Lehmer - 3 Mean Labeling of Graphs

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Abstract

A graph $G=(V,E)$ with $P$ vertices and $q$ edges is called Lehmer - 3 mean graph, if it is possible to label vertices $x \in V$ with distinct label $f(x)$ from $1,2,3,\ldots,q+1$ in such a way that when each edge $e=uv$ is labeled with

$$f(e=uv)=\left\lfloor \frac{f(u)^3+f(v)^3}{f(u)^2+f(v)^2} \right\rfloor \quad \text{or} \quad \left\lceil \frac{f(u)^3+f(v)^3}{f(u)^2+f(v)^2} \right\rceil,$$

then the edge labels are distinct. In this case $f$ is called Lehmer - 3 mean labeling of $G$. In this paper we investigate Lehmer - 3 mean labeling of some standard graphs.

Keywords: Path, Cycle, Comb, Ladder, Crown, Triangular snake, Quadrilateral snake

1. Introduction

A graph considered here are finite, undirected and simple. The vertex set and edge set of a graph are denoted by $V(G)$ and $E(G)$ respectively. A path of length $n$ is denoted by $P_n$. A cycle of length $n$ is denoted by $C_n$. For standard terminology and notations we follow Harary [2] and for the detailed survey of graph labeling we follow J.A. Gallian [1]. S. Somasundaram and S.S. Sandhya introduced the concept of Harmonic Mean Labeling of Graphs in [3] and its basic results was proved in [3]. In this paper we investigate Lehmer – 3 Mean Labeling behavior of some standard graphs. The following definitions are used here:
Definition 1.1 A graph $G=(V,E)$ with $p$ vertices and $q$ edges is called Lehmer - 3 mean graph. If it is possible to label vertices $x \in V$ with distinct labels $f(x)$ from $1,2,3,\ldots,q+1$ in such a way that when each edge $e=uv$ is labeled with $f(e=uv)=\left\lceil \frac{f(u)^3+f(v)^3}{f(u)^2+f(v)^2} \right\rceil$ (or) $\left\lfloor \frac{f(u)^3+f(v)^3}{f(u)^2+f(v)^2} \right\rfloor$, then the edge labels are distinct. In this case $f$ is called Lehmer - 3 mean labeling of $G$.

Definition 1.2 A path $P_n$ is obtained by joining $u_i$ to the consecutive vertices $u_{i+1}$ for $1 \leq i \leq n$.

Definition 1.3 Comb is a graph obtained by joining a single pendant edge to each vertex of a path.

Definition 1.4 A closed path is called a cycle of $G$.

Definition 1.5 A product graph $P_m \times P_n$ is called a planar grid $P_2 \times P_n$ is called a ladder.

Definition 1.6 Crown is a graph obtained by joining a single pendant edge to each vertex of a cycle.

Definition 1.7 A star graph is a graph obtained from a complete bipartite graph $K_{1,n}$.

Definition 1.8 A triangular snake $T_n$ is obtained from a path $v_n, v_{n-1}, \ldots, v_1$ by joining $v_i$ to a new vertex $w_i$ for $1 \leq i \leq n-1$. That is every edge of a path is replaced by a triangle $C_3$.

Definition 1.9 A quadrilateral snake $Q_n$ is obtained from a path $u_n, u_{n-1}, \ldots, u_1$ by joining $u_i, u_{i+1}$ to a new vertices $v_i$ and $w_i$ respectively and joining $v_i$ and $w_i$. That is every edge of a path is replaced by a cycle $C_4$.

2. Main Results

Theorem 2.1 Any path is a Lehmer -3 mean graph.

Proof: Let $P_n$ be a path $v_n, v_{n-1}, \ldots, v_1$. Define a function $f: (V(P_n)) \rightarrow \{1,2,\ldots,q+1\}$ by $f(v_i)=i$; $1 \leq i \leq n$.

Then we get distinct edge labels clearly $f$ is a Lehmer -3 mean labeling of $G$.

Example 2.2 A Lehmer -3 mean labeling of $P_n$ is given below.
Theorem 2.3 Any cycle is a Lehmer -3 mean graph.

Proof: Let $C_n$ be a cycle of length $n$. Let the vertices of $C_n$ be $u_1, u_2, \ldots, u_n, u_1$. Define a function $f: V(C_n) \rightarrow \{1, 2, \ldots, q+1\}$ by $f(u_i)=i; 1 \leq i \leq n$. The edges are labeled with $f(u_i, u_{i+1})=i; 1 \leq i \leq n$. Hence $f$ is a Lehmer -3 mean labeling of graph $G$.

Example 2.4 A Lehmer -3 mean labeling of $C_{10}$ is given below.
Theorem 2.5 Combs are Lehmer-3 mean graph.

Proof: Let $C_n$ be a comb with $V(G) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. Let $P_n$ be a path. Let us label $P_n = v_1, v_2, \ldots, v_n$ and join a vertex $u_i$ to $v_i$, $1 \leq i \leq n$.

Define a function $f: V(G) \rightarrow \{1, 2, 3, \ldots, q+1\}$ by $f(v_i) = 2i-1$; $1 \leq i \leq n$ and $f(u_i) = 2i$, $1 \leq i \leq n$.

The label of the edge $u_iv_i$ is $2i-1$; $1 \leq i \leq n$ and the label of the edge $u_iv_{i+1}$ is $2i$, $1 \leq i \leq n$.

Clearly the edge labels are distinct and hence $f$ is a Lehmer-3 mean labeling of graph $G$.

Example 2.6 A Lehmer-3 mean labeling of $P_6 \circ k_{1,1}$ is given below.

![Figure 3](image)

Theorem 2.7 Any Triangular snake $T_n$ is a Lehmer-3 mean graph.

Proof: Let $T_n$ be a Triangular snake.

Define a function $f$: $V(T_n) \rightarrow \{1, 2, 3, \ldots, q+1\}$ by

$f(v_i) = 1$

$f(v_i) = 3i-3$; $2 \leq i \leq n$

$f(w_i) = 3i-1$; $1 \leq i \leq n$

The label of the edges $f(v_1v_2) = 2$

The label of the edges $f(v_iv_{i+1})$ is $3i-1$, $2 \leq i \leq n$

The label of the edge $f(v_iw_i)$ is $3i-2$, $1 \leq i \leq n$

The label of the edge $f(w_iv_{i+1}) = 3i$, $1 \leq i \leq n$
Hence this $T_n$ forms a Lehmer -3 mean graph.

**Example 2.8** The Lehmer -3 mean labeling of $T_3$ is given below.

![Figure 4]

**Theorem 2.9** Any Quadrilateral snake $Q_n$ is a Lehmer -3 mean graph.

**Proof:** Let $Q_n$ be the Quadrilateral snake as in definition.
Define $f: V(Q_n)\rightarrow \{1,2,\ldots,q+1\}$ by

- $f(u_1)=1$,
- $f(u_i)=4i-4; \ 2\leq i\leq n$
- $f(v_i)=4i-2; \ 1\leq i\leq n$
- $f(w_i)=4i-1; \ 1\leq i\leq n$

The label of the edge $u_1u_2$ is 3
The label of the edge $u_iu_{i+1}$ is $4i-1, \ 2\leq i\leq n$
The label of the edge $u_iv_i$ is $4i-3, \ 1\leq i\leq n$
The label of the edge $v_iw_i$ is $4i-2, \ 1\leq i\leq n$
The label of the edge $u_iw_i$ is $4i, \ 1\leq i\leq n$

This gives a Lehmer -3 mean labeling $Q_n$.

**Example 2.10** The Lehmer -3 mean labeling of $Q_4$ is given below.

![Figure 5]
Theorem 2.11 Any Ladder is a Lehmer -3 mean graph.

Proof: Let Lₙ denote a ladder graph. Define f: V(Lₙ) → {1, 2, …, q+1} by f(u₁) = 1
f(uᵢ) = 3i - 3; i = 2, 3, …, n and f(vᵢ) = f(uᵢ) + 1; i ≤ i ≤ n
The label of the edge uᵢuᵢ₊₁ is 3i - 1, 1 ≤ i ≤ n

The label of the edge uᵢvᵢ is 3i - 2, 1 ≤ i ≤ n

The label of the edge vᵢvᵢ₊₁ is 3i, 1 ≤ i ≤ n

This makes Lₙ as a Lehmer -3 mean graph.

Example 2.12 Lehmer -3 mean labeling of L = PₚxP₂ is given below.

Theorem 2.13 A Crown CₙOK₁ is a Lehmer 3 mean graph for all n ≥ 3

Proof: Let Cₙ be a cycle uᵢuᵢ,…, u₀u₁ and vᵢ be the pendent vertices adjacent to uᵢ, 1 ≤ i ≤ n.

Define a function f: V(CₙOK₁) → {1, 2, …, q+1} by
f(uᵢ) = 2i - 1; 1 ≤ i ≤ n
f(vᵢ) = 2i; 1 ≤ i ≤ n

Then the edge labels are all distinct.

Obviously f is a Lehmer -3 mean labeling.

Example 2.14 A Lehmer -3 mean labeling of C₅OK₁ is given below.
References


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