ M. Then we have the following statements:

1. $J_a(M)$ is the largest annihilator small submodule of M.
2. $J_a(M) = \bigcap\{W | W$ is a maximal submodule of M with $J_a(M) \subseteq W\}$.

Proof:

1. Clear by the definition of $J_a(M)$.
2. Let a $\in \bigcap\{W|W$ is a maximal submodule of M with $J_a(M) \subseteq W\}$. Claim that *Ra* a$\ll$ M, if not then M=*Ra*+X where X is a submodule of M and $l_R(X) = l_R(M)$. Since $J_a(M)$ a$\ll$ M then $J_a(M)$+X≠M. But M is finitely generated, thus there exist a maximal submodule B of M such that $J_a(M) + X \subseteq$ B. Now, if a $\in$ B then B=M a contradiction! But a $\in \bigcap\{W|W$ is a maximal submodule of M with $J_a(M) \subseteq W\}$ a contradiction! Thus *Ra* a$\ll$M and hence a$\in J_a(M)$. Hence, $J_a(M) = \bigcap\{W|W$ is a maximal submodule of M with $J_a(M) \subseteq W\}$. ■

3. Fully annihilator small stable modules

Definition 3.1: An R-module M is called fully annihilator small stable; (briefly FASS-module), if every annihilator small submodule of it is stable.

Characterizations of FASS-modules are given in the following.

Proposition 3.2: Let M be an R-module. Then the following statements are equivalent:

1- M is a FASS-module.
2- Each a-small cyclic submodule of M is stable.
3- For each $x \in AS_M$, $y \in$ M if $l_R(x) \subseteq l_R(y)$ then $Ry \subseteq Rx$.
4- M satisfies Baer's criterion on a-small cyclic submodules.
5- $r_M(l_R(Rx)) = Rx$ for each $x \in AS_M$.

Proof:

(1)$\Longrightarrow$ (2) Obvious

(2)$\Longrightarrow$ (3) Let $x \in AS_M$, $y \in M$ such that $l_R(x) \subseteq l_R(y)$. Define $\theta: Rx \longrightarrow M$ by $\theta(rx) = ry$ if *rx=0* then $r \in l_R(x)$, hence $r \in l_R(y)$ and *ry=0*, this shows that θ is well-defined which is clear a homo. Now, since $x \in AS_M$ then Rx a$\ll$ M by definition of AS_M. Thus $\theta(Rx) \subseteq Rx$ implies that $Ry \subseteq Rx$.

(3)$\Longrightarrow$ (1) Let N be an a-small submodule of M and let $\alpha: N \longrightarrow M$ be an R-homomorphism. Now, let $y = \alpha(x) \in \alpha(N)$ then $x \in N$ and hence $Rx \subseteq N$ implies that Rx is a-small by proposition (2.4) and $x \in AS_M$. Now, let $r \in l_R(x) \Longrightarrow rx = 0 \Longrightarrow \alpha(rx) = 0 \Longrightarrow r(\alpha(x)) = 0 \Longrightarrow ry = 0 \Longrightarrow r \in l_R(y) \Longrightarrow l_R(x) \subseteq l_R(y) \Longrightarrow Ry \subseteq Rx \subseteq N$ and since in particular $y = 1.y \in Ry \subseteq Rx \subseteq N$ then $\alpha(N) \subseteq N$.

(2)$\Longrightarrow$ (4) Let Rx be a-small cyclic in M and let $\alpha: Rx \longrightarrow M$ be an R-homo. Then by (2) $\alpha(Rx) \subseteq Rx \Longrightarrow \forall\, n \in Rx, \alpha(n) \in Rx \Longrightarrow \forall\, n \in Rx\ \exists\, r \in R\ such\ that\ \alpha(n) = rn.$

(4)$\Longrightarrow$ (5) Let $y \in r_M(l_R(Rx))$, define $\theta: Rx \longrightarrow M$ by $\theta(rx) = ry$ if $r_1x = r_2x \Longrightarrow (r_1 - r_2)x = 0 \Longrightarrow (r_1 - r_2) \in l_R(x) \Longrightarrow (r_1 - r_2)y = 0 \Longrightarrow r_1y = r_2y \Longrightarrow \theta$ is well-defined and clearly a homo. Now, by assumption there exists $t \in R$ such that $\theta(w) = tw\ \forall\, w \in Rx$ since Rx a$\ll$ M. In particular, $\theta(x) = y = tx \in Rx \Longrightarrow y \in Rx \Longrightarrow r_M(l_R(Rx)) = Rx$.

(5)$\Longrightarrow$ (1) Let N be an a-small submodule of M and $\alpha: N \longrightarrow M$ be an R-homo. Suppose $y = \alpha(x) \in \alpha(N) \Longrightarrow x \in N \Longrightarrow Rx \subseteq N \Longrightarrow Rx\ a \ll M\ by\ (2.4) \Longrightarrow x \in AS_M \Longrightarrow let\ s \in l_R(Rx) \Longrightarrow s\alpha(x) = \alpha(sx) = \alpha(0) = 0 \Longrightarrow \alpha(x) \in r_M(l_R(Rx)) \Longrightarrow \alpha(x) \in Rx\ by\ assumption \Longrightarrow y = \alpha(x) \in N\ since\ Rx \subseteq N$, which implies that M is a FASS module. ∎

Proposition 3.3: Let M be an R-module such that $l_R(N \cap K) = l_R(N) + l_R(K)$ for every finitely generated a-small submodules N and K of M. Then M is a FASS module if and only if M satisfies baer's criterion on finitely generated a-small submodules of M.

Proof: $\Longrightarrow$) Let N be a finitely generated a-small submodule of M and let $f: N \longrightarrow M$ be an R-homomorphism. Now, $N = Rx_1 + Rx_2 + \cdots + Rx_n$ for some $x_1, \dots, x_n$ in N. Now, the proof goes by induction if n=1 then it is the same as for proposition (3.2). Assume that Baer's criterion holds for all a-small submodules generated by m elements for $m \leq n\text{-}1$, there exists two elements r, s in R such that $f(x)=rx$ for each $x \in Rx_1 + Rx_2 + \cdots + Rx_{n-1}$ and $f(x^*) = sx^*$ for each $x^* \in Rx_n$. Now, for each $y \in ((Rx_1 + Rx_2 + \cdots + Rx_{n-1}) \cap Rx_n)$ we have $ry=sy$ and hence $r\text{-}s \in l_R((Rx_1 + Rx_2 + \cdots + Rx_{n-1}) \cap Rx_n)$, thus by hypothesis there exists $u + v \in l_R(Rx_1 + \cdots + Rx_n) + l_R(Rx_n)$ such that $r\text{-}s=u+v$ and then $r\text{-}u=v+s=t$. For each $z \in N$, $z = \sum_{i=1}^{n} r_i x_i$ for some $r_i \in R$, $i=1,\dots,n$ and $f(z) = f(\sum_{i=1}^{n} r_i x_i) = f(\sum_{i=1}^{n-1} r_i x_i) + f(r_n x_n) = r(\sum_{i=1}^{n-1} r_i x_i) + s(r_n x_n) = r(\sum_{i=1}^{n-1} r_i x_i) - u(\sum_{i=1}^{n-1} r_i x_i) + s(r_n x_n) + v(r_n x_n) = (r-u)(\sum_{i=1}^{n-1} r_i x_i) + (s+v)(r_n x_n) = t(\sum_{i=1}^{n-1} r_i x_i) + t(r_n x_n) = t(\sum_{i=1}^{n} r_i x_i) = tz.$

$\Longleftarrow$) If Baer's criterion holds for a-small finitely generated submodules then it holds for a-small cyclic submodules and proposition (3.2) ends the discussion. ■

Proposition 3.4: Let M be a FASS R-module such that for each x in AS_M and left ideal I of R, every R-homo $\theta: Ix \longrightarrow M$ can be extended to an R-homomorphism $\alpha: Rx \longrightarrow M$. If any a-small submodule N of M satisfies the double annihilator condition; that is, $r_M\big(l_R(N)\big) = N$ then so does N+Rx.

Proof: Denote $l_R(N)$ and $l_R(Rx)$ by A and B respectively. Then by our assumption $r_M(A) = N$, and since M is a FASS module then $r_M(B) = Rx$. The proof of $N + Rx \subseteq r_M(l_R(N + Rx))$ is obvious, since $l_R(N + Rx) = l_R(N) \cap l_R(Rx) = A \cap B$. It is enough to show that $r_M(l_R(N + Rx) \subseteq N + Rx$. Now, let $y \in l_R(A \cap B)$ and define $\theta: Ax \longrightarrow M$ by $\theta(ax) = ay$ for each $a \in A$, if $ax=0$ then $a \in l_R(x) = B$ hence $a \in A \cap B$ and since $y \in r_M(A \cap B)$ then $ay=0$. Therefore, θ is a well-defined clearly a homo. The use of our assumption implies that there exists an extension $\alpha: Rx \longrightarrow M$ of θ, and $\alpha(Rx) \subseteq Rx$ since M is a FASS module implies that $a\alpha(x) = \alpha(ax) = ay$ for each a in A. Then $a(\alpha(x) - y) = 0$ implies that $\alpha(x) - y \in r_M(A) = N$; that is, there exists $n \in N$ such that $\alpha(x) - y = n$ or $y = n + \alpha(x) \in N + Rx$. Thus $N + Rx = r_M(l_R(N + Rx)$. ■

Proposition 3.5: Let M be an R-module such that for each $x \in AS_M$ and left ideal I of R, every R-homomorphism $\theta: Ix \longrightarrow M$ can be extended to an R-homomorphism $\alpha: Rx \longrightarrow M$. Then M is a FASS module if and only if each finitely generated a-small submodule of M satisfies the double annihilator condition.

Proof: The proof goes by induction as for n=1 it implies from proposition (3.2), and for n=m+1 it implies from proposition (3.4). ■

The following proposition gives properties of FASS modules.

Proposition 3.6: Let M be an R-module. consider the following statements:

1. M is a FASS module.
2. Every submodule N of M with $l_R(N) = l_R(M)$ is a FASS module.

3. Every 2-generated a-small submodule B of M with $l_R(B) = l_R(M)$ is a FASS module.
4. If N,K ⊆ M, K a≪ M and N is an epimorphic image of K then N ⊆ K.

Then (1)$\Leftrightarrow$ (2) $\Rightarrow$ (3), and (1)$\Leftrightarrow$ (4)

Proof: (1)$\Leftrightarrow$ (2) Necessity, Let N be a submodule of M such that $l_R(N) = l_R(M)$, let K a≪ N and $\alpha: K \longrightarrow N$ be an R-homo. Proposition (2.6) implies that K a≪ M and hence $i \circ \alpha(K) \subseteq K$ by M being a FASS module, where $i: N \longrightarrow M$ is the inclusion homomorphism. Thus $\alpha(K) \subseteq K$ and N is a FASS module.
Sufficiency, clear.
(2)$\Rightarrow$ (3) Obvious.
(1)$\Rightarrow$ (4) Let $x \in N$ and $\alpha: K \longrightarrow N$ be an R-epimorphism. Then $i \circ \alpha: K \longrightarrow M$ is an R-homomorphism, since K a≪ M then by (1) $i \circ \alpha(K) \subseteq K$ where $i: N \longrightarrow M$ is the inclusion homomorphism. Since N is an epimorphic image of K then for each x in N there exist y in K such that α(y)=x and hence N⊆ α(K)⊆K implies that N⊆K.
(4)$\Rightarrow$ (1) Let N a≪ M and $\alpha: N \longrightarrow M$ be an R-homo. Now, $\alpha: N \longrightarrow \alpha(N)$ is an epimorphism and using (4) we get $\alpha(N) \subseteq N$. ■

Next, the relation between the property of an R-module M being FASS and the commutativity of its endomorphism ring is discussed.

Proposition 3.7: Let R be a commutative ring and M a FASS R-module. Then $End_R(M)$ is commutative over elements in AS_M. Moreover, for each $x \in AS_M$ and $f \in End_R(M)$ there exists an element *r* in R (depends on *x*) such that *f(x)=rx*.

Proof: Let *f* and *g* be any two elements in $End_R(M)$ and let *x* belongs to AS_M, then R*x* is annihilator small in M and hence stable. But every stable submodule is fully invariant [1], which leads to that there exists two elements *r, s* in R such that *f(x)=rx* and *g(x)=sx* [8]. Now, *(f∘g)(x)=f(g(x)=f(sx)=r(sx)=s(rx) =g(rx)=g(f(x)=*$(g\circ f)(x)$*;* that is, $End_R(M)$ is commutative over elements in AS_M. ■

A natural question to ask is whether there exist conditions under which the converse true? Such a question leads us to define the concept of annihilator small regular modules as shown below.

Definition 3.8: An R-module M is called *annihilator small regular* if given any element in AS_M, then there exists *f*∈ $M^* = Hom_R(M, R)$ such that *m=f(m)m.* A ring R is called annihilator small regular if it is annihilator small module on itself.

There are bilinear functions:

$$\theta: M \times M^* \longrightarrow R$$
$$\psi: M^* \times M \longrightarrow End_R$$

Where $\theta(m, \alpha) = \alpha(m)\ \forall\ m \in AS_M, \alpha \in M^*$ and $\psi(\alpha, m) = \alpha(m)m\ \forall\ \alpha \in M^*, m \in AS_M$.

Proposition 3.9: Every commutative a-small regular ring R is a FASS ring.

Proof: For each a-small principal ideal $< r >$ of R and each R-homomorphism $f: < r > \longrightarrow R$, there exists an element $t \in R$ such that *r=rtr* and hence *f(r)=f(rtr)=rf(tr)=rtf(r)* implies that *f(< r >)⊆< r >*. ∎

Next, S will denote the ring of endomorphisms of M and Z_S denotes the center of *S.*

Lemma 3.10: Let $\alpha \in AS_{Z_S}$. Then there exists an element $\beta \in Z_S$ with $\alpha = \alpha\beta\alpha$ if and only if $M = \alpha(M) \oplus \ker(\alpha)$.

Proof: ⟹) Suppose that such a β exists. Set $\pi = \alpha\beta = \beta\alpha$ then $\pi^2 = (\beta\alpha)(\beta\alpha) = \beta(\alpha\beta\alpha) = \beta\alpha = \pi$. Now, $\alpha(M) \subseteq M$ thus $\pi(\alpha(M) \subseteq \pi(M)$ but $\pi(\alpha(M)) = \alpha\beta(\alpha(M)) = (\alpha\beta\alpha)(M) = \alpha(M)$ which implies that $\alpha(M) \subseteq \pi(M)$ and since $\beta(M) \subseteq M$, then $\alpha(\beta(M)) \subseteq \alpha(M)$ and $\pi(M) \subseteq \alpha(M)$ implies that $\pi(M) = \alpha(M)$. Moreover, $\ker(\alpha) = \ker(\pi)$ since $\pi = \alpha\beta$ *and* $\alpha = \pi\alpha$. Clearly, $M = \pi(M) \oplus (1-\pi)(M) = \pi(M) \oplus ker(\pi)$ and hence $M = \alpha(M) \oplus \ker(\alpha)$.
⟸) Suppose that $\alpha \in Z_S$ and $M = \alpha(M) \oplus ker(\alpha)$. Given any $m \in M$ then $m = \alpha(n) + k$ with $n \in M, k \in \ker(\alpha)$ and so we write $n = \alpha(n_1) + k_1$ with $n_1 \in M, k_1 \in \ker(\alpha)$. Thus $m = \alpha(\alpha(n_1) + k_1) + k$ and $\alpha(m) = \alpha(\alpha(\alpha(n_1) + k_1) + k) = \alpha(\alpha^2(n_1) + \alpha(k_1) + k) = \alpha^3(n_1) + \alpha(k) = \alpha^2(\alpha(n_1))$. Set $x_m = \alpha(n_1) \in \alpha(M)$, observe that x_m is the unique element of M such that $\alpha(m) = \alpha^2(x_m)$, for if $y \in \alpha(M)$ with $\alpha(m) = \alpha^2(y)$ then $\alpha^2(x_m - y) = 0$ so that $\alpha(x_m - y) \in \alpha(M) \cap \ker(\alpha) = (0)$ and hence $x_m - y \in \alpha(M) \cap \ker(\alpha) = (0)$ implies that $x_m = y$. It is easy to check that $x_{rm} = rx_m, x_{n+m} = x_n + x_m$ and $x_{\gamma(m)} = \gamma(x_m)$ for any $n, m \in M, r \in R$ *and* $\gamma \in S$. Consequently, there is a homomorphism $\beta \in S$ defined by $\beta(m) = x_m$. For any $m \in M$, $\alpha\beta\alpha(m) = \alpha(\beta\alpha(m)) = \alpha(x_{\alpha(m)}) = \alpha(\alpha(x_m)) = \alpha^2(x_m) = \alpha(m)$ so that $\alpha = \alpha\beta\alpha$. It remains only to show that $\beta \in Z_S$ which easy to show for given any $\gamma \in S$ and $m \in M, \beta(\gamma(m)) = x_{\gamma(m)} = \gamma(x_m) = \gamma(\beta(m))$. ∎

Proposition 3.11: Let M be an annihilator small regular R-module. Then Z_S is an annihilator small regular ring.

Proof: Let $\alpha \in AS_{Z_S}$ and given any $m \in AS_M$ choose $f \in M^*$ with $m = f(m)m$, hence $\alpha(m) = f(m)\alpha(m)$ *and* $\alpha(m) = \psi(f, \alpha(m))\alpha(m) = f(m)\alpha^2(m)$. Now, $m = f(m)\alpha(m) + (m - f(m)\alpha(m))$ then $f(m)\alpha(m) \in \alpha(M)$ *and* $m - f(m)\alpha(m) \in \ker(\alpha)$, since $\alpha(m - f(m)\alpha(m)) = \alpha(m) - f(m)\alpha^2(m) = 0$. So $M = \alpha(M) + \ker(\alpha)$. Now, let $\alpha(m) \in \alpha(M) \cap \ker(\alpha)$ then from above we get that $\alpha(m) = f(m)\alpha^2(m) = 0$ since $\alpha(\alpha(m)) = 0$. Hence, $\alpha(M) \cap \ker(\alpha) = (0)$ and thus $M = \alpha(M) \oplus \ker(\alpha)$ and by lemma (3.10) we get that Z_S is a small regular ring. ∎

Proposition 3.12: Let M be an R-module such that every a-small cyclic submodule is a direct summand. If S is commutative over elements in AS_M, then M is a FASS module.

Proof: let N be any a-small cyclic submodule of M and $f: N \longrightarrow M$ be an R-homomorphism, then there exists $m \in AS_M$ such that $N=Rm$. By our assumption Rm is a direct summand of M; that is, there exists a submodule L of M such that $M = Rm \oplus L$. Now, f can be extended to an R-endomorphism of M, $g: M \longrightarrow M$ by putting $g(l) = 0$ for each $l \in L$. Define $h: M \longrightarrow M$ by $h(x, y) = x$ for each $x \in Rm$ and $y \in L$. Let $f(x)=n_1 + l_1$ for some $n_1 \in N$ and $l_1 \in L$. Now, let $z \in AS_M$ then $z=x+l$ for some $x \in N$ and $l \in L$. Thus we have $(g \circ h)(z) = (g \circ h)(x + l) = g(x) = f(x) = n_1 + l_1$ and $(h \circ g)(z) = (h \circ g)(x + l) = h\big(f(x)\big) = h(n_1 + l_1) = n_1$,but S is commutative for each $z \in AS_M$ and hence $g \circ h = h \circ g$ implies that $l_1 = 0$ and $f(x) \in N$ $\forall\, x \in N$. Therefore, $f(N) \subseteq N$ and then M is a FASS module. ■

Lemma 3.13: Let M be an annihilator small regular R-module. Then every a-small cyclic submodule of M is a direct summand.

Proof: Let Rm be an a-small cyclic submodule of M, then $m \in AS_M$ and hence $m=\psi(f,m)m$ for some f in M^*, which implies that $M = Rm \oplus ker\,(\psi)$ [9]. ■

The proof of the following corollary is immediate.

Corollary 3.14: Let M be an annihilator small regular R-module. Then M is a FASS module if and only S is commutative over elements in AS_M.

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