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On Mildly - Regular Space

Afraa R. Sadek[1] and Jalal H. Hussein[2]

[1] Department of Mathematics, College of Science
Baghdad University, Iraq

[2] Department of Mathematics, College Science of Woman
Baghdad University, Iraq

Abstract

In this paper mildly-regular topological space was introduced via the concept of mildly g-open sets. Many properties of mildly - regular space are investigated and the interactions between mildly-regular space and certain types of topological spaces are considered. Also the concept of strong mildly-regular space was introduced and a main theorem on this space was proved.

Keywords: g-closed set, mildly closed set, mildly closure of set, mildly Hausdorff space, irresolute function, M-open function

1. Introduction and Preliminaries

Throughout this paper X is topological space and no separation axioms are assumed. For a subset A of a space X the symbols of closure, interior and the complement of A with respect to a topological space (X,τ) will be denoted by *cl A*, *int A* and A^c, respectively**.** By $cl_{\tau A}(B)$ and $int_{\tau A}(B)$ we mean the closure and the interior operators for a subset B with respect to the relative topology τ_A on a subspace (A,τ_A) of a topological space (X,τ_X). The concept of generalized closed set introduced by Levine [1], recall that a subset A of a space *X* is called g-closed if cl(A) subset of every open set in X containing A. the complement of g-closed set is called g-open. In 2004 J. K. Park and J. H. Park [4] introduce the concept of mildly closed set and this concept was developed by P.G. Palanimani and R.parimelazhagan [2, 3]. A subset A of a space X is called mildly closed set if $cl(int(A))\subset U$, whenever $A\subset U$ and U is g-open, the complement of mildly closed set is called mildly open.

2. Mildly - regular space

Beginning this section with the definition of mildly- regular space.

Definition 2.1. A space X is called mildly-regular if for each $a \in X$ and every closed set F in X such that $a \notin F$, there exists disjoint mildly open sets U and V in X such that $F \subset U$, $a \in V$.

Now we will show that the mildly open set in any space still mildly open in any closed subspace but before that we need following lemma.

Lemma 2.2. [1] Let (A, τ_A) be a subspace of space X and suppose B be a subset of A such that B is a g-closed set relative to τ_A and A is a g-closed subset of X. Then B is g-closed relative to X.

Theorem 2.3. Let (A, τ_A) be a closed subspace of a space X then every mildly open set of X is mildly open set relative to τ_A.

Proof. Let $B \subset A \subset X$, such that A is closed in X and B is mildly open in X. .By duality of definition of mildly closed, a set B is mildly open in X if and only if for each g-closed set F in X such that $F \subset B$ then $F \subset int(cl(B))$. Let $G \subset B$ and G is g-closed set relative to τ_A, since A is closed then A is g-closed so by lemma 2.2. G is g-closed relative to X, but B is mildly open relative to X, hence $G \subset int_{\tau X}(cl_{\tau X}(B)) \subset int_{\tau X}(cl_{\tau X}(B)) \cap A \subset int_{\tau A}(cl_{\tau X}(B)) \subset int_{\tau A}(cl_{\tau X}(B \cap A)) \subset int_{\tau A}((cl_{\tau X}(B) \cap cl_{\tau X}(A)) \subset int_{\tau A}((cl_{\tau X}(B) \cap (A)) \subset int_{\tau A}((cl_{\tau A}(B))$ and this complete the proof.

Definition2.4. [5] The mildly closure of a set A, denoted by mcl(A), is the intersection of all mildly closed sets containing A.

In the following theorems we give a characterization of mildly-regular space.

Theorem 2.5. A topological space X is mildly-regular if and only if for every $x \in X$ and every open set U such that $x \in U$, there exists a mildly open set V such that $x \in V \subset mcl(V) \subset U$.

Proof. Suppose X is mildly regular and U an open set containing $x \in X$. Thus $x \notin U^c$ so there exist disjoint mildly open sets V and W containing x and U^c respectively, also $V \subset W^c$ and W^c is mildly closed hence $mcl(V) \subset mcl(W^c)$ and hence $mcl(V) \subset W^c \subset U$, then we have $x \in V \subset mcl(V) \subset U$. For sufficiency let $x \notin F$ where F is closed set in X then $x \in F^c$ and F^c is open, so there exist mildly g-open set V such that $x \in V \subset mcl(V) \subset F^c$ implies $F \subset (mcl(V))^c$ and $(mcl(V))^c$ is mildly g-open set. But $V \cap (mcl(V))^c = \phi$ hence $x \in V$, $F \subset (mcl(V))^c$ and we have done.

Theorem 2.6. A topological space X is a mildly-regular space if and only if for every $x \in X$ and every open set V belong to finite base B such that $x \in V$, there exists a mildly open set U such that $x \in U$ and $cl(int(U)) \subset V$.

Proof. Since $x \notin V^c$ and V^c closed, so there exist disjoint mildly open sets U_1,U_2 such that $x \in U_1$ and $V^c \subset U_2$ then we have $U_1 \subset (U_2)^c \subset V$. Furthermore, $(U_2)^c$ is mildly closed set and V is open set so V is g-open set, then by definition of mildly closed set we have that $cl(int(U_1)) \subset cl(int(U_2)^c) \subset V$. For sufficiency let $x \in X$ and F be a closed subset of X such that $x \notin F$ and suppose $V_1, V_2, \ldots V_k \in B$ such that $x \in \bigcap_{i=1}^{k} Vi \subset F^c$. Thus there exist a mildly open set U_i such that $x \in U_i$ and $cl(int(U_i)) \subset V_i$ for $i=1,2,\ldots,k$. Then the mildly open sets $U_1 = \bigcap_{i=1}^{k} Vi$ and $U_2 = (\bigcap_{i=1}^{k} cl\, int(Ui))^c$ are disjoint such that $x \in U_1$ and $F \subset (\bigcap_{i=1}^{k} Vi)^c \subset (\bigcap_{i=1}^{k} cl\, int(Ui))^c = U_2$

Definition2.7. A space X is called weakly mildly- Hausdorff if for each distinct points $x,y \in X$ such that $x \notin cl(V_y)$ where V_y any mildly open set containing y then there exist a disjoint mildly open sets V, W containing x and y respectively.

Propostion2.8. Every mildly regular topological space is weakly mildly-Hausdorff.

Proof. Let X be a mildly regular space and let x,y be any distinct points in X suppose $x \notin cl(V_y)$ where V_y be any mildy open set containing x . Now since X is mildly regular then there exist disjoint mildly open sets U_x and U_y such that $x \in U_x$ and $y \in cl(V_y) \subset U_y$ implies X is weakly Hausdorff space. If $y \notin cl(V_x)$ the proof is similar.

Definition2.9. [7] A space X is called mildly- Hausdorff if for each distinct points $x,y \in X$ there exist two disjoint mildly-open sets U_x and U_y containing x and y respectively.

The following example showed that the quotient topology of the mildly regular space can be mildly Hausdorff space.

Example 2.10. Let X be mildly regular space and F be a closed subset of X. Define a relation on X as follow, $x \Re y$ if and only if either x,y in F or if $x,y \notin F$ then x=y. We can see that $\Re$ is an equivalents relation, to prove that $X/\Re$ with the quotient topology is mildly Hausdorff space take [x],[y] in $X/\Re$ such that $[x] \neq [y]$, in this case we can see that either x or y belong to F and the other may not so let x in F and $y \notin F$, but F is closed in the mildly regular space X then there exists disjoint mildly open sets U and V such that $[x] \subset F \subset U$ and $[y] \subset V$ therefore X is mildly Hausdorff space.

In the following example we show that the quotient space of mildly regular space need not be mildly regular.

Example 2.11. Take the real line with usual topology and let P: R $\to$ A, where A ={a,b,c} defined as follow:

$$P(x) = \begin{cases} a & if\ x>0 \\ b & if\ x<0 \\ c & if x = 0 \end{cases}$$

Then the quotient topology on A will be T={ϕ,{a},{b},{a,b},{a,b,c}} and A with the topology T is not mildly regular space but R with usual topology is mildly regular.

Definition 2.12. [5] A function f: X$\to$Y is called mildly irresolute if the inverse image of each mildly open set in Y is mildly open set in X.

The following proposition showed that the property of mildly regular space invers preserving under injective closed mildly irresolute function

Proposition 2.13. Let f: X $\to$ Y be a closed mildly irresolute injective function and Y is mildly regular space then X is mildly regular space.

Proof. Let x be a point in X and A be any closed subset of X such that x$\notin$A, then f(A) is closed in a space Y such that f(x)$\notin$f(A) and by mildly regularity of Y there exists disjoint mildly-open sets U,V such that f(A) $\subset$ U and f(x)$\in$V. Clear that A$\subset$f $^{-1}$(f(A))$\subset$f $^{-1}$(U) and x$\in$f $^{-1}$(V) also f $^{-1}$(U) $\cap$f $^{-1}$(V) =ϕ for instance see [6] , but f is mildly irresolute implies f $^{-1}$(U), f $^{-1}$(V) are mildly open subsets of X which complete the proof.

Definition 2.14. [7**]** A function f: X$\to$Y is called M- open if the image of each mildly-open set in X is mildly open set in Y.

Proposition 2.15. Let f: X $\to$ Y be a bijective, M- open and continuous function and X is mildly regular space then Y is mildly- regular space.

Proof. Let B be a closed set in Y and y $\notin$ B, then f $^{-1}$ (B) $\subset$ X and f $^{-1}$ (y) $\notin$f $^{-1}$ (B). Now f $^{-1}$ (B) is closed in X, X is mildly regular space then there exists disjoint mildly open sets U and V such that f $^{-1}$(B) $\subset$ U and f $^{-1}$(y) $\in$V, but f is M – open then it is very easy to see that f(U) and f(V) are disjoint mildly open sets in Y containing B and y respectively.

3. Strongly mildly regular

Definition 3.1. A space is called strong mildly-regular space if for any $y\notin M$, M mildly closed there exist two disjoint open sets U and V such that $y\in U$ and $M\subset V$.
It is very easy to see that every strong mildly-regular is mildly-regular.
The following example showed that the convers need not necessarily true.

Example 3.2. The space $X=\{a,b,c\}$ with topology $T=\{X,\phi,\{a\},\{b,c\}\}$ is mildly-regular but not strong mildly-regular
We can characterize strongly mildly regular space as follows:

Theorem.3.3. Let X be a topological space then the following are equivalent:
(1) X is strongly mildly regular
(2) for each $y\in U$, U is mildly open, there exist open set V such that $y\in V\subset cl(V)\subset U$
(3) For each $y\in X$ and mildly closed set U, such that $y\notin U$, there exist an open set V such that $y\in V$ and $cl\,(V)\subset U^c$.

Proof. (1)$\rightarrow$(2) since $y\notin U^c$ and U^c is mildly closed then there exist V and W such that $y\in V$ and $U^c\subset W$ where W and V are disjoint open sets and it is very easy to see that $cl(V)\subset W^c$ hence $cl(V)\cap U^c\subset cl(V)\cap W=\phi$ and hence $cl\,(V)\subset U$.
(2) $\rightarrow$(3) We can apply (2) on y and U^c to find open set V such that $y\in V\subset cl(V)\subset U^c$
(3) $\rightarrow$(1) let y be any point in X and U any mildly closed subset of X such that $y\notin U$, then we can find two disjoint open sets V and $(cl\,(V))^c$ such that $y\in V$ and $U\subset(cl\,(V))^c$.

Definition3.4. [7] A function $f{:}X\rightarrow Y$ is said to be M- closed if the image of mildly closed set in X is mildly closed in Y.
In the following theorem we can apply proposition 2.13 to show that the strongly mildly regular property invers preserved under continuous M-closed function

Theorem 3.5. If f: $X\rightarrow Y$ injective, continuous, M- closed function and Y is strongly mildly regular then X is strongly mildly regular.

Proof. Let A be a mildly closed subset of a topological space X and let x be any point belong to X and $x\notin A$, then f(A) is mildly closed set and $f(x)\notin f(A)$, but Y is strongly mildly regular so there exist disjoint open sets U,V in Y such that $f(x)\in V$ and $f(A)\subset U$, but $A\subset f^{-1}(f(A))\subset f^{-1}(U)$ and $x\in f^{-1}(V)$ hence $f^{-1}(U)$,$f^{-1}(V)$ are disjoint open sets containing x and A respectively. Thus X is strongly mildly regular.

Lemma 3.6.[5] Suppose that $B \subset A \subset X$, and B is mildly closed set relative to A and A is g-open and mildly closed set relative to X, then B is mildly closed relative to X.

The following theorem showed that the property of being strongly mildly regular space is hereditary under some conditions

Theorem 3.7. Let X be a mildly regular space and A be a g-open and mildly closed subspace of X then A is mildly regular space.

Proof. Let F_A be a mildly closed set relative to A and $x \in A$ such that $x \notin F_A$. By the above lemma F_A is mildly closed set relative to X so there exist disjoint open sets U,V such that $y \in V$ and $F_A \subset U$. Now it is clear that $A \cap U$ and $A \cap V$ are disjoint open sets relative to A containing F_A and x respectively.

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