Fully Annihilator Small Stable Modules

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Abstract

Let $R$ be an associative ring with non-zero identity and $M$ be a left $R$-module. A submodule $N$ of $M$ is called annihilator small (briefly $a$-small), if for every submodule $L$ of $M$ with $N+L=M$, then $l_R(L)=l_R(M)$. The properties of $a$-small submodules have been studied and characterizations of $a$-small cyclic submodules have been investigated. The sum of $a$-small submodules is studied. Moreover, we shall introduce fully annihilator small stable module (briefly FASS module) where $M$ is called a FASS module if every annihilator small submodule of $M$ is stable. Characterizations of FASS modules are proven.

Keywords: Annihilator small submodules, Fully stable modules, Annihilator small regular modules

1. Introduction

Throughout this work $R$ will denote an associative ring with non-zero identity, $M$ a left $R$-module. A submodule $N$ of $M$ is called small, if for every submodule $K$ of $M$ with $N+K=M$, then $l_R(K)=l_R(M)$ [5]. Recently, many authors have been interested in studying different kinds of $a$-small submodules as in [3] and [4], where the authors in [3] introduced the concept of $R$-annihilator small submodules, that is; a submodule $N$ of an $R$-module $M$ is called $R$-annihilator small, if whenever $N+K=M$, where $K$ a submodule of $M; then l_R(K)=0$. This has motivated us in turn to introduce the concept of annihilator small submodules, in way that a submodule $N$ of $M$ is called annihilator small (briefly $a$-small) in case $l_R(K)=l_R(M)$, where $K$ is a submodule of $M; whenever N+K=M. It is clear that every small submodule is $a$-small, but the converse is not true generally as examples can show next, while the two definitions become equal if $M$ is faithful, recalling that $M$ is called faithful in case $l_R(M) = 0. Remember that singular submodule of an $R$-module $M$ denoted by $Z(M) = \{m \in M \mid l_R(m) \text{ is essential in } R \}$ [5]. We shall study the properties of $a$-small
submodules, and define a subset of $M$ that consists of all annihilator small elements (denoted by $\text{AS}_M$), as well as; we shall denote the sum of all annihilator small submodules of $M$ by $J_\alpha(M)$, and study its properties and the relation between it and the Jacobson radical. Finally, we shall introduce the concept of fully annihilator small stable modules as a generalization of fully stable modules [1]. Recall that a submodule $N$ of an $R$-module $M$ is called stable in case for every $R$-homomorphism $\alpha: N \to M$ we have $\alpha(N) \subseteq N$ and $M$ is called fully stable if every submodule of $M$ is stable. Characterizations and properties of this concept is studied involving the satisfaction of Baer’s criterion on a-small cyclic submodules and its effect on $M$ being a FASS module. Recall that, a submodule $N$ of $M$ is said to satisfy Baer’s criterion if for each $\beta: N \to M$ there exists an element $r \in R$ such that $\beta(n) = rn$ for each $n \in N$ [1]. In this paper, we are also interested to study the relation between $M$ being a FASS module and $\text{End}_R(M)$ being commutative.

2. Annihilator small submodules

**Definition 2.1:** A submodule $N$ of an $R$-module $M$ is called annihilator small (briefly $a$-small) in $M$, and denoted by $N \ll M$; if whenever $N + K = M$ for each submodule $K$ of $M$, then $l_R(K) = l_R(M)$. Where $l_R$ denotes the left annihilators in $R$. A left ideal $I$ of $R$ is annihilator small if for each left ideal $J$ of $R$ with $I + J = R$, implies that $l_R(J) = 0$.

**Examples and remarks 2.2:**
1. It is clear that every small submodule is annihilator small, but the converse is not true generally. For example, in the $\mathbb{Z}$-module $\mathbb{Z}$, $(0)$ is the only small submodule while for every $n > 1$, there exists $m$ such that $n\mathbb{Z} + m\mathbb{Z} = \mathbb{Z}$ and $l_\mathbb{Z}(m\mathbb{Z}) = 0 = l_\mathbb{Z}(\mathbb{Z})$.
2. If $M$ is a faithful $R$-module then the concepts of annihilator small submodules and $R$-annihilator small submodules are equivalent.
3. There are annihilator small submodules that are direct summands as in the $\mathbb{Z}_2$-module $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, where it is clear that $A = \mathbb{Z}_2 \oplus (0)$ is a direct summand of $M$, $M = A \oplus \mathbb{Z}_2 = A \oplus (\bar{1}, \bar{1})$ and $l_{\mathbb{Z}_2}(\mathbb{Z}_2) = 0 = l_{\mathbb{Z}_2}(\bar{1}, \bar{1})$.

Recall that, $M$ is called **prime** if $l_R(N) = l_R(M)$ for every non-zero submodule $N$ of $M[5]$. $M$ is called **quasi-Dedekind** if $\text{Hom}(M/N, M) = 0$ for every proper submodule $N$ of $M[6]$, it is mentioned in [6] that every quasi-Dedekind module is prime.

The proof of the following proposition is obvious.

**Proposition 2.3:** Let $M$ be a prime $R$-module. Then every proper submodule of $M$ is annihilator small. In particular, every proper submodule of a quasi-Dedekind $R$-module is annihilator small.

It is mentioned in [6, p.25] that $\mathbb{Q}$ as $\mathbb{Z}$-module is quasi-Dedekind, and hence by the use of proposition (2.3) we get that every proper submodule of $\mathbb{Q}$ is annihilator small, but only finitely generated submodules of $\mathbb{Q}$ are small.
**Proposition 2.4:** Let M be an R-module with submodules $A \subseteq N$. If $N \triangleleft M$ then $A \triangleleft M$.

**Proof:** Let X be a submodule of M such that $A + X = M$, since $A \subseteq N$ hence $N + X = M$. By N being a-small in M then $l_R(X) = l_R(M)$ and hence $A \triangleleft M$. ■

**Proposition 2.5:** Let M be an R-module with submodules $A \subseteq N$, if $A \triangleleft N$ and $l_R(N) = l_R(M)$ then $A \triangleleft M$.

**Proof:** Let X be any submodule of M such that $A + X = M$, now $N \cap M = N \cap (A + X)$ implies that $N = A + (N \cap X)$ by the modular law. Since $A \triangleleft N$, thus $l_R(N \cap X) = l_R(N)$. But $l_R(X) \subseteq l_R(N \cap X) = l_R(N) = l_R(M)$ implies that $l_R(X) \subseteq l_R(M)$ and then $l_R(X) = l_R(M)$, hence $X \triangleleft M$. ■

**Proposition 2.6:** Let M and N be R-modules and $\alpha: M \rightarrow N$ an R-monomorphism if $W \triangleleft M$ then $\alpha(W) \triangleleft \alpha(M)$.

**Proof:** Let U be a submodule of N such that $\alpha(W) + U = \alpha(M)$, now $U \subseteq N$ implies $\alpha^{-1}(U) \subseteq \alpha^{-1}(N) = M$ and $\alpha(\alpha^{-1}(U)) = U \cap \text{Im}(\alpha) = U \cap \alpha(M) = U$. Now, $\alpha^{-1}(\alpha(W)) \subseteq \alpha^{-1}(\alpha(M))$ and then $W + \alpha^{-1}(U) = M$ this implies that $l_R(\alpha^{-1}(U)) = l_R(M)$ since $W \triangleleft M$. Let $X = \alpha^{-1}(U)$ then $l_R(X) = l_R(M)$. Let $r \in l_R(U) = l_R(\alpha(X))$, thus $r \alpha(X) = 0 \implies rX = 0 \implies r \in l_R(X) \implies l_R(U) \subseteq l_R(X) = l_R(M) \implies l_R(U) = l_R(M) \subseteq l_R(\alpha(M)) \implies l_R(U) = l_R(\alpha(M))$. Hence, $\alpha(W) \triangleleft \alpha(M)$. ■

**Corollary 2.7:** Let M and N be R-modules and $\alpha: M \rightarrow N$ an R-monomorphism such that $l_R(\alpha(M)) = l_R(N)$, if $W \triangleleft M$ then $\alpha(W) \triangleleft N$.

In the same manner of the definition of Jacobson radical related to small submodules, we will state a definition related to annihilator small submodules in the following. But first we need this definition.

**Definition 2.8:** Let M be an R-module and $a \in M$. We say that an element a in M is annihilator small if $Ra$ is annihilator small submodule of M. let $AS_M = \{a \in M | Ra \triangleleft M\}$.

Note that $AS_M$ is not a submodule of M. In fact, it is not closed under addition, for example in the $\mathbb{Z} - module$ $\mathbb{Z}$ we have that $3, -2 \in AS_\mathbb{Z}$ but $3 - 2 = 1 \notin AS_\mathbb{Z}$

We can see by the use of proposition (2.4) that if M is an R-module and $a \in AS_M$, then $Ra \subseteq AS_M$. Moreover, if $A \triangleleft M$ then $A \subseteq AS_M$.

**Definition 2.9:** Let M be an R-module. Denote $I_a(M)$ for the sum of all annihilator small submodules of M.
It is clear that $AS_M \subseteq J_a(M)$ for every $R$-module $M$. The $\mathbb{Z} - module \mathbb{Z}$ is an example of this inclusion being proper, where $n\mathbb{Z}$ is a-small for each $n \neq 1, -1$ in $\mathbb{Z}$, hence $J_a(\mathbb{Z}) = \sum_{n \mathbb{Z} \ll \mathbb{Z}} n\mathbb{Z} = \mathbb{Z}$, but $AS_{\mathbb{Z}} = \{n \in \mathbb{Z}| n\mathbb{Z} \ll \mathbb{Z}\} = \{n\mathbb{Z}| n \neq 1, -1\}$.

Recall that, if $T$ is an arbitrary proper submodule of a right $R$-module $M$ and $N$ a submodule of $M$, then $N$ is called $T$-essential provided that $N \not\subseteq T$ and for each submodule $K$ of $M$, $N \cap K \subseteq T$ implies that $K \subseteq T$ [8].

We introduce the following singularity of modules.

**Definition 2.10:** Let $M$ be an $R$-module and $J$ be an arbitrary left ideal of $R$. define the subset $Z(J,M)$ of $M$ by $Z(J,M)=\{x \in M| l_R(x) \text{ is } J\text{-essential in } R\}$, it is easy to show that $Z(J,M)=\{x \in M| l_R(x) \text{ is } J\text{-essential in } R\}$. It is clear that $Z(0,M)=Z(\mathbb{M})$ for any $R$-module $M$.

**Proposition 2.11:** Let $M$ be an $R$-module and $J$ an arbitrary proper left ideal of $R$. Then $Z(J,M)$ is a submodule of $M$, and it is called the singular submodule of $M$ relative to $J$.

**Proof:** It is clear that $Z(J,M)$ is non-empty. Let $x,y \in Z(J,M)$, then there exist two $J$-essential left ideals $A$ and $B$ of $R$ with $Ax=0$ and $By=0$. Now, $A \cap B$ is $J$-essential and $(A\cap B)(x-y)=0$ [7] and thus $x-y \in Z(J,M)$. For each $r \in R$, since $l_R(x) \subseteq l_R(rx)$ and $l_R(x)$ is $J$-essential in $R$ hence $rx \in Z(J,M)$.

**Lemma 2.12:** Let $M$ be a non-zero $R$-module and $N$ a submodule of $M$. If $l_R(N)$ is $l_R(M)$-essential in $R$, then $r_M(l_R(N))$ is a-small in $M$; in particular, $N$ is a-small in $M$.

**Proof:** Let $X$ be a submodule of $M$ with $X+r_M(l_R(N))=M$. Then $l_R(X) \cap l_R(\ell_M(l_R(N))) = l_R(X) \cap l_R(N) = l_R(M)$, since $l_R(N)$ is $l_R(M)$-essential in $R$ then $l_R(X) \subseteq l_R(M)$ and hence $r_M(l_R(N))$ is a-small in $M$. The last assertion follows from proposition (2.4).

**Corollary 2.13:** Let $M$ be a non-zero $R$-module. If $m \in Z(l_R(M),M)$, then $Rm$ is a-small in $M$.

**Proof:** Let $m \in Z(l_R(M),M)$. Then $l_R(m)$ is $l_R(M)$-essential in $R$, and by lemma (2.12) we have $Rm$ is a-small in $M$.

Note that the converse of lemma(2.12) is true if $r_M(A \cap B) = r_M(A) + r_M(B)$ for each left ideals $A$ and $B$ of $R$. For this, let $T$ be a left ideal of $R$ with $l_R(N) \cap T \subseteq l_R(M)$. Then

$$M \subseteq r_M(l_R(M)) \subseteq r_M(l_R(N) \cap T) = r_M(l_R(N)) + r_M(T).$$

Since $r_M(l_R(N))$ is a-small in $M$, then $T \subseteq l_R(r_M(T)) \subseteq l_R(M)$. This shows that $l_R(N)$ is $l_R(M)$-essential in $R$. 


Proposition 2.14: Let M be a non-zero finitely generated R-module and K a submodule of M. If K is a-small in M, then so is K+J(M)+Z(J,M) where J=l_R(M).

Proof: Let X be a submodule of M such that K+J(M)+Z(J,M)+X=M. Since M is finitely generated, then \{m_i\}_{i=1}^n is a set of generators of M and M=\sum_{i=1}^n Rm_i, and J(M) is small in M; that is, K+Z(J,M)+X=M. Now, for each m_i \in M we have m_i = k_i + z_i + x_i where k_i \in K, z_i \in Z(J,M) and x_i \in X for each i=1,\ldots,n. Thus M=K+\sum_{i=1}^n Rz_i+X and since K is a-small in M by our assumption.

Thus \(l_R(M)=l_R(\sum_{i=1}^n Rz_i+X)=l_R(\sum_{i=1}^n Rz_i) \cap l_R(X)=(\cap_{i=1}^n l_R(Rz_i)) \cap (l_R(X))\). But z_i \in Z(J,M), thus l_R(z_i) is l_R(M)-essential in R for each i=1,\ldots,n, and hence \(\cap_{i=1}^n l_R(Rz_i)\) is l_R(M)-essential in R [2]. Thus \(l_R(X) \subseteq l_R(M)\), and hence K+J(M)+Z(J,M) is a-small submodule of M.

Corollary 2.15: Let M be a finitely generated R-module. Then J(M)+Z(J,M) is a-small in M where J=l_R(M).

The proof of the following proposition is as that in lemma (2.12).

Proposition 2.16: let M be an R-module such that Z(J,M) is finitely generated. If K is an a-small submodule of M, then so is K+Z(J,M).

In the following we give a characterization of cyclic annihilator small submodules.

Theorem 2.17: Let M be an R-module and m \in M. Then the following statements are equivalent:
1. Rm \triangleleft\triangleleft M.
2. \(\cap_{i \in I} l_R(m_i-r_i m) = l_R(M)\) for each \(r_i \in R\).
3. There exists \(j \in I\) such that \(rm_j \notin Rrm\) for all \(r \notin l_R(M)\).

Proof: (1)⇒ (2) For each \(i \in I\), \(m_i = m_i - r_i m + r_i m\) and hence \(M=\sum_{i \in I} R(m_i - r_i m) + Rm\). By (1) we have \(l_R(M) = l_R(\sum_{i \in I} R(m_i - r_i m)) = \cap_{i \in I} l_R(m_i - r_i m)\).

(2)⇒ (1) Let X be a submodule of M with X+Rm=M. Then for each \(i \in I\), \(m_i = x_i + r_i m\), \(r_i \in R\) and \(x_i \in X\). Let \(t \in l_R(X)\), then \(tm_i = tr_i m + tx_i = l_R(M)\).

(2)⇒ (3) Let \(r \notin l_R(M)\) and assume that \(rm_i \in Rrm\) for all \(i \in I\). Then \(r m_i = r i m = r m_i\) for all \(i \in I\), so by (1) \(r \in \cap_{i \in I} l_R(m_i - r_i m) = l_R(M)\) which is a contradiction.

(3)⇒ (2) Let \(r \in \cap_{i \in I} l_R(m_i - r_i m)\) and hence \(r \in l_R(m_i - r_i m)\) for all \(i \in I\). Thus \(rm_i = r_i m\) for all \(i \in I\), so \(r m_i \in Rrm\). By (2) \(r \in l_R(M)\) and hence \(\cap_{i \in I} l_R(m_i - r_i m) \subseteq l_R(M)\) and \(\cap_{i \in I} l_R(m_i - r_i m) = l_R(M)\) for all \(r_i \in R\).

Theorem 2.18: Let R be a commutative ring, M=\(\sum_{i \in I} Rm_i\) and K a submodule of M. Then the following statements are equivalent:
1. K \triangleleft\triangleleft M.
2. \(\cap_{i \in I} l_R(m_i - k_i) = l_R(M)\) for all \(k_i \in K\).
Proof: (1) \(\Rightarrow\) (2) For each \(i \in I\), let \(k_i \in K\). Then \(m_i = m_i - k_i + k_i\) for each \(i \in I\). Then \(M = \sum_{i \in I} R(m_i - k_i) + K\), by (1) we obtain \(l_R(M) = l_R(\sum_{i \in I} R(m_i - k_i)) = l_R(R(m_i - k_i))\).

(2)\(\Rightarrow\) (1) Let \(A\) be a submodule of \(M\) with \(M = A + K\). Then for each \(i \in I\) \(m_i = a_i + k_i\) where \(a_i \in A\) and \(k_i \in K\). Hence \(a_i = m_i - k_i\) for each \(i \in I\) and \(M = \sum_{i \in I} R(m_i - k_i) + K\). Now, let \(t \in l_R(A)\) then \(ta_i = t(m_i - k_i)\) for each \(i \in I\) and hence \(t \in l_R(R(m_i - k_i)) = l_R(M)\) by (2), so \(l_R(A) \subseteq l_R(M)\). Thus \(K \triangleleft M\).

Next, properties and characterization of \(J_a(M)\) are given.

Proposition 2.19: Let \(M\) be an \(R\)-module such that \(AS_M \neq \emptyset\), then we have the following:
1. \(J_a(M)\) is a submodule of \(M\) and contains every annihilator small submodule of \(M\).
2. \(J_a(M) = \{a_1 + a_2 + \ldots + a_n; a_i \in AS_M\ for each i, n \geq 1\}\).
3. \(J_a(M)\) is generated by \(AS_M\).
4. If \(M\) is finitely generated, then \(J(M) \subseteq J_a(M)\).

Proof:
1. Let \(\{N_\lambda | \lambda \in \Lambda\}\) be the set of all annihilator small submodules of \(M\), thus \(J_a(M) = \sum_{\lambda \in \Lambda} N_\lambda\). Let \(x, y \in J_a(M)\), this means that \(x = \sum_{\lambda \in \Lambda} x_\lambda \) and \(y = \sum_{\lambda \in \Lambda} y_\lambda \) where \(x_\lambda, y_\lambda \in N_\lambda\) for each \(\lambda \in \Lambda\) and \(x_\lambda, y_\lambda \neq 0\) for at most a finite number of \(\lambda \in \Lambda\). Then \(x + y = \sum_{\lambda \in \Lambda} (x_\lambda + y_\lambda)\) such that \(x_\lambda + y_\lambda \in N_\lambda\) for each \(\lambda \in \Lambda\), \(x + y \in J_a(M)\). Hence, \(J_a(M) \subseteq \text{a submodule of } M\). it is clear from the definition of \(J_a(M)\) that it contains every a-small submodule of \(M\).
2. Follows from (1) and \(AS_M \subseteq J_a(M)\).
3. Since \(AS_M \subseteq J_a(M)\), then \(<AS_M> \subseteq J_a(M)\). Clearly, \(J_a(M) \subseteq <AS_M>\).
4. Hence, \(J_a(M)\) is generated by \(AS_M\).
4. Since \(M\) is finitely generated then \(J(M) \triangleleft M\), hence \(J(M) \triangleleft M\) and by (1) \(J(M) \subseteq J_a(M)\).

Proposition 2.20: Let \(M\) be an \(R\)-module such that \(AS_M \neq \emptyset\). Then the following statements are equivalent:
1. \(AS_M\) is closed under addition; that is, a finite sum of a-small elements is a-small.
2. \(J_a(M) = AS_M\).

Proof:
(1)\(\Rightarrow\) (2) Let \(a_1 + a_2 + \ldots + a_n \in J_a(M)\), \(a_i \in A_i\ i = 1, \ldots, n\), \(A_i\) is a-small in \(M\) for each \(i = 1, \ldots, n\). Then \(RA_i \triangleleft M\) by proposition (2.4). Hence \(a_i \in AS_M\) for each \(i = 1, \ldots, n\), by the assumption in (1) we get that \(a_1 + \cdots + a_n \in AS_M\). thus \(J_a(M) \subseteq AS_M\) and hence \(J_a(M) = AS_M\).
(2)\(\Rightarrow\) (1) Let \(x, y \in AS_M\), since \(AS_M \subseteq J_a(M)\) then \(x, y \in J_a(M)\) and by using proposition (2.19) we have \(x + y \in J_a(M)\). Hence, \(x + y \in AS_M\) (by our assump-
tion); that is, $\text{AS}_M$ is closed under addition. We can prove that a finite sum of annihilator small elements is annihilator small by the use of induction.

**Proposition 2.21:** Let $M$ be an $R$-module such that $\text{AS}_M \neq \emptyset$. If considering the following statements:

1. $J_a(M)$ is an annihilator small submodule of $M$.
2. If $K$ and $L$ are annihilator small submodules of $M$, then $K+L$ is an annihilator small submodule of $M$.
3. $\text{AS}_M$ is closed under addition; that is, sum of annihilator small elements of $M$ is annihilator small.
4. $J_a(M) = \text{AS}_M$.

Then $(1) \implies (2) \implies (3) \iff (4)$. If $M$ is finitely generated, then $(1) \iff (2)$.

**Proof:**

$(1) \implies (2)$ Let $K,L$ be a-small in $M$, then $K+L \subseteq J_a(M)$ which is a-small by assumption. Thus by using proposition (2.4) we get $K+L \ll M$.

$(2) \implies (3)$ Let $x,y \in AS_M$, then $Rx, Ry$ are a-small in $M$, and hence by $(2) Rx + Ry$ is annihilator small in $M$. But $R(x+y) \subseteq Rx + Ry$ and by using proposition (2.4) we get $R(x+y) \ll M$. Hence, $x+y \in AS_M$.

$(3) \iff (4)$ By proposition (2.20).

Now, let $M$ be finitely generated to prove $(2) \implies (1)$. Consider

$\{m_1,m_2,\ldots,m_n\}$ to be the set of generators of $M$. Let $X$ be a submodule of $M$ such that $J_a(M)+X=M$, then $m_i = a_i + x_i$ such that $a_i \in J_a(M)$ and $x_i \in X$ for each $i=1,\ldots,n$. Thus $\sum_{i=1}^n Rm_i = \sum_{i=1}^n Ra_i + \sum_{i=1}^n Rx_i$ and hence $M=\sum_{i=1}^n Ra_i + X$.

Now, since $a_i \in J_a(M)$ and since $(2) \implies (3) \iff (4)$ we get $J_a(M) = AS_M$; that is, $a_i \in AS_M$ and hence $Ra_i \ll M$ thus $l_R(X) = l_R(M)$ implies that $J_a(M) \ll M$.

**Proposition 2.22:** Let $M$ be a finitely generated $R$-module and $J_a(M) \ll M$. Then we have the following statements:

1. $J_a(M)$ is the largest annihilator small submodule of $M$.
2. $J_a(M) = \cap\{W | W$ is a maximal submodule of $M$ with $J_a(M) \subseteq W\}$.

**Proof:**

1. Clear by the definition of $J_a(M)$.
2. Let $a \in \cap\{W | W$ is a maximal submodule of $M$ with $J_a(M) \subseteq W\}$. Claim that $Ra \ll M$, if not then $M=Ra+X$ where $X$ is a submodule of $M$ and $l_R(X) = l_R(M)$. Since $J_a(M) \ll M$ then $J_a(M)+X \neq M$. But $M$ is finitely generated, thus there exist a maximal submodule $B$ of $M$ such that $J_a(M) + X \subseteq B$. Now, if $a \in B$ then $B=M$ a contradiction! But $a \in \cap\{W | W$ is a maximal submodule of $M$ with $J_a(M) \subseteq W\}$ a contradiction! Thus $Ra \ll M$ and hence $a \in J_a(M)$. Hence, $J_a(M) = \cap\{W | W$ is a maximal submodule of $M with $J_a(M) \subseteq W\}$. ■
3. Fully annihilator small stable modules

Definition 3.1: An R-module M is called fully annihilator small stable; (briefly FASS-module), if every annihilator small submodule of it is stable.

Characterizations of FASS-modules are given in the following.

Proposition 3.2: Let M be an R-module. Then the following statements are equivalent:
1- M is a FASS-module.
2- Each a-small cyclic submodule of M is stable.
3- For each $x \in AS_M$, $y \in M$ if $l_R(x) \subseteq l_R(y)$ then $Ry \subseteq Rx$.
4- M satisfies Baer’s criterion on a-small cyclic submodules.
5- $r_M(l_R(Rx)) = Rx$ for each $x \in AS_M$.

Proof:
(1)$\implies$ (2) Obvious
(2)$\implies$ (3) Let $x \in AS_M$, $y \in M$ such that $l_R(x) \subseteq l_R(y)$. Define $\theta: Rx \to M$ by $\theta(rx) = ry$ if $rx=0$ then $r \in l_R(x)$, hence $r \in l_R(y)$ and $ry=0$, this shows that $\theta$ is well-defined which is a hom. Now, since $x \in AS_M$ then $Rx \trianglelefteq M$ by definition of $AS_M$. Thus $\theta(Rx) \subseteq Rx$ implies that $Ry \subseteq Rx$.
(3)$\implies$ (1) Let N be an a-small submodule of M and let $\alpha: N \to M$ be an R-homomorphism. Now, let $y = \alpha(x) \in \alpha(N)$ then $x \in N$ and hence $Rx \subseteq N$ implies that $Rx$ is a-small by proposition (2.4) and $x \in AS_M$. Now, let $r \in l_R(x) \implies rx = 0 \implies \alpha(rx) = 0 \implies r(\alpha(x)) = 0 \implies ry = 0 \implies r \in l_R(y) \implies l_R(x) \subseteq l_R(y) \implies Ry \subseteq Rx \subseteq N$ and since in particular $y = 1 \cdot y \in Ry \subseteq Rx \subseteq N$ then $\alpha(N) \subseteq N$.
(2)$\implies$ (4) Let Rx be a small cyclic in M and let $\alpha: Rx \to M$ be an R-homo. Then by (2) $\alpha(Rx) \subseteq Rx \implies \forall n \in Rx, \alpha(n) \in Rx \implies \forall n \in Rx \exists r \in R$ such that $\alpha(n) = rn$.
(4)$\implies$ (5) Let $y \in r_M(l_R(Rx))$, define $\theta: Rx \to M$ by $\theta(rx) = ry$ if $r_1x = r_2x \implies (r_1 - r_2)x = 0 \implies r_1 - r_2) \in l_R(x) \implies (r_1 - r_2)y = 0 \implies r_1y = r_2y \implies \theta$ is well-defined and clearly a homo. Now, by assumption there exists $t \in R$ such that $\theta(w) = tw \forall w \in Rx$ since $Rx \trianglelefteq M$. In particular, $\theta(x) = y = tx \in Rx \implies y \in Rx \implies r_M(l_R(Rx)) = Rx$.
(5)$\implies$ (1) Let N be an a-small submodule of M and $\alpha: N \to M$ be an R-homo. Suppose $y = \alpha(x) \in \alpha(N) \implies x \in N \implies Rx \subseteq N \implies Rx \trianglelefteq M$ by (2.4) $\implies x \in AS_M \implies$ let $s \in l_R(Rx) \implies s(x) = s(0) = 0 \implies \alpha(x) \in r_M(l_R(Rx)) \implies \alpha(x) \in Rx$ by assumption $\implies y = \alpha(x) \in N$ since $Rx \subseteq N$, which implies that M is a FASS module.

Proposition 3.3: Let M be an R-module such that $l_R(N \cap K) = l_R(N) + l_R(K)$ for every finitely generated a-small submodules N and K of M. Then M is a FASS module if and only if M satisfies baer’s criterion on finitely generated a-small submodules of M.
Proof: $\implies$) Let $N$ be a finitely generated a-small submodule of $M$ and let $f: N \to M$ be an $R$-homomorphism. Now, $N = Rx_1 + Rx_2 + \cdots + Rx_n$ for some $x_1, \ldots, x_n$ in $N$. Now, the proof goes by induction if $n=1$ then it is the same as for proposition (3.2). Assume that Baer’s criterion holds for all a-small submodules generated by $m$ elements for $m \leq n-1$, there exists two elements $r, s$ in $R$ such that $f(x) = rx$ for each $x \in Rx_1 + Rx_2 + \cdots + Rx_{n-1}$ and $f(x^*) = sx^*$ for each $x^* \in Rx_n$. Now, for each $y \in ((Rx_1 + Rx_2 + \cdots + Rx_{n-1}) \cap Rx_n)$ we have $ry = sy$ and hence $r-s \in l_R((Rx_1 + Rx_2 + \cdots + Rx_{n-1}) \cap Rx_n)$, thus by hypothesis there exists $u + v \in l_R((Rx_1 + \cdots + Rx_n) + l_R(Rx_n)$ such that $r-s = u + v$ and then $r-u + v = t$. For each $z \in l_R(N) = \sum_{i=1}^n r_i x_i$ for some $r_i \in R$, $i=1,\ldots, n$ and $f(z) = f(\sum_{i=1}^n r_i x_i) = f(\sum_{i=1}^{n-1} r_i x_i) + f(r_n x_n) = r(\sum_{i=1}^{n-1} r_i x_i) + s(r_n x_n) = r(\sum_{i=1}^{n-1} r_i x_i) - u(\sum_{i=1}^{n-1} r_i x_i) + s(r_n x_n) + v(r_n x_n) = (r - u)(\sum_{i=1}^{n-1} r_i x_i) + (s + v)(r_n x_n) = t(\sum_{i=1}^{n-1} r_i x_i) + t(r_n x_n) = t(\sum_{i=1}^n r_i x_i) = tz$.

$\impliedby$) If Baer’s criterion holds for a small finitely generated submodules then it holds for small cyclic submodules and proposition (3.2) ends the discussion.

Proposition 3.4: Let $M$ be a FASS $R$-module such that for each $x$ in $AS_M$ and left ideal $I$ of $R$, every R-homo $\theta : Ix \to M$ can be extended to an $R$-homomorphism $\alpha : Rx \to M$. If any a-small submodule $N$ of $M$ satisfies the double annihilator condition; that is, $r_M(l_R(N)) = N$ then so does $N+Rx$.

Proof: Denote $l_R(N)$ and $l_R(Rx)$ by $A$ and $B$ respectively. Then by our assumption $r_M(A) = N$, and since $M$ is a FASS module then $r_M(B) = Rx$. The proof of $N + Rx \subseteq l_R(N + Rx)$ is obvious, since $l_R(N + Rx) = l_R(N) \cap l_R(Rx) = A \cap B$. It is enough to show that $r_M(l_R(N + Rx) \subseteq N + Rx$. Now, let $y \in l_R(A \cap B)$ and define $\theta : Ax \to M$ by $\theta(ax) = ay$ for each $a \in A$, if $a = 0$ then $a \in l_R(x) = B$ hence $a \in A \cap B$ and since $y \in r_M(A \cap B)$ then $ay = 0$. Therefore, $\theta$ is a well-defined clearly a homo. The use of our assumption implies that there exists an extension $\alpha : Rx \to M$ of $\theta$, and $\alpha(Rx) \subseteq Rx$ since $M$ is a FASS module implies that $a\alpha(x) = \alpha(ax) = ay$ for each $a$ in $A$. Then $a(\alpha(x) - y) = 0$ implies that $\alpha(x) - y \in r_M(A) = N$; that is, there exists $n \in N$ such that $\alpha(x) - y = n + \alpha(x) \in N + Rx$. Thus $N + Rx = r_M(l_R(N + Rx)$.

Proposition 3.5: Let $M$ be an $R$-module such that for each $x \in AS_M$ and left ideal $I$ of $R$, every $R$-homomorphism $\theta : Ix \to M$ can be extended to an $R$-homomorphism $\alpha : Rx \to M$. Then $M$ is a FASS module if and only if each finitely generated a-small submodule of $M$ satisfies the double annihilator condition.

Proof: The proof goes by induction as for $n=1$ it implies from proposition (3.2), and for $n=m+1$ it implies from proposition (3.4).

The following proposition gives properties of FASS modules.

Proposition 3.6: Let $M$ be an $R$-module. consider the following statements:

1. $M$ is a FASS module.
2. Every submodule $N$ of $M$ with $l_R(N) = l_R(M)$ is a FASS module.
3. Every 2-generated a-small submodule B of M with \( l_R(B) = l_R(M) \) is a FASS module.
4. If \( N, K \subseteq M \), \( K \ll M \) and \( N \) is an epimorphic image of \( K \) then \( N \subseteq K \).
Then \( (1) \implies (2) \implies (3) \), and \( (1) \iff (4) \)

**Proof:** (1)\( \iff \) (2) Necessity, Let \( N \) be a submodule of \( M \) such that \( l_R(N) = l_R(M) \), let \( K \ll N \) and \( \alpha : K \to N \) be an R-homo. Proposition (2.6) implies that \( K \ll M \) and hence \( i \circ \alpha(K) \subseteq K \) by \( M \) being a FASS module, where \( i : N \to M \) is the inclusion homomorphism. Thus \( \alpha(K) \subseteq K \) and \( N \) is a FASS module.
Sufficiency, clear.
(2)\( \implies \) (3) Obvious.
(1)\( \implies \) (4) Let \( x \in N \) and \( \alpha : K \to N \) be an R-epimorphism. Then \( i \circ \alpha : K \to M \) is an R-homomorphism, since \( K \ll M \) then by (1) \( i \circ \alpha(K) \subseteq K \) where \( i : N \to M \) is the inclusion homomorphism. Since \( N \) is an epimorphic image of \( K \) then for each \( x \) in \( N \) there exist \( y \) in \( K \) such that \( \alpha(y) = x \) and hence \( N \subseteq \alpha(K) \subseteq K \) implies that \( N \subseteq K \).
(4)\( \implies \) (1) Let \( N \ll M \) and \( \alpha : N \to M \) be an R-homo. Now, \( \alpha : N \to \alpha(N) \) is an epimorphism and using (4) we get \( \alpha(N) \subseteq N \).

Next, the relation between the property of an \( R \)-module \( M \) being FASS and the commutativity of its endomorphism ring is discussed.

**Proposition 3.7:** Let \( R \) be a commutative ring and \( M \) a FASS \( R \)-module. Then \( \text{End}_R(M) \) is commutative over elements in \( AS_M \). Moreover, for each \( x \in AS_M \) and \( f \in \text{End}_R(M) \) there exists an element \( r \) in \( R \) (depends on \( x \)) such that \( f(x) = rx \).

**Proof:** Let \( f \) and \( g \) be any two elements in \( \text{End}_R(M) \) and let \( x \) belongs to \( AS_M \), then \( Rx \) is annihilator small in \( M \) and hence stable. But every stable submodule is fully invariant \([1]\), which leads to that there exists two elements \( r, s \) in \( R \) such that \( f(x) = rx \) and \( g(x) = sx \) \([8]\). Now, \( (f \circ g)(x) = f(g(x)) = f(sx) = r(sx) = s(rx) = g(rx) = g(f(x) = (g \circ f)(x) \); that is, \( \text{End}_R(M) \) is commutative over elements in \( AS_M \).

A natural question to ask is whether there exist conditions under which the converse true? Such a question leads us to define the concept of annihilator small regular modules as shown below.

**Definition 3.8:** An \( R \)-module \( M \) is called **annihilator small regular** if given any element in \( AS_M \), then there exists \( f \in M^* = \text{Hom}_R(M, R) \) such that \( m = f(m)m \). A ring \( R \) is called annihilator small regular if it is annihilator small module on itself.

There are bilinear functions:
\[
\theta : M \times M^* \to R \\
\psi : M^* \times M \to \text{End}_R
\]
Where \( \theta(m, \alpha) = \alpha(m) \forall \ m \in AS_M, \alpha \in M^* \) and \( \psi(\alpha, m) = \alpha(m)m \forall \ \alpha \in M^*, m \in AS_M \).

**Proposition 3.9:** Every commutative a-small regular ring \( R \) is a FASS ring.
Fully annihilator small stable modules

Proof: For each a-small principal ideal \(< r >\) of \(R\) and each \(R\)-homomorphism \(f: < r >\to R\), there exists an element \(t \in R\) such that \(r = rt\) and hence \(f(r) = f(rt) = rt f(r)\) implies that \(f(< r >) \subseteq < r >\).

Next, \(S\) will denote the ring of endomorphisms of \(M\) and \(Z_S\) denotes the center of \(S\).

Lemma 3.10: Let \(\alpha \in AS_{Z_S}\). Then there exists an element \(\beta \in Z_S\) with \(\alpha = \alpha \beta \alpha\) if and only if \(M = \alpha(M) \oplus \ker(\alpha)\).

Proof: \(\implies\) Suppose that such a \(\beta\) exists. Set \(\pi = \alpha \beta = \beta \alpha\) then \(\pi^2 = (\beta \alpha)(\beta \alpha) = \beta (\alpha \beta \alpha) = \beta \alpha = \pi\). Now, \(\alpha(M) \subseteq M\) thus \(\pi(\alpha(M)) \subseteq \pi(M)\) but \(\pi(\alpha(M)) = \alpha\beta(\alpha(M)) = (\alpha \beta \alpha)(M) = \alpha(M)\) which implies that \(\alpha(M) \subseteq \pi(M)\) and since \(\beta(M) \subseteq M\), then \(\beta(\alpha(M)) \subseteq \alpha(M)\) and \(\pi(M) \subseteq \alpha(M)\) implies that \(\pi(M) = \alpha(M)\). Moreover, \(\ker(\alpha) = \ker(\pi)\) since \(\pi = \alpha \beta\) and \(\alpha = \pi \alpha\). Clearly, \(M = \pi(M) \oplus (1 - \pi)(M) = \pi(M) \oplus \ker(\pi)\) and hence \(M = \alpha(M) \oplus \ker(\alpha)\).

\(\impliedby\) Suppose that \(\alpha \in Z_S\) and \(M = \alpha(M) \oplus \ker(\alpha)\). Given any \(m \in M\) then \(m = (\alpha(n) + k)\) with \(n \in M, k \in \ker(\alpha)\) and so we write \(n = \alpha(n_1) + k_1\) with \(n_1 \subseteq M, k_1 \subseteq \ker(\alpha)\). Thus \(m = \alpha(n_1) + k + \alpha(n) = \alpha(\alpha(n_1) + k_1) + k = \alpha(\alpha(n_1) + k_1) + k = \alpha^2(n_1) + \alpha(k_1) + k = \alpha^2(n_1) + \alpha(k) = \alpha^2(\alpha(n_1))\). Set \(x_m = \alpha(n_1) \subseteq \alpha(M)\), observe that \(x_m\) is the unique element of \(M\) such that \(\alpha(m) = \alpha^2(x_m)\), for if \(y \subseteq \alpha(M)\) with \(\alpha(m) = \alpha^2(y)\) then \(\alpha^2(x_m - y) = 0\) so that \(\alpha(x_m - y) \subseteq \alpha(M) \cap \ker(\alpha) = (0)\) and hence \(x_m - y \subseteq \alpha(M) \cap \ker(\alpha) = (0)\) implies that \(x_m = y\). It is easy to check that \(x_{rm} = rx_m, x_{n+m} = x_n + x_m\) and \(x_{y(m)} = y(x_m)\) for any \(n, m \in M, r \subseteq R\) and \(y \subseteq S\). Consequently, there is a homomorphism \(\beta \subseteq S\) defined by \(\beta(m) = x_m\). For any \(m \subseteq M\), \(\alpha \beta \alpha(m) = \alpha(\beta \alpha(m)) = \alpha(x_{\alpha(m)}) = \alpha(\alpha(x_m)) = \alpha^2(x_m) = \alpha(m)\) so that \(\alpha = \alpha \beta \alpha\). It remains only to show that \(\beta \subseteq Z_S\) which easy to show for given any \(\gamma \subseteq S\) and \(m \subseteq M\) then \(\beta(\gamma(m)) = x_{\gamma(m)} = y(x_m) = y(\beta(m))\).

Proposition 3.11: Let \(M\) be an annihilator small regular \(R\)-module. Then \(Z_S\) is an annihilator small regular ring.

Proof: Let \(\alpha \in AS_{Z_S}\) and given any \(m \in AS_M\) choose \(f \subseteq M^*\) with \(m = f(m)m\), hence \(\alpha(m) = f(m)\alpha(m)\) and \(\alpha(m) = \psi(f, \alpha(m))\). Now, \(m = f(m)\alpha(m) + (m - f(m)\alpha(m))\) then \(f(m)\alpha(m) \subseteq \alpha(M)\) and \(m - f(m)\alpha(m) \subseteq \ker(\alpha)\), since \(\alpha(m - f(m)\alpha(m)) = \alpha(m) - f(m)\alpha^2(m) = 0\). So \(M = \alpha(M) + \ker(\alpha)\). Now, let \(\alpha(m) \subseteq \alpha(M) \cap \ker(\alpha)\) then from above we get that \(\alpha(m) = f(m)\alpha^2(m) = 0\) since \(\alpha(\alpha(m)) = 0\). Hence, \(\alpha(M) \cap \ker(\alpha) = (0)\) and thus \(M = \alpha(M) \oplus \ker(\alpha)\) and by lemma (3.10) we get that \(Z_S\) is a small regular ring.
Proposition 3.12: Let $M$ be an $R$-module such that every a-small cyclic submodule is a direct summand. If $S$ is commutative over elements in $AS_M$, then $M$ is a FASS module.

Proof: let $N$ be any a-small cyclic submodule of $M$ and $f: N \rightarrow M$ be an $R$-homomorphism, then there exists $m \in AS_M$ such that $N = Rm$. By our assumption $Rm$ is a direct summand of $M$; that is, there exists a submodule $L$ of $M$ such that $M = Rm \oplus L$. Now, $f$ can be extended to an $R$-endomorphism of $M$, $g: M \rightarrow M$ by putting $g(l) = 0$ for each $l \in L$. Define $h: M \rightarrow M$ by $h(x, y) = x$ for each $x \in Rm$ and $y \in L$. Let $f(x) = n_1 + l_1$ for some $n_1 \in N$ and $l_1 \in L$. Now, let $z \in AS_M$ then $z = x + l$ for some $x \in N$ and $l \in L$. Thus we have $(g \circ h)(z) = (g \circ h)(x + l) = g(x) = f(x) = n_1 + l_1$ and $(h \circ g)(z) = (h \circ g)(x + l) = h(f(x)) = h(n_1 + l_1) = n_2$, but $S$ is commutative for each $z \in AS_M$ and hence $g \circ h = h \circ g$ implies that $l_1 = 0$ and $f(x) \in N \forall x \in N$. Therefore, $f(N) \subseteq N$ and then $M$ is a FASS module. $\blacksquare$

Lemma 3.13: Let $M$ be an annihilator small regular $R$-module. Then every a-small cyclic submodule of $M$ is a direct summand.

Proof: Let $Rm$ be an a-small cyclic submodule of $M$, then $m \in AS_M$ and hence $m = \psi(f, m)m$ for some $f$ in $M^*$, which implies that $M = Rm \oplus \ker(\psi)$ [9]. $\blacksquare$

The proof of the following corollary is immediate.

Corollary 3.14: Let $M$ be an annihilator small regular $R$-module. Then $M$ is a FASS module if and only if $S$ is commutative over elements in $AS_M$.

References


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