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A Generalized Fermat Equation with an Emphasis on Non-Primitive Solutions

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Abstract

The equation $x^m + y^n = z^r$ is considered under the condition that the given integers values for m, n, and r are greater than one. Solutions to this equation are given for cases in which $\gcd(mn, r) = 1$, $\gcd(mr, n) = 1$, or $\gcd(nr, m) = 1$.

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1 Introduction

It is not surprising that, although Fermat's Last Theorem has been established [5], [6], variations of the original Fermat equation, $x^n + y^n = z^n$, continue to be studied. We will consider equations of the form

$$x^m + y^n = z^r \tag{1}$$

such that x, y, and z are nonzero integers, and m, n, and r are integers that are greater than one. Furthermore, we will assume that the values of the exponents m, n, and r are given. As usual, $\gcd(x, y, z)$ represents the greatest common divisor of x, y, and z, and $\operatorname{lcm}(m, n)$ denotes the least common multiple of m and n. A solution to equation (1) is said to be *primitive* if $\gcd(x, y, z) = 1$, and is called *non-primitive* otherwise. When studying equation (1), many authors are focused on the primitive solutions [2], [3]. In the current note, we reveal formulas that generate all solutions for specified cases of equation (1). If $\gcd(m, n, r) > 2$, then there are no solutions to this equation due to Fermat's Last Theorem. Many cases in which $\gcd(m, n, r) = 2$ have yet to be resolved. Presently, we are concerned with solutions to equation (1) when $\gcd(m, n, r) = 1$; specifically, we are examining the cases in which $\gcd(mn, r) = 1$, $\gcd(mr, n) = 1$, or $\gcd(nr, m) = 1$.

Recently, M. B. Nathanson [4] constructed the solution set for each equation of the form

$$x^n - y^n = z^{n+1} \tag{2}$$

such that x, y, z, and n are positive integers, and the value of n is given. We will extend his methods to the other cases of equation (1) that we are concerned with. We start by generalizing a couple of the definitions that Professor Nathanson uses. We say that an (ordered) triple (a, b, c) of nonzero integers is $\langle m, n, r\rangle$-*addition-powerful* if $a^m + b^n \neq 0$ and c^r divides $a^m + b^n$. Furthermore, we define the function

$$t^+{}_{\langle m,n,r\rangle}(a, b, c) = \frac{a^m + b^n}{c^r}.$$

Similarly, we say that (a, b, c) is $\langle m, n, r\rangle$-*subtraction-powerful* if $a^m - b^n \neq 0$ and c^r divides $a^m - b^n$. We define the function

$$t^-{}_{\langle m,n,r\rangle}(a, b, c) = \frac{a^m - b^n}{c^r}.$$

Now let $l_1 = \operatorname{lcm}(m, n)$. Due to results from elementary number theory, if mn is relatively prime to r, then there exist positive integers j_1 and k_1 such that $j_1 l_1 + 1 = k_1 r$.

2 Main Results

The proof of the following theorem is not difficult.

Theorem 2.1. *Suppose that the integer values of m, n, and r are given, and that $m \geq 2$, $n \geq 2$, and $r \geq 2$.*

(A) *Consider the case in which mn is relatively prime to r; let $l_1 = \operatorname{lcm}(m, n)$ and let j_1 and k_1 be positive integers such that $j_1 l_1 + 1 = k_1 r$. If (a, b, c) is a $\langle m, n, r\rangle$-addition-powerful triple of nonzero integers and we let $t_1 = t^+{}_{\langle m,n,r\rangle}(a, b, c)$, then*

$$(x, y, z) = (a{t_1}^{(j_1 l_1)/m}, b{t_1}^{(j_1 l_1)/n}, c{t_1}^{k_1}) \tag{3}$$

is a solution to equation (1). *In this case, every solution to equation* (1) *can be expressed in the form given in* (3) *(remembering that x, y, and z are nonzero integers).*

(B) *For the case in which* $\gcd(mr, n) = 1$*, let* $l_2 = \text{lcm}(m, r)$ *and let* j_2 *and* k_2 *be positive integers such that* $j_2l_2 + 1 = k_2n$*. If* (c, a, b) *is a* $\langle r, m, n\rangle$*-subtraction-powerful triple of nonzero integers and we let* $t_2 = t^-{}_{\langle r,m,n\rangle}(c, a, b)$*, then*

$$(x, y, z) = (a{t_2}^{(j_2l_2)/m}, b{t_2}^{k_2}, c{t_2}^{(j_2l_2)/r}) \tag{4}$$

is a solution to equation (1)*. In this case, every solution to equation* (1) *can be expressed in the form given in* (4)*.*

(C) *Finally, when* $\gcd(nr, m) = 1$*, let* $l_3 = \text{lcm}(n, r)$ *and let* j_3 *and* k_3 *be positive integers such that* $j_3l_3+1 = k_3m$*. If* (c, b, a) *is a* $\langle r, n, m\rangle$*-subtraction-powerful triple of nonzero integers and we let* $t_3 = t^-{}_{\langle r,n,m\rangle}(c, b, a)$*, then*

$$(x, y, z) = (a{t_3}^{k_3}, b{t_3}^{(j_3l_3)/n}, c{t_3}^{(j_3l_3)/r}) \tag{5}$$

is a solution to equation (1)*. Furthermore, every solution to equation* (1) *can be expressed in the form given in* (5) *in this case.*

In formulas (3), (4), and (5), the exponents on t_1, t_2, and t_3, that is $(j_1l_1)/m$, $(j_1l_1)/n$, etc., are obviously positive integers and they are treated as such. There are some similarities between theorem 2.1 and a result given by M. Bennett, P. Mihăilescu, and S. Siksek [3] (see pages 194-195).

Proof. **(A)** Assume that $\gcd(mn, r) = 1$, $l_1 = \text{lcm}(m, n)$, j_1 and k_1 are positive integers with the property that $j_1l_1 + 1 = k_1r$, and (a, b, c) is a $\langle m, n, r\rangle$-addition-powerful triple; let $t_1 = t^+{}_{\langle m,n,r\rangle}(a, b, c)$. Substituting $(a{t_1}^{(j_1l_1)/m}, b{t_1}^{(j_1l_1)/n})$ in for (x, y) in the left-hand side of equation (1), we see that

$$x^m + y^n = a^m{t_1}^{j_1l_1} + b^n{t_1}^{j_1l_1} = (a^m + b^n){t_1}^{j_1l_1}$$

$$= c^r \cdot \frac{(a^m + b^n)}{c^r} \cdot {t_1}^{j_1l_1} = c^r{t_1}^{j_1l_1+1} = (c{t_1}^{k_1})^r.$$

Therefore, the statement in (3) is a solution to equation (1).

Now suppose that (a_0, b_0, c_0) is any (integer) solution to equation (1) such that $a_0b_0c_0 \neq 0$. Let $t_{10} = t^+{}_{\langle m,n,r\rangle}(a_0, b_0, c_0)$, which is equal to one in this case. Thus, the solution (a_0, b_0, c_0) can be written as

$$(a_0{t_{10}}^{(j_1l_1)/m}, b_0{t_{10}}^{(j_1l_1)/n}, c_0{t_{10}}^{k_1})$$

which is in the form expressed in formula (3).

(B) Assume that $\gcd(mr, n) = 1$, $l_2 = \operatorname{lcm}(m, r)$, j_2 and k_2 are positive integers with the property that $j_2l_2 + 1 = k_2n$, and (c, a, b) is a $\langle r, m, n\rangle$-subtraction-powerful triple; let $t_2 = t^-{}_{\langle r,m,n\rangle}(c, a, b)$. If we set x equal to $at_2{}^{(j_2l_2)/m}$ and z equal to $ct_2{}^{(j_2l_2)/r}$, then

$$z^r - x^m = c^r t_2{}^{j_2l_2} - a^m t_2{}^{j_2l_2} = (c^r - a^m)t_2{}^{j_2l_2}$$

$$= b^n \cdot \frac{(c^r - a^m)}{b^n} \cdot t_2{}^{j_2l_2} = b^n t_2{}^{j_2l_2+1} = (bt_2{}^{k_2})^n.$$

Therefore, the statement in (4) is a solution to equation (1).

Once again, suppose that (a_0, b_0, c_0) is any solution to equation (1) such that $a_0b_0c_0 \neq 0$. Let $t_{20} = t^-{}_{\langle r,m,n\rangle}(c_0, a_0, b_0)$, which is equal to one. Thus, the solution (a_0, b_0, c_0) can be written as

$$(a_0t_{20}{}^{(j_2l_2)/m}, b_0t_{20}{}^{k_2}, c_0t_{20}{}^{(j_2l_2)/r})$$

which is in the form expressed in formula (4).

(C) The proof of part C of this theorem is similar to the proof of part B. □

Note that formulas (3), (4), and (5), when they apply, generate infinitely many non-primitive solutions to equation (1). For example, assume that $\gcd(mn, r) = 1$ and let a_1 and a_2 represent any two positive integers such that $a_1 \neq a_2$. Then $(a_1, 1, 1)$ and $(a_2, 1, 1)$ are $\langle m, n, r\rangle$-addition-powerful triples, and the (clearly non-primitive) solutions to equation (1) generated by these triples, using equation (3), are

$$(a_1(a_1{}^m + 1)^{(j_1l_1)/m}, (a_1{}^m + 1)^{(j_1l_1)/n}, (a_1{}^m + 1)^{k_1})$$

and

$$(a_2(a_2{}^m + 1)^{(j_1l_1)/m}, (a_2{}^m + 1)^{(j_1l_1)/n}, (a_2{}^m + 1)^{k_1})$$

respectively; these solutions are not equal.

An (ordered) triple of nonzero integers (a, b, c) (that may, or may not, be a solution to equation (1)) is said to be *relatively prime* if $\gcd(a, b, c) = 1$. M. B. Nathanson [4] showed that, for any given value of n, every positive integer solution to equation (2) can be constructed from a relatively prime $\langle n, n, n+1\rangle$-subtraction-powerful triple. However, it is possible to find solutions to equation (1) that cannot be generated by a relatively prime $\langle m, n, r\rangle$-addition-powerful triple using formula (3) when $\gcd(mn, r) = 1$, and cannot be generated by applying formula (4) to a relatively prime $\langle r, m, n\rangle$-subtraction-powerful triple when $\gcd(mr, n) = 1$, etc. To see this, note the following example.

Example 2.2. *Consider the equation*

$$x^3 + y^5 = z^2. \tag{6}$$

Note that $(2,2,2)$ *is a* $\langle 3,5,2\rangle$*-addition-powerful triple due to the fact that* $t^+{}_{\langle 3,5,2\rangle}(2,2,2) = 10$*. Obviously,* $3 \cdot 5$ *is relatively prime to* 2*,* $l_1 = \mathrm{lcm}(3,5) = 15$*, and we can let* $j_1 = 1$ *and* $k_1 = 8$ *because* $1 \cdot 15 + 1 = 8 \cdot 2$*. Thus, formula* (3) *yields the solution* $(2 \cdot 10^5, 2 \cdot 10^3, 2 \cdot 10^8)$ *to equation* (6) *when* $(a,b,c) = (2,2,2)$*. Suppose that* (a_1, b_1, c_1) *is a relatively prime* $\langle 3,5,2\rangle$*-addition-powerful triple that generates this solution; let* $t_{11} = t^+{}_{\langle 3,5,2\rangle}(a_1, b_1, c_1)$*. Then, applying formula* (3)*, there exist positive integers* j_{11} *and* k_{11}*, satisfying* $15j_{11}+1 = 2k_{11}$*, for which*

$$(a_1 {t_{11}}^{5j_{11}}, b_1 {t_{11}}^{3j_{11}}, c_1 {t_{11}}^{k_{11}}) = (2 \cdot 10^5, 2 \cdot 10^3, 2 \cdot 10^8). \tag{7}$$

Thus, ${t_{11}}^{3j_{11}}$ *divides* $2 \cdot 10^3$*, and it is easy to show that* t_{11} *divides* 10*. Furthermore, if* t_{11} *is neither negative one nor one, then* $j_{11} = 1$ *and* $k_{11} = 8$*. But neither* $t_{11} = -10, -5, -2, -1, 1, 2, 5,$ *nor* 10 *yields a solution to equation* (7) *that has the property that* $\gcd(a_1, b_1, c_1) = 1$*. Therefore, there is no relatively prime* $\langle 3,5,2\rangle$*-addition-powerful triple that can be utilized in formula* (3) *to generate the solution* $(2 \cdot 10^5, 2 \cdot 10^3, 2 \cdot 10^8)$ *to equation* (6)*.*

We have yet to exclude the possibility that a relatively prime $\langle 2,3,5\rangle$*-subtraction-powerful triple may generate the given solution to equation* (6)*. Obviously,* $2 \cdot 3$ *is relatively prime to* 5 *and* $l_2 = \mathrm{lcm}(2,3) = 6$*. Suppose that* (c_2, a_2, b_2) *is a relatively prime* $\langle 2,3,5\rangle$*-subtraction-powerful triple that generates this solution; let* $t_{22} = t^-{}_{\langle 2,3,5\rangle}(c_2, a_2, b_2)$*. Applying formula* (4)*, there exist positive integers* j_2 *and* k_2*, satisfying* $6j_2 + 1 = 5k_2$*, for which*

$$(a_2 {t_{22}}^{2j_2}, b_2 {t_{22}}^{k_2}, c_2 {t_{22}}^{3j_2}) = (2 \cdot 10^5, 2 \cdot 10^3, 2 \cdot 10^8). \tag{8}$$

If $j_2 = 1, 2,$ *or* 3*, then* k_2 *is not an integer; it follows that* $j_2 \geq 4$*. We see that* ${t_{22}}^{2j_2}$ *divides* $2 \cdot 10^5$*, and it follows that* t_{22} *is equal to negative one or one. But neither* $t_{22} = -1$ *nor* $t_{22} = 1$ *yields a solution to equation* (8) *such that* $\gcd(c_2, a_2, b_2) = 1$*. Therefore, there is no relatively prime* $\langle 2,3,5\rangle$*-subtraction-powerful triple that can be utilized in formula* (4) *to generate the solution* $(2 \cdot 10^5, 2 \cdot 10^3, 2 \cdot 10^8)$*. Proceeding in a similar fashion, it is easy to verify that no relatively prime* $\langle 2,5,3\rangle$*-subtraction-powerful triple can be used in formula* (5) *to generate the given solution to equation* (6)*.*

It is plain to see that theorem 2.1, when applicable, does not exclude primitive solutions to equation (1), when they exist. For example, $t^+{}_{\langle 3,5,2\rangle}(2,1,3) = 1$; thus, due to formula (3), $(2,1,3)$ is a solution to equation (6). In fact, $(2,1,3)$ is a well-known primitive solution to $x^3 + y^n = z^2$ for each value of n.

3 Additional Comments

Clearly, not all cases of equation (1), such that $\gcd(m,n,r)=1$, are addressed by theorem 2.1. For example, consider the equation $x^{15}+y^{21}=z^{35}$. Although $\gcd(15,21,35)=1$, it is easy to see that $\gcd(15\cdot 21,35)>1$, $\gcd(15\cdot 35,21)>1$, and $\gcd(21\cdot 35,15)>1$.

We would be remiss if we did not mention the Beal Prize, which is funded by D. A. Beal, a famous banker and mathematics enthusiast [1]. For some cases in which $m=2$, $n=2$, or $r=2$, primitive solutions to equation (1) have been found [2], [3]. Currently, no primitive solution to equation (1) has been found such that $m>2$, $n>2$, and $r>2$. To qualify for the Beal Prize, one must find a primitive solution to equation (1) under the conditions that x, y, z, m, n, and r are positive integers, $m>2$, $n>2$, and $r>2$, or prove that no primitive solution exists under these conditions. Presently, the Beal Prize is worth one million U.S. dollars. Theorem 2.1 will probably not be helpful in the search for a prize-winning example, unless an ingenious method for separating any prize-winning solution from the others is discovered.

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