Fractional Laplace Transform and Fractional Calculus

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Abstract

In this work we study the action of the Fractional Laplace Transform introduced in [6] on the Fractional Derivative of Riemann-Liouville. The properties of the transformation in the convolution product defined as Miana were also presented. As an example we calculate the solution of a differential equation.

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1 Introduction and Preliminaries

We start by recalling some elementary definitions of page. 103 of [7].

Definition 1. Let $f = f(t)$ be a function of $\mathbb{R}_0^+$. The Laplace transform $\tilde{f}(s)$ is given by the integral

$$\tilde{f}(s) = \mathcal{L}[f(t)](s) = \int_0^{\infty} e^{-st} f(t) dt \quad (1.1)$$
for $s \in \mathbb{R}$

**Definition 2.** Let $A(\mathbb{R}_0^+)$ a function of the space:

i) $f$ is piecewise continuous in the interval $0 \leq t \leq T$ for any $T \in \mathbb{R}_0^+$.

ii) $f$ it is of exponential order, 

\[ |f(t)| \leq Ke^{at} \]

for $t \geq M$ where $M, K$ y $a$ are real positive constants.

The parameter $a$ is called the *abscissa of convergence* of the Laplace transform. Therefore we have the next classic

**Definition 3.** Let $f = f(t)$ a function defined in $\mathbb{R}_0^+$

The incomplete Laplace transform $\tilde{f}(s)$ is given by the integral

\[ \mathfrak{L}[f(t), b](s) = \int_0^b e^{-st} f(t) dt \]  \hspace{1cm} (1.2)

for $b, s \in \mathbb{R}$

Medina, Ojeda, Pereira and Romero (cf.[6]) has introduced the following

**Definition 4.** Let $f = f(t)$ by a function of $\mathbb{R}_0^+$. The $\alpha$-Integral Laplace Transform $\tilde{f}_\alpha(s)$ of order $\alpha \in \mathbb{R}^+$ is given the integral

\[ \tilde{f}_\alpha(s) = \mathfrak{L}_\alpha[f(t)](s) = \int_0^\infty e^{-s^{1/\alpha} t} f(t) dt \]  \hspace{1cm} (1.3)

for $s \in \mathbb{R}$

The $\alpha$-Integral Laplace Transform it is a generalization of the Laplace transform so that when $\alpha \to 1$. That is to say

\[ \mathfrak{L}_1[f(t)](s) = \mathfrak{L}[f(t)](s) \]  \hspace{1cm} (1.4)

Then we can generalize

**Theorem 2.** If $f(t) \in A(\mathbb{R}_0^+)$, then $\tilde{f}_\alpha(s) = \mathfrak{L}_\alpha[f(t)](s)$ for $s > a^\alpha$

Note that it is natural to enunciate the following

**Lemma 2.** Let $f$ be a sufficiently well-behaved function and let $\alpha$ be a real number, $0 < \alpha < 1$. The fractional Laplace transform of the $f$ function is given by

\[ \mathfrak{L}_\alpha[f](s) = \mathfrak{L}[f](\mu), \mu = s^{\frac{1}{\alpha}} \]

**Proof** Follow from the definition (1.3)
Properties: If \( f^{(k)}(t) \in A(\mathbb{R}_0^+) \) con \( k = 1, 2, ..., n \) and \( n \in \mathbb{N} \) then

\[
\mathcal{L}_\alpha \left[ \left( \frac{df(t)}{dt} \right)^n \right](s) = s^n \mathcal{L}_\alpha[f(t)](s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)
\]  

(1.5)

Proof: Recall

\[
\mathcal{L} \left[ \left( \frac{df(t)}{dt} \right)^n \right](\mu) = \mu^n \mathcal{L}_\alpha[f(t)](s) - \sum_{k=1}^{n} \mu^{n-k} f^{(k-1)}(0)
\]

(1.6)

and how

\[ \mathcal{L}_\alpha[f](s) = \mathcal{L}[f](\mu), \mu = s^{\frac{1}{\alpha}} \]

we obtained

\[
\mathcal{L}_\alpha \left[ \left( \frac{df(t)}{dt} \right)^n \right](s) = s^n \mathcal{L}_\alpha[f(t)](s) - \sum_{k=1}^{n} s^{\frac{n-k}{\alpha}} f^{(k-1)}(0) \]

(1.7)

Now, we are able to find the inversion formula for the k-TL.

\[ \mathcal{L}_\alpha[f](s) = \mathcal{L}[f](\mu), \mu = s^{\frac{1}{\alpha}} \]

then

\[ f(t) = \mathcal{L}^{-1}_\alpha[\mathcal{L}_\alpha[f](s)] = \mathcal{L}^{-1}(g_1(\mu))(t) \]

applying the Laplace inverse transform gives

\[
\mathcal{L}^{-1}(g_1(\mu))(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\mu t} g_1(\mu) d\mu = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\mu t} \mathcal{L}[f](\mu) d\mu
\]

(1.8)

and making the change of variable \( \mu = s^{\frac{1}{\alpha}} \), where \( d\mu = \frac{1}{\alpha} s^{\frac{1}{\alpha}-1} ds \)

\[
\mathcal{L}^{-1}(g_1(\mu))(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{s^{\alpha} t} \mathcal{L}_\alpha[f](s) s^{\frac{1}{\alpha}-1} ds
\]

(1.9)

From this expression we have the following

Definition 5. Let \( f \) be a sufficiently well-behaved function and let \( \alpha \) be a real number, \( 0 < \alpha < 1 \). The inverse \( \alpha \)-Integral Laplace Transform is given by

\[
\mathcal{L}^{-1}_\alpha[\tilde{f}_\alpha(s)](t) = \frac{1}{2\pi i \alpha} \int_{a-i\infty}^{a+i\infty} e^{s^{\alpha} t} \tilde{f}_\alpha(s) s^{\frac{1}{\alpha}-1} ds
\]

(1.10)

Remark. Making the change of variable \( \mu = s^{\frac{1}{\alpha}} \), and taking into account the formula that establish the relationship between the conventional and the fractional Laplace transform, easily we can prove that
\[ \mathcal{L}_\alpha \left[ \mathcal{L}_\alpha^{-1} \right] = \text{Id} \]

where \( \text{Id} \) denote the identity operator.

**Definition 6.** Let \( f \) and \( g \) functions belonging to \( L^1(\mathbb{R}^+) \), the usual or classic convolution product is given by

\[ (f \ast t)(t) = \int_0^t f(\tau)g(t - \tau)d\tau \quad t > 0 \quad (1.11) \]

**Definition 7.** Let \( f \) and \( g \) functions belonging to \( L^1(\mathbb{R}^+) \), Miana in [2] introduce the convolution product \( \circ \) as the integral

\[ (f \circ g)(t) = \int_t^{\infty} f(\tau - t)g(\tau)d\tau \quad t > 0 \quad (1.12) \]

**Theorem 5.** If \( f(t), g(t) \in A(\mathbb{R}^+) \) such that \( \tilde{f}_\alpha(s) = \mathcal{L}_\alpha[f(t)](s) \) and \( \tilde{g}_\alpha(s) = \mathcal{L}_\alpha[g(t)](s) \), then

\[ \mathcal{L}_\alpha[f(t) \ast g(t)](s) = \tilde{f}_\alpha(s).\tilde{g}_\alpha(s) \quad (1.13) \]

2 Main Results

**Properties** Let \( \lambda \in \mathbb{R}^+ \), \( f \) and \( g \) functions belonging to \( L^1(\mathbb{R}^+) \) and the exponential function \( e_{\lambda^{1/\alpha}} := e^{\lambda^{1/\alpha}t} \) then:

i) \( f \circ e_{\lambda^{1/\alpha}} = \mathcal{L}_\alpha[f](\lambda).e_{\lambda^{1/\alpha}} \)

ii) \( e_{\lambda^{1/\alpha}} \circ f = \mathcal{L}_\alpha[f](\lambda^\alpha)e_{-\lambda^{1/\alpha}} - (e_{-\lambda} \ast f)(t) \)

iii) \( \mathcal{L}_\alpha(f \circ g)(s) = \mathcal{L}_\alpha(g\mathcal{L}_\alpha(f, .)(-s^{1/\alpha}))(s) \)

**Proof**

i) From definition 7 we have

\[ (f \circ e_{\lambda^{1/\alpha}})(t) = \int_t^{\infty} f(\tau - t)e^{-\lambda^{1/\alpha}\tau}d\tau \]

if \( u = \tau - t \), then \( du = d\tau \)

\[ (f \circ e_{\lambda^{1/\alpha}})(t) = \int_0^{\infty} f(u)e^{-\lambda^{1/\alpha}(u+t)}du \]

\[ = \left[ \int_0^{\infty} f(u)e^{-\lambda^{1/\alpha}u}du \right].e_{\lambda^{1/\alpha}} \]

\[ = \mathcal{L}_\alpha[f](\lambda).e_{\lambda^{1/\alpha}} \]
ii) From definition 7 we have

\[(e_{\lambda^{1/\alpha}} \circ f)(t) = \int_{t}^{\infty} e^{-\lambda^{1/\alpha}(\tau-t)} f(\tau) d\tau \quad (2.1)\]

as \( f \) y \( e^{-\lambda^{1/\alpha}} \) are functions belonging to \( L^1(\mathbb{R}^+) \), then \( e^{-\lambda^{1/\alpha} * f} \in L^1(\mathbb{R}^+) \) we obtain

\[(e_{\lambda^{1/\alpha}} \circ f)(t) = \left( \int_{0}^{\infty} e^{-\lambda^{1/\alpha}(\tau-t)} f(\tau) d\tau \right) - (e_{-\lambda} * f)(t)\]

\[= \left( \int_{0}^{\infty} e^{-\lambda^{1/\alpha} \tau} f(\tau) d\tau \right) e^{-\lambda^{1/\alpha}} - (e_{-\lambda} * f)(t)\]

\[= \mathcal{L}_\alpha[f](\lambda) e^{-\lambda^{1/\alpha}} - (e_{-\lambda^{1/\alpha}} * f)(t)\]

iii) Let \( f \) and \( g \) functions belonging to \( L^1(\mathbb{R}^+) \), from definition 7 we have

\[(f \circ g)(t) = \int_{t}^{\infty} f(\tau - t) g(\tau) d\tau, \quad t > 0\]

applying definition 4 we obtain

\[\mathcal{L}_\alpha[(f \circ g)(t)](s) = \int_{0}^{\infty} e^{-s^{1/\alpha} t} (f \circ g)(t) dt\]

\[= \int_{0}^{\infty} e^{-s^{1/\alpha} \tau} \left( \int_{t}^{\infty} f(\tau - t) g(\tau) d\tau \right) dt\]

Applying Fubini's Theorem we have

\[\int_{0}^{\infty} e^{-s^{1/\alpha} \tau} \left( \int_{t}^{\infty} f(\tau - t) g(\tau) d\tau \right) dt = \int_{0}^{\infty} g(\tau) \left( \int_{0}^{\tau} e^{-s^{1/\alpha} u} f(\tau - u) du \right) d\tau\]

If \( \tau < t < \infty \), \( 0 < \tau < \infty \) and we consider changing the variable \( u = \tau - t \), then \( \tau = u + t \), \( 0 < u < \infty \) and the differential \( dt = du \)

\[\mathcal{L}_\alpha[(f \circ g)(t)](s) = \int_{0}^{\infty} g(\tau) \left( \int_{0}^{\tau} e^{-s^{1/\alpha} (\tau - u)} f(u) du \right) d\tau\]

\[= \int_{0}^{\infty} e^{s^{1/\alpha} u} g(\tau) \left( \int_{0}^{\tau} e^{s^{1/\alpha} u} f(u) du \right) d\tau\]

\[= \mathcal{L}_\alpha[g \mathcal{L}(f, .)(-s^{1/\alpha})](s)\]

3 \( \alpha \)-Laplace Transform of Fractional Riemann-Liouville Operator

In this last section we consider the Riemann-Liouville fractional operators and we show the results of applies our \( \alpha \)-Laplace Transform to them.
Previously we need some elementary definitions and results.

**Definition 8.** Let $f$ be a locally integrable function on $(a, +\infty)$. The Riemann-Liouville integral of order $\alpha$, of the function $f$ is given by

$$I^\alpha_x f(t) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha - 1} f(t) dt \quad (3.1)$$

Here $\Gamma(\alpha)$ denotes the Gamma function of Euler

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \quad (3.2)$$

For $\alpha > 1$, and $t > 0$, let $j_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, be the singular kernel of Riemann-Liouville.

It can be proved that the Riemann-Liouville fractional integral may be expressed as the convolution

$$I^\alpha_x f(t) = (\frac{t^{\alpha-1}}{\Gamma(\alpha)} * f)(x) \quad (3.3)$$

The Riemann-Liouville fractional derivative of order $\alpha$, is defined inverse

$$D^\alpha_x I^\alpha_x = id$$

Another way to defined this fractional derivative is as follows.

**Definition 9.** Let be a real number, and let $m$ be an integer. Then the Riemann-Liouville fractional derivative of order $\alpha$ is given by

$$D^\alpha_x f(t) = \left( \frac{d}{dx} \right)^m I^{m-\alpha}_x f(t) \quad (3.4)$$

**Lemma 1.** Let $f$ be a sufficiently well-behaved function and let $\alpha$ be a real number, $0 < \alpha < 1$. The Laplace transform of the Riemann-Liouville fractional integral of the $f$ function is given by

$$\mathcal{L}[I^\alpha_x f](s) = (s)^{-\alpha} \mathcal{L}[f](s) \quad (3.5)$$

**Lemma 2.** Let $f$ be a sufficiently well-behaved function and let $\alpha$ be a real number, $0 < \alpha < 1$. The Laplace transform of the Riemann-Liouville fractional derivative of the $f$ function is given by

$$\mathcal{L}[D^\alpha_x f(t)](s) = s^\alpha \mathcal{L}[f(t)](s) - I^\alpha_x f(t)|_{t=0} \quad (3.6)$$

**Lemma 3.** Let $f$ be a sufficiently well-behaved function and let $\alpha$ be a real number, $0 < \alpha < 1$. The Laplace transform of the Riemann-Liouville fractional integral of the $f$ function is given by
\[ \mathcal{L}_\alpha[I_\beta^f](s) = (s)^{\beta/\alpha}\mathcal{L}_\alpha[f](s) \] (3.7)

**Proof** Remember that is \( t > 0 \) y \( \beta \in \mathbb{R} \) for \( \) of table 6, page 61

\[ \mathcal{L}_\alpha[t^\beta] = \frac{\Gamma(\beta + 1)}{s^{\beta+1}} \] (3.8)

From definition 4 and (3.8) we have

\[ \mathcal{L}_\alpha[j_\beta(t)](s) = s^{-\beta/\alpha} \] (3.9)

recall (3.3)

\[ I_\alpha^\beta f(x) = j_\beta(t) * f(t) \] (3.10)

applying definition 4 to (3.10) and (3.8) propertie

\[ \mathcal{L}_\alpha(I_\beta^f(x)) = \mathcal{L}_\alpha[j_\beta(t) * f(t)](s) = \mathcal{L}_\alpha[j_\beta(t)](s)\mathcal{L}_\alpha[f](s) = s^{-\beta/\alpha}\mathcal{L}_\alpha[f](s) \]

**Lemma 4.** Let \( f \) be a sufficiently well-behaved function and let \( \alpha \) be a real number, \( 0 < \alpha < 1 \). The Laplace transform of the Riemann-Liouville fractional derivative of the \( f \) function is given by

\[ \mathcal{L}_\alpha[D_\beta^\alpha f(t)](s) = s^{\beta/\alpha}\mathcal{L}_\alpha[f(t)](s) - I_\alpha^1 - \beta x f(t) \big|_{t=0} \] (3.11)

**Proof** by definition 9 we have that if \( 0 < \beta \leq 1 \), \( m = 1 \) y

\[ \mathcal{L}_\alpha[D_\beta^\alpha f(t)](s) = \mathcal{L}_\alpha[\frac{d}{dx}I_\beta^{-1} f(t)](s) \] (3.12)

by Lemma 2 we have

\[ \mathcal{L}_\alpha[\frac{d}{dx}I_\beta^{-1} f(t)](s) = s^{\beta/\alpha}\mathcal{L}_\alpha[I_\beta^{-1} f] - I_\beta^{-1} f \big|_{t=0} \]

we get the thesis
4 Mittag-Leffler

The called functions of the Mittag-Leffler type, play an important role in the theory of fractional differential equations (FDEs). First we introduce a two-parameter Mittag-Leffler function defined by formula (4.1)

\[ E_{\alpha,\beta}(\lambda t^\alpha) = \sum_{k=0}^{\infty} \frac{(\lambda t^\alpha)^k}{\Gamma(\alpha k + \beta)} \]  

As we will see later, classical derivatives of the Mittag-Leffler function appear in solution of FDEs. Since the series (4.1) is uniformly convergent we may differentiate term by term and obtain

\[ E_{\alpha,\beta}^{(m)}(\lambda t^\alpha) = \sum_{k=0}^{\infty} \frac{(k + m)!}{k!} \frac{(\lambda t^\alpha)^k}{\Gamma(\alpha k + am + \beta)} \]  

**Theorem 6.** Let \( \gamma, \beta \in \mathbb{C}, R(\gamma) > 0, R(\beta) > 0, \lambda \in \mathbb{R} \). Then hold

\[ \mathcal{L}_s \left( t^{\gamma m + \beta - 1} E_{\gamma,\beta}^{(m)}(\lambda t^\gamma) \right) = s^{-\gamma/\alpha} \frac{s^{-\beta}}{(s^{\gamma/\alpha} - \lambda)^{m+1}} \]

**Proof** Remember the next series convergence

\[ \sum_{k=0}^{\infty} \frac{(k + m)!}{k!} x^k = \frac{m!}{(1 - x)^{m+1}} \]

Then

\[ \mathcal{L}_s \left( t^{\gamma m + \beta - 1} E_{\gamma,\beta}^{(m)}(\lambda t^\gamma) \right) = \sum_{k=0}^{\infty} \frac{(k + m)! \lambda^k}{k!} \frac{\mathcal{L}_s \left[ t^{\gamma k + \gamma m + \beta - 1} \right]}{\Gamma(\gamma k + \gamma m + \beta)} (s) \]

\[ = \sum_{k=0}^{\infty} \frac{(k + m)! \lambda^k}{k!} \frac{\Gamma(\gamma k + \gamma m + \beta)}{\Gamma(\gamma k + \gamma m + \beta)} s^{-\gamma k - \gamma m - \beta + 1} \]

\[ = \sum_{k=0}^{\infty} \frac{(k + m)! \lambda^k}{k!} \frac{\lambda^k}{s^{-\gamma k - \gamma m - \beta}} \]

\[ = s^{-\gamma m - \beta} \sum_{k=0}^{\infty} \frac{(k + m)!}{k!} (\lambda s^{-\gamma/\alpha})^k \]

\[ = s^{-\gamma m - \beta} \frac{m!}{(1 - \lambda s^{-\gamma/\alpha})^{m+1}} \]

\[ = s^{-\gamma m - \beta} \frac{s^{-m+1}\gamma/\alpha (s^{\gamma/\alpha} - \lambda)^{m+1}}{s^{\gamma/\alpha} - \lambda} \]

\[ = \frac{s^{-\beta}}{(s^{\gamma/\alpha} - \lambda)^{m+1}} \]
5 Example

A slight generalization of an equation solved in [4, page 157]

\[ D^{\frac{1}{2}} f(t) + af(t) = 0; \quad I^{\frac{1}{2}} f(t)|_{t=0} = C \]  \hfill (5.1)

applying the The $\alpha$–Integral Laplace Transform, with $\alpha = \frac{1}{2}$, we obtained

\[ \mathcal{L}_{\frac{1}{2}} \left( D^{\frac{1}{2}} f(t) + af(t) \right) = 0 \]  \hfill (5.2)

\[ s \mathcal{L}_{\frac{1}{2}} [f(t)](s) - I^{\frac{1}{2}} f(t)|_{t=0} + a \mathcal{L}_{\frac{1}{2}} = 0 \]  \hfill (5.3)

\[ \mathcal{L}_{\frac{1}{2}} [f(t)](s) = \frac{C}{s + a} \]  \hfill (5.4)

and applying definition (1.10) gives the solution of (5.1)

\[ \mathcal{L}^{-1}_{\frac{1}{2}} \left( \mathcal{L}_{\frac{1}{2}} [f(t)](s) \right) = \mathcal{L}^{-1}_{\frac{1}{2}} \left( \frac{C}{s + a} \right) \]  \hfill (5.6)

\[ f(t) = Ct^{-\frac{1}{2}}E_{\frac{1}{2}, \frac{1}{2}}(-at^{\frac{1}{2}}) \]  \hfill (5.7)

is identical to solution obtained in [8, page 139]

References


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