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Fractional Laplace Transform and Fractional Calculus

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Abstract

In this work we study the action of the Fractional Laplace Transform introduced in [6] on the Fractional Derivative of Riemann-Liouville. The properties of the transformation in the convolution product defined as Miana were also presented. As an example we calculate the solution of a differential equation.

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1 Introduction and Preliminaries

We start by recalling some elementary definitions of page. 103 of [7].

Definition 1. Let $f = f(t)$ be a function of $\mathbb{R}_0^+$ .The Laplace transform $\tilde{f}(s)$ is given by the integral

$$\tilde{f}(s) = \mathfrak{L}[f(t)]_{(s)} = \int_0^\infty e^{-st} f(t)dt \tag{1.1}$$

for $s \in \mathbb{R}$

Definition 2. Let $A(\mathbb{R}_0^+)$ a function of the space:

i) f is piecewise continuous in the interval $0 \leq t \leq T$ for any $T \in \mathbb{R}_0^+$.

ii) f it is of exponential order ,

$$|f(t)| \leq Ke^{at}$$

for $t \geq M$ where M, K y a are real positive constants.

The parameter a is called the *abscissa of convergence* of the Laplace transform.Therefore we have the next classic
Definition 3. Let $f = f(t)$ a function defined in $\mathbb{R}_0^+$
The incomplete Laplace transform $\tilde{f}(s)$ is given by the integral

$$\mathfrak{L}[f(t), b]_{(s)} = \int_0^b e^{-st} f(t)dt \tag{1.2}$$

for $b, s \in \mathbb{R}$
Medina, Ojeda, Pereira and Romero (cf.[6]) has introduced the following
Definition 4. Let $f = f(t)$ by a function of $\mathbb{R}_0^+$ .The $\alpha-$Integral Laplace Transform $\tilde{f}_\alpha(s)$ of order $\alpha \in \mathbb{R}^+$ is given the integral

$$\tilde{f}_\alpha(s) = \mathfrak{L}_\alpha[f(t)](s) = \int_0^\infty e^{-s^{1/\alpha}t} f(t)dt \tag{1.3}$$

for $s \in \mathbb{R}$
The $\alpha-$Integral Laplace Transform it is a generalization of the Laplace transform so that when $\alpha \to 1$. That is to say

$$\mathfrak{L}_1[f(t)](s) = \mathfrak{L}[f(t)](s) \tag{1.4}$$

Then we can generalize
Theorem 2. *If $f(t) \in A(\mathbb{R}_0^+)$, then there $\tilde{f}_\alpha(s) = \mathfrak{L}_\alpha[f(t)](s)$ for $s > a^\alpha$*
Note that it is natural to enunciate the following
Lemma 2. *Let f be a sufficiently well-behaved function and let α be a real number,$0 < \alpha < 1$. The fraccional Laplace transform of the f function is given by*

$$\mathfrak{L}_\alpha[f](s) = \mathfrak{L}[f](\mu), \mu = s^{\frac{1}{\alpha}}$$

Proof Follow from the definition (1.3)

Properties If $f^{(k)}(t) \in A(\mathbb{R}_0^+)$ con $k = 1, 2, ...n$ y $n \in \mathbb{N}$ then

$$\mathfrak{L}_\alpha \left[\left(\frac{df(t)}{dt} \right)^n \right](s) = s^{\frac{n}{\alpha}} \mathfrak{L}_\alpha[f(t)](s) - \sum_{k=1}^{n} s^{\frac{n-k}{\alpha}} f^{k-1}(0) \tag{1.5}$$

Proof: Recall

$$\mathfrak{L} \left[\left(\frac{df(t)}{dt} \right)^n \right](\mu) = \mu^n \mathfrak{L}_\alpha[f(t)](s) - \sum_{k=1}^{n} \mu^{n-k} f^{k-1}(0) \tag{1.6}$$

and how

$$\mathfrak{L}_\alpha[f](s) = \mathfrak{L}[f](\mu), \mu = s^{\frac{1}{\alpha}}$$

we obtained

$$\mathfrak{L}_\alpha \left[\left(\frac{df(t)}{dt} \right)^n \right](s) = s^{\frac{n}{\alpha}} \mathfrak{L}_\alpha[f(t)](s) - \sum_{k=1}^{n} s^{\frac{n-k}{\alpha}} f^{k-1}(0) \blacksquare \tag{1.7}$$

Now, we are able to find the inversion formula for the k-TL.

$$\mathfrak{L}_\alpha[f](s) = \mathfrak{L}[f](\mu) = g_1(\mu), \mu = s^{\frac{1}{\alpha}}$$

then

$$f(t) = \mathfrak{L}_\alpha^{-1}[\mathfrak{L}_\alpha[f](s)] = \mathfrak{L}^{-1}(g_1(\mu))(t)$$

applying the Laplace inverse transform gives

$$\mathfrak{L}^{-1}(g_1(\mu))(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\mu t} g_1(\mu) d\mu = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\mu t} \mathfrak{L}[f](\mu) d\mu \tag{1.8}$$

and making the change of variable $\mu = s^{\frac{1}{\alpha}}$, where $d\mu = \frac{1}{\alpha} s^{\frac{1}{\alpha}-1} ds$

$$\mathfrak{L}^{-1}(g_1(\mu))(t) = \frac{1}{2\pi i} \int_{a^\alpha - i\infty}^{a^\alpha + i\infty} e^{s^\alpha t} \mathfrak{L}_\alpha[f](s) \frac{1}{\alpha} s^{\frac{1}{\alpha}-1} ds \tag{1.9}$$

From this expression we have the following

Definition 5. Let f be a sufficiently well-behaved function and let α be a real number,$0 < \alpha < 1$. The inverse $\alpha-$Integral Laplace Transform is given by

$$\mathfrak{L}_\alpha^{-1}[\tilde{f}_\alpha(s)](t) = \frac{1}{2\pi i \alpha} \int_{a^\alpha - i\infty}^{a^\alpha + i\infty} e^{s^\alpha t} \tilde{f}_\alpha(s) s^{\frac{1-\alpha}{\alpha}} ds \tag{1.10}$$

Remark. Making the change of variable $\mu = s^{\frac{1}{\alpha}}$, and taking into account the formula that establish the relationship between the conventional and the fractional Laplace transform, easily we can prove that

$$\mathfrak{L}_\alpha[\mathfrak{L}_\alpha^{-1}] = \mathrm{Id}$$

where Id denote the identity operator.

Definition 6. Let f and g functions belonging to $L^1(\mathbb{R}^+)$, the ussual or classic convolution product is given by

$$(f * t)(t) = \int_0^t f(\tau)g(t-\tau)d\tau \ , \ t > 0 \tag{1.11}$$

Definition 7. Let f and g functions belonging to $L^1(\mathbb{R}^+)$, Miana in [2] introduce the convolution product $\circ$ as the integral

$$(f \circ g)(t) = \int_t^\infty f(\tau - t)g(\tau)d\tau \ , \ t > 0 \tag{1.12}$$

Theorem 5. If $f(t),\ g(t) \in A(\mathbb{R}_0^+)$ such that $\tilde{f}_\alpha(s) = \mathfrak{L}_\alpha[f(t)](s)$ and $\tilde{g}_\alpha(s) = \mathfrak{L}_\alpha[g(t)](s)$, then

$$\mathfrak{L}_\alpha[f(t) * g(t)](s) = \tilde{f}_\alpha(s).\tilde{g}_\alpha(s) \tag{1.13}$$

2 Main Results

Properties Let $\lambda \in \mathbb{R}^+$, f and g functions belonging to $L^1(\mathbb{R}^+)$ and the exponential function $e_{\lambda^{1/\alpha}} := e^{\lambda^{1/\alpha}t}$ then:

i) $f \circ e_{\lambda^{1/\alpha}} = \mathfrak{L}_\alpha[f](\lambda).e_{\lambda^{1/\alpha}}$

ii) $e_{\lambda^{1/\alpha}} \circ f = \mathfrak{L}_\alpha[f](\lambda^\alpha)e_{-\lambda^{1/\alpha}} - (e_{-\lambda} * f)(t)$

iii) $\mathfrak{L}_\alpha(f \circ g)(s) = \mathfrak{L}_\alpha(g\mathfrak{L}_\alpha(f,.)(-s^{1/\alpha}))(s)$

Proof

$i)$ From definition 7 we have

$$(f \circ e_{\lambda^{1/\alpha}})(t) = \int_t^\infty f(\tau - t)e^{-\lambda^{1/\alpha}\tau}d\tau$$

if $u = \tau - t$, then $du = d\tau$

$$\begin{aligned}(f \circ e_{\lambda^{1/\alpha}})(t) &= \int_0^\infty f(u)e^{-\lambda^{1/\alpha}(u+t)}du \\ &= \left[\int_0^\infty f(u)e^{-\lambda^{1/\alpha}u}du\right].e_{\lambda^{1/\alpha}} \\ &= \mathfrak{L}_\alpha[f](\lambda).e_{\lambda^{1/\alpha}}\end{aligned}$$

ii) From definition 7 we have

$$(e_{\lambda^{1/\alpha}} \circ f)(t) = \int_t^\infty e^{-\lambda^{1/\alpha}(\tau - t)} f(\tau) d\tau \tag{2.1}$$

as f y $e_{-\lambda^{1/\alpha}}$ are functions belonging to $L^1(\mathbb{R}^+)$, then $e_{-\lambda^{1/\alpha}} * f \in L^1(\mathbb{R}^+)$ we obtain

$$\begin{aligned}
(e_{\lambda^{1/\alpha}} \circ f)(t) &= \left(\int_0^\infty e^{-\lambda^{1/\alpha}(\tau - t)} f(\tau) d\tau \right) - (e_{-\lambda} * f)(t) \\
&= \left(\int_0^\infty e^{-\lambda^{1/\alpha}\tau} f(\tau) d\tau \right) e_{-\lambda^{1/\alpha}} - (e_{-\lambda} * f)(t) \\
&= \mathfrak{L}_\alpha[f](\lambda) e_{-\lambda^{1/\alpha}} - (e_{-\lambda^{1/\alpha}} * f)(t)
\end{aligned}$$

iii) Let f and g functions belonging to $L^1(\mathbb{R}^+)$, from definition 7 we have

$$(f \circ g)(t) = \int_t^\infty f(\tau - t) g(\tau) d\tau \ , \ t > 0$$

applying definition 4 we obtain

$$\begin{aligned}
\mathfrak{L}_\alpha[(f \circ g)(t)](s) &= \int_0^\infty e^{-s^{1/\alpha}t} (f \circ g)(t) dt \\
&= \int_0^\infty e^{-s^{1/\alpha}t} \left(\int_t^\infty f(\tau - t) g(\tau) d\tau \right) dt
\end{aligned}$$

Applying Fubini's Theorem we have

$$\int_0^\infty e^{-s^{1/\alpha}t} \left(\int_t^\infty f(\tau - t) g(\tau) d\tau \right) dt = \int_0^\infty g(\tau) \left(\int_0^\tau e^{-s^{1/\alpha}t} f(\tau - t) dt \right) d\tau$$

If $\tau < t < \infty$,$0 < \tau < \infty$ and we consider changing the variable $u = \tau - t$, then $\tau = u + t$, $0 < u < \infty$ and the differential $dt = du$

$$\begin{aligned}
\mathfrak{L}_\alpha[(f \circ g)(t)](s) &= \int_0^\infty g(\tau) \left(\int_0^\tau e^{-s^{1/\alpha}(\tau - u)} f(u) du \right) d\tau \\
&= \int_0^\infty e^{s^{1/\alpha}\tau} g(\tau) \left(\int_0^\tau e^{s^{1/\alpha}u} f(u) du \right) d\tau \\
&= \mathfrak{L}_\alpha(g\mathfrak{L}(f, .)(-s^{1/\alpha}))(s)
\end{aligned}$$

3 α-Laplace Transform of Fractional Riemann-Liouville Operator

In this last section we consider the Riemann-Liouville fractional operators and we show the results of applies our α-Laplace Transform to them.

Previously we need some elementary definitions and results.

Definition 8. Let f be a locally integrable function on $(a, +\infty)$. The Riemann-Liouville integral of order α, of the function f is given by

$$I_x^\alpha f(t) \doteq \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt \tag{3.1}$$

here $\Gamma(\alpha)$ denotes the Gamma function of Euler

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{3.2}$$

For $\alpha > 1$, and $t > 0$, let $j_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, be the singular kernel of Riemann-Liouville.

It can be proved that the Riemann-Liouville fractional integral may be expressed as the convolution

$$I_x^\alpha f(t) = \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)} * f \right)(x) \tag{3.3}$$

The Riemann-Liouville fractional derivative of order α , is defined inverse

$$D_x^\alpha I_x^\alpha = id$$

Another way to defined this fractional derivative is as follows.

Definition 9. Let be a real number, and let m be an integer. Then the Riemann-Liouville fractional derivative of order α is given by

$$D_x^\alpha f(t) = \left(\frac{d}{dx} \right)^m I_x^{m-\alpha} f(t) \tag{3.4}$$

Lemma 1. *Let f be a sufficiently well-behaved function and let α be a real number,$0 < \alpha < 1$. The Laplace transform of the Riemann-Liouville fractional integral of the f function is given by*

$$\mathfrak{L}[I^\alpha f](s) = (s)^{-\alpha} \mathfrak{L}[f](s) \tag{3.5}$$

Lemma 2. *Let f be a sufficiently well-behaved function and let α be a real number,$0 < \alpha < 1$. The Laplace transform of the Riemann-Liouville fractional derivative of the f function is given by*

$$\mathfrak{L}[D^\alpha f(t)](s) = s^\alpha \mathfrak{L}[f(t)](s) - I^\alpha f(t)|_{t=0} \tag{3.6}$$

Lemma 3. *Let f be a sufficiently well-behaved function and let α be a real number,$0 < \alpha < 1$. The Laplace transform of the Riemann-Liouville fractional integral of the f function is given by*

$$\mathfrak{L}_\alpha[I_x^\beta f](s) = (s)^{\beta/\alpha}\mathfrak{L}_\alpha[f](s) \tag{3.7}$$

Proof Remember that is $t > 0$ y $\beta \in \mathbb{R}$ for [table of 6, page 61]

$$\mathfrak{L}_\alpha[t^\beta] = \frac{\Gamma(\beta+1)}{s^{\frac{\beta+1}{\alpha}}} \tag{3.8}$$

From definition 4 and (3.8) we have

$$\mathfrak{L}_\alpha[j_\beta(t)](s) = s^{-\beta/\alpha} \tag{3.9}$$

recall (3.3)

$$I_x^\alpha f(x) = j_\beta(t) * f(t) \tag{3.10}$$

applying definition 4 to (3.10) and (3.8) propertie

$$\begin{aligned}\mathfrak{L}_\alpha(I^\beta f(x)) &= \mathfrak{L}_\alpha[j_\beta(t) * f(t)](s)\\ &= \mathfrak{L}_\alpha[j_\beta(t)](s).\mathfrak{L}_\alpha[f](s)\\ &= s^{-\beta/\alpha}.\mathfrak{L}_\alpha[f](s)\end{aligned}$$

Lemma 4. *Let f be a sufficiently well-behaved function and let α be a real number,$0 < \alpha < 1$. The Laplace transform of the Riemann-Liouville fractional derivative of the f function is given by*

$$\mathfrak{L}_\alpha[D^\alpha f(t)](s) = s^{\beta/\alpha}\mathfrak{L}_\alpha[f(t)](s) - I^{1-\alpha} f(t)|_{t=0} \tag{3.11}$$

Proof by definition 9 we have that if $0 < \beta \leq 1$, $m = 1$ y

$$\mathfrak{L}_\alpha[D_x^\beta f(t)](s) = \mathfrak{L}_\alpha[\frac{d}{dx} I_x^{1-\beta} f(t)](s) \tag{3.12}$$

by Lemma 2 we have

$$\begin{aligned}\mathfrak{L}_\alpha[\frac{d}{dx} I_x^{1-\beta} f(t)](s) &= s^\beta \mathfrak{L}_\alpha[I_x^{1-\beta} f] - I_x^{1-\beta}\\ &= s^{1/\alpha} s^{-(1-\beta)/\alpha}\mathfrak{L}_\alpha[f] - I_x^{1-\beta}|_{t=0}\\ &= s^{1/\alpha} s^{-(1-\beta)/\alpha}\mathfrak{L}_\alpha[f] - I_x^{1-\beta}|_{t=0}\\ &= s^{\beta/\alpha}\mathfrak{L}_\alpha[f] - I_x^{1-\beta}|_{t=0}\end{aligned}$$

we get the thesis

4 Mittag-Leffler

The called functions of the Mittag-Leffler type, play an important role in the theory of fractional differential equations (FDEs). First we introduce a two-parameter Mittag-Leffler function defined by formula (4.1)

$$E_{\alpha,\beta}(\lambda t^{\alpha}) = \sum_{k=0}^{\infty} \frac{(\lambda t^{\alpha})^k}{\Gamma(\alpha k + \beta)} \tag{4.1}$$

As we will see later, classical derivatives of the Mittag-Leffler function appear in so- lution of FDEs. Since the series (4.1) is uniformly convergent we may differentiate term by term and obtain

$$E^{(m)}_{\alpha,\beta}(\lambda t^{\alpha}) = \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(\lambda t^{\alpha})^k}{\Gamma(\alpha k + \alpha m + \beta)} \tag{4.2}$$

Theorem 6. Let $\gamma, \beta \in \mathbb{C},\ \mathbb{R}(\gamma) > 0,\ \mathbb{R}(\beta) > 0,\ \lambda \in \mathbb{R}$. Then hold

$$\mathfrak{L}_{\alpha}\left(t^{\gamma m+\beta-1} E^{(m)}_{\gamma,\beta}(\lambda t^{\gamma})\right) = \frac{s^{\frac{\gamma-\beta}{\alpha}}}{(s^{\gamma/\alpha} - \lambda)^{m+1}} \tag{4.3}$$

Proof Remember the next series convergence

$$\sum_{k=0}^{\infty} \frac{(k+m)!}{k!} x^k = \frac{m!}{(1-x)^{m+1}} \tag{4.4}$$

Then

$$\begin{aligned}
\mathfrak{L}_{\alpha}\left(t^{\gamma m+\beta-1} E^{(m)}_{\gamma,\beta}(\lambda t^{\gamma})\right) &= \sum_{k=0}^{\infty} \frac{(k+m)!\lambda^k}{k!} \frac{\mathfrak{L}_{\alpha}[t^{\gamma k+\gamma m+\beta-1}]}{\Gamma(\gamma k + \gamma m + \beta)}(s) \\
&= \sum_{k=0}^{\infty} \frac{(k+m)!\lambda^k}{k!} \frac{\Gamma(\gamma k + \gamma m + \beta)}{\Gamma(\gamma k + \gamma m + \beta) s^{\frac{\gamma k+\gamma m+\beta-1+1}{\alpha}}} \\
&= \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{\lambda^k}{s^{\frac{\gamma k+\gamma m+\beta}{\alpha}}} \\
&= s^{\frac{-\gamma m-\beta}{\alpha}} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} (\lambda s^{-\gamma/\alpha})^k \\
&= s^{\frac{-\gamma m-\beta}{\alpha}} \frac{m!}{(1-\lambda s^{-\gamma/\alpha})^{m+1}} \\
&= s^{\frac{-\gamma m-\beta}{\alpha}} \frac{m!}{s^{-(m+1)\gamma/\alpha}(s^{\gamma/\alpha} - \lambda)^{m+1}} \\
&= \frac{s^{\frac{\gamma-\beta}{\alpha}}}{(s^{\gamma/\alpha} - \lambda)^{m+1}}
\end{aligned}$$

5 Example

A slight generalization of an equation solved in [4, page 157]

$$D^{\frac{1}{2}}f(t) + af(t) = 0; \qquad I^{\frac{1}{2}}f(t)|_{t=0} = C \tag{5.1}$$

applying the The $\alpha-$Integral Laplace Transform, with $\alpha = \frac{1}{2}$, we obtained

$$\mathfrak{L}_{\frac{1}{2}}\left(D^{\frac{1}{2}}f(t) + af(t)\right) = 0 \tag{5.2}$$

$$s\mathfrak{L}_{\frac{1}{2}}[f(t)](s) - I^{\frac{1}{2}}f(t)|_{t=0} + a\mathfrak{L}_{\frac{1}{2}} = 0 \tag{5.3}$$

$$\mathfrak{L}_{\frac{1}{2}}[f(t)](s) = \frac{C}{s+a} \tag{5.4}$$

$$\tag{5.5}$$

and applying definition (1.10) gives the solution of (5.1)

$$\mathfrak{L}_{\frac{1}{2}}^{-1}\left(\mathfrak{L}_{\frac{1}{2}}[f(t)](s)\right) = \mathfrak{L}_{\frac{1}{2}}^{-1}\left(\frac{C}{s+a}\right) \tag{5.6}$$

$$f(t) = Ct^{-\frac{1}{2}}E_{\frac{1}{2},\frac{1}{2}}(-at^{\frac{1}{2}}) \tag{5.7}$$

is identical to solution obtained in [8,page 139]

References

[1] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Vol. 204, Elsevier Science, 2006.
https://doi.org/10.1016/s0304-0208(06)x8001-5

[2] P.J. Miana, Convolution products in $L^1(R^+)$, integral transforms and fractional calculus, *Fractional Calculus and Applied Analysis*, **8** (2005), no. 4, 361-370.

[3] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Willey, 1993.

[4] K. Oldham, J. Spanier, *The Fractional Calculus*, Academic Press, 1992.

[5] L.G. Romero, R.A. Cerutti, L.L. Luque, A New Fractional Fourier Transform and Convolution Products, *International Journal of Pure and Applied Mathematics*, **66** (2011), no. 4, 397-408.

[6] L.G. Romero, G.D. Medina, N.R. Ojeda, J.H. Pereira, A new alfa-Integral Laplace Transform, *Asian Journal of Current Engineering and Maths.*, **5** (2016), 59-62.

[7] I. Podlubny, *Fractional Differential Equations*, Academic Press, United States, 1999.

[8] S. Samko, A. Kilbas, O. Marichev, *Fractional Integrals and Derivatives*, Gordon and Breach, 1993.

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