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Two Topics in Number Theory: Sum of Divisors of the Factorial and a Formula for Primes

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Abstract

A asymptotic formula for the sum of the divisors of the factorial is obtained. A formula for primes is also obtained.

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1 Sum of Divisors of the Factorial

In this article, as usual, $d(n)$ denotes the number of positive divisors of n, $\sigma(n)$ denotes the sum of the positive divisors of n and p denotes a positive prime.

Let $E(p)$ be the multiplicity (exponent) of the prime p in the prime factorization of $n!$. Therefore

$$n! = \prod_{p \leq n} p^{E(p)} \tag{1}$$

The logarithm of the number of divisors of $n!$, namely

$$\log(d(n!)) = \sum_{p \leq n} \log(E(p) + 1)$$

was studied by P. Erdős, S. W. Graham, A. Ivić and C. Pomerance in [2]. These authors obtained (among another results) the asymptotic formula

$$\log\left(d(n!)\right) \sim c_0 \frac{n}{\log n} \tag{2}$$

where the constan c_0 is

$$c_0 = \sum_{k=2}^{\infty} \frac{\log k}{(k-1)k} \tag{3}$$

R. Jakimczuk [4] studied the sum of the logarithms of the exponents of $n!$, namely

$$\sum_{p \leq n} \log E(p)$$

and obtained the formula

$$\sum_{p \leq n} \log E(p) \sim c_1 \frac{n}{\log n}$$

where the constan c_1 is

$$c_1 = \sum_{k=2}^{\infty} \frac{\log k}{(k+1)k}$$

A simple change in the reasoning given in [4] proves equation (2) with the constant c_0 (3).

In this section we study the sum of the divisors of $n!$, namely $\sigma(n!)$ and the logarithm of $\sigma(n!)$.

We shall need the following lemmas.

Lemma 1.1 *The following two power series are well-known*

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad |x| < 1 \tag{4}$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad |x| < 1 \tag{5}$$

We also have the following asymptotic formulae.

$$\frac{1}{1-x} = 1 + f(x)x \tag{6}$$

where $f(x) \to 1$ *when* $x \to 0$ *(use the L'Hospital's rule).*

$$\log(1+x) = f(x)x \tag{7}$$

where $f(x) \to 1$ *when* $x \to 0$ *(use the L'Hospital's rule).*

$$e^x = 1 + f(x)x \tag{8}$$

where $f(x) \to 1$ *when* $x \to 0$ *(use the L'Hospital's rule).*

Lemma 1.2 *The following asymptotic formulae hold*

$$\sum_{j=1}^{n} \log j = n \log n - n + \frac{1}{2} \log n + \log \sqrt{2\pi} + O\left(\frac{1}{n}\right), \tag{9}$$

$$n! = \frac{\sqrt{2\pi} n^n \sqrt{n}}{e^n} \left(1 + O\left(\frac{1}{n}\right)\right) \tag{10}$$

Proof. The proof of equation (9) is as follows

$$\sum_{j=1}^{n} \log j = \int_1^n \log x + \log n - \frac{1}{2} \log n - \sum_{i=1}^{n-1} c_i = n \log n - n + \frac{1}{2} \log n + 1$$

$$- \sum_{i=1}^{\infty} c_i + \sum_{i=n}^{\infty} c_i = n \log n - n + \frac{1}{2} \log n + C + O\left(\frac{1}{n}\right),$$

where $C = 1 - \sum_{i=1}^{\infty} c_i$ and $\sum_{i=n}^{\infty} c_i = O\left(\frac{1}{n}\right)$. Note that the area $\int_j^{j+1} \log x dx$ is the sum of three areas, the area of the rectangle of basis 1 and height $\log j$, the area of the rectangle triangle of basis 1 and height $\log(j+1) - \log j$ and the area c_j between the choord and the curve $\log x$. Note also that the derivative of $\log x$, namely $1/x$ is strictly decreasing and consequently the area $\sum_{i=n}^{\infty} c_i$ is contained in the rectangle triangle of basis 1 and height $1/n$.

The value of the constant C is obtained from the Stirling's formula $n! \sim \sqrt{2\pi}\frac{n^n\sqrt{n}}{e^n}$. Finally, equation (10) is an immediate consequence of equations (9) and (8). The lemma is proved.

Lemma 1.3 *The following two Mertens's formulae are well-known (see, for example, [5])*

$$\sum_{p \le x} \frac{1}{p} = \log \log x + M + O\left(\frac{1}{\log x}\right) \tag{11}$$

where M is called Mertens's constant.

$$\prod_{p \le x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x} \left(1 + O\left(\frac{1}{\log x}\right)\right) \tag{12}$$

where $\gamma = 0.5772156649\ldots$ is Euler's constant.

The following is a well-known Chevyshev inequality (see, for example, [1], [3] or [5])

$$\pi(x) \le c \frac{x}{\log x} \tag{13}$$

where $\pi(x)$ is the prime counting function and c is a positive constant.

The following Legendre's theoren is well-known (see for example [1], [3] or [5])

$$E(p) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor \tag{14}$$

where, as usual, $\lfloor x \rfloor$ *denotes the integer part of* x.

Theorem 1.4 *The following asymptotic formulae hold.*

$$\sigma(n!) = e^{\gamma} n! \log n \left(1 + O\left(\frac{1}{\log n}\right)\right) \tag{15}$$

$$\sigma(n!) = e^{\gamma} \sqrt{2\pi} \frac{n^n \sqrt{n}}{e^n} \log n \left(1 + O\left(\frac{1}{\log n}\right)\right) \tag{16}$$

$$\log\left(\sigma(n!)\right) = n \log n - n + \frac{1}{2} \log n + \log\log n + \gamma + \log\sqrt{2\pi} + O\left(\frac{1}{\log n}\right) \tag{17}$$

Proof. We have

$$\sigma(n!) = \prod_{p \leq n} \frac{p^{E(p)+1} - 1}{p - 1} = \frac{1}{\prod_{p \leq n}\left(1 - \frac{1}{p}\right)} \prod_{p \leq n} p^{E(p)} \prod_{p \leq n} \left(1 - \frac{1}{p^{E(p)+1}}\right) \tag{18}$$

If $0 < x < 1$ then we have (see equations (5) and (4))

$$\begin{aligned} & 0 < -\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \leq x + x^2 + x^3 + \cdots \\ = \quad & \frac{x}{1-x} = \frac{1}{\frac{1}{x} - 1} \end{aligned} \tag{19}$$

Substituting $x = \frac{1}{p^{E(p)+1}}$ into (19) we obtain (see (14))

$$\begin{aligned} & 0 < -\log(1 - \frac{1}{p^{E(p)+1}}) \leq \frac{1}{p^{E(p)+1} - 1} = \frac{1}{p^{E(p)+1}\left(1 - \frac{1}{p^{E(p)+1}}\right)} \leq 2\frac{1}{p^{E(p)+1}} \\ \leq \quad & \frac{2}{p^{E(p)}} \leq \frac{2}{pE(p)} \leq \frac{2}{p\left\lfloor \frac{n}{p} \right\rfloor} \end{aligned} \tag{20}$$

Note that $p^{E(p)+1} \geq 2$ implies $\frac{1}{1 - \frac{1}{p^{E(p)+1}}} \leq 2$ and if $k \geq 1$ and $n \geq 2$ then by mathematical induction we can prove easily the inequality $n^k \geq kn$.

On the other hand, we have

$$\begin{aligned} 0 \quad &\leq \quad \sum_{p\leq n}\frac{1}{p\left\lfloor\frac{n}{p}\right\rfloor} = \sum_{p\leq\frac{n}{2}}\frac{1}{p\left\lfloor\frac{n}{p}\right\rfloor} + \sum_{\frac{n}{2}<p\leq n}\frac{1}{p} \leq \sum_{p\leq\frac{n}{2}}\frac{1}{p\left(\frac{n}{p}-1\right)} + \sum_{\frac{n}{2}<p\leq n}\frac{1}{p} \\ &= \quad \frac{1}{n}\sum_{p\leq\frac{n}{2}}\frac{1}{1-\frac{p}{n}} + \sum_{\frac{n}{2}<p\leq n}\frac{1}{p} \end{aligned} \tag{21}$$

Note that if $\frac{n}{2} < p \leq n$ then $\left\lfloor\frac{n}{p}\right\rfloor = 1$, besides $\lfloor x\rfloor \geq x-1$.

Equation (11) and the mean value theorem give us

$$\sum_{\frac{n}{2}<p\leq n}\frac{1}{p} = \log\log n - \log\log\left(\frac{n}{2}\right) + O\left(\frac{1}{\log n}\right) = O\left(\frac{1}{\log n}\right) \tag{22}$$

On the other hand, we have (see equation (13))

$$0 \leq \frac{1}{n}\sum_{p\leq\frac{n}{2}}\frac{1}{1-\frac{p}{n}} \leq \frac{2}{n}\sum_{p\leq\frac{n}{2}}1 = \frac{2}{n}\pi\left(\frac{n}{2}\right) \leq c'\frac{1}{\log n}$$

where c' is a positive constant. Note that if $p \leq \frac{n}{2}$ then $\frac{1}{1-\frac{p}{n}} \leq 2$. Therefore

$$\frac{1}{n}\sum_{p\leq\frac{n}{2}}\frac{1}{1-\frac{p}{n}} = O\left(\frac{1}{\log n}\right) \tag{23}$$

Equations (23), (22), (21) and (20) give us

$$\sum_{p\leq n}\log\left(1-\frac{1}{p^{E(p)+1}}\right) = O\left(\frac{1}{\log n}\right) \tag{24}$$

and consequently (see (8))

$$\prod_{p\leq n}\left(1-\frac{1}{p^{E(p)+1}}\right) = 1 + O\left(\frac{1}{\log n}\right) \tag{25}$$

Finally, substituting equations (1), (12) and (25) into (18) and using equation (6) we obtain (15). Equation (16) is an immediate consequence of (15) and (10). Equation (17) is an immediate consequence of equations (16) and (7). The theorem is proved.

Now, note the following observation. Among the divisors of $n!$ are the n divisors $\frac{n!}{k}$ $(k = 1, \ldots, n)$. The contribution of these n divisors to $\sigma(n!)$ is (see (15))

$$\frac{\sum_{k=1}^{n}\frac{n!}{k}}{\sigma(n!)} \to \frac{1}{e^{\gamma}} = 0.561459\ldots$$

where we have used the well-known formula (see, for example, [5])

$$\sum_{k=1}^{n} \frac{1}{k} \sim \log n$$

On the other hand, the contribution of the rest of the divisors to $\sigma(n!)$ is

$$\frac{\sigma(n!) - \sum_{k=1}^{n} \frac{n!}{k}}{\sigma(n!)} = 1 - \frac{\sum_{k=1}^{n} \frac{n!}{k}}{\sigma(n!)} \to \left(1 - \frac{1}{e^{\gamma}}\right) = 0.438540\ldots$$

That is, a less contribution than the former n.

2 A Formula for Primes

The following theorem is sometimes called either the principle of cross-classification or the inclusion-exclusion principle. We now enunciate the principle.

Theorem 2.1 *(Inclusion-exclusion principle)Let S be a set of N distinct elements, and let $S_1, \ldots, S_r$ be arbitrary subsets of S containing $N_1, \ldots, N_r$ elements, respectively. For $1 \leq i < j < \ldots < l \leq r$, let $S_{ij\ldots l}$ be the intersection of $S_i, S_j, \ldots, S_l$ and let $N_{ij\ldots l}$ be the number of elements of $S_{ij\ldots l}$. Then the number K of elements of S not in any of $S_1, \ldots, S_r$ is*

$$K = N - \sum_{1\leq i\leq r} N_i + \sum_{1\leq i<j\leq r} N_{ij} - \sum_{1\leq i<j<k\leq r} N_{ijk} + \ldots + (-1)^r N_{12\ldots r}$$

Proof. See, for example, [1], [3] or [5].

In the following theorem we establish a formula for the k-th prime.

Theorem 2.2 *Let p_n be the n-th prime and let k be an arbitrary but fixed positive integer. The following formula holds*

$$\lim_{x\to\infty} x^{\frac{1}{A_k(x)}} = p_k \tag{26}$$

where the function $A_k(x)$ is

$$A_k(x) = \sum_{n'\leq x} \mu(n') \left\lfloor \frac{x}{n'} \right\rfloor$$

and where n' denotes a squarefree not multiple of p_k (including 1), $\mu(n)$ is the Möbius function and $\lfloor x \rfloor$ denotes the integer part of x.

Therefore, if we know the primes different of p_k then we can determinate the prime p_k using limit (26). That is, if we have as information all primes except one, we can determinate this one.

Proof. The number of prime powers of p_k not exceeding x is clearly

$$\left\lfloor \frac{\log x}{\log p_k} \right\rfloor + 1.$$

By the inclusion-exclusion principle this number is

$$\lfloor x \rfloor - \sum_{\substack{p_i \leq x \\ p_i \neq p_k}} \left\lfloor \frac{x}{p_i} \right\rfloor + \sum_{\substack{p_i < p_j \leq x \\ p_i, p_j \neq p_k}} \left\lfloor \frac{x}{p_i p_j} \right\rfloor - \cdots = \sum_{n' \leq x} \mu(n') \left\lfloor \frac{x}{n'} \right\rfloor = A_k(x)$$

Therefore we have

$$A_k(x) = \left\lfloor \frac{\log x}{\log p_k} \right\rfloor + 1 = \frac{\log x}{\log p_k} - \left\{ \frac{\log x}{\log p_k} \right\} + 1$$

where $\{y\} = y - \lfloor y \rfloor$ is the fractional part of y and consequently

$$\lim_{x \to \infty} \frac{A_k(x)}{\log x} = \frac{1}{\log p_k}$$

The theorem is proved.

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References

[1] T. M. Apostol, *Introduction to Analytic Number Theory,* Springer, 1976. https://doi.org/10.1007/978-1-4757-5579-4

[2] P. Erdős, S. W. Graham, A. Ivić and C. Pomerance, On the number of divisors of $n!$, Chapter in *Analytic Number Theory*, Springer, 1996, 337-355. https://doi.org/10.1007/978-1-4612-4086-0_19

[3] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers,* Oxford, 1960.

[4] R. Jakimczuk, Logarithm of the exponents in the prime factorization of the factorial, *International Mathematical Forum,* **12** (2017), no. 13, 643-649. https://doi.org/10.12988/imf.2017.7543

[5] W. J. LeVeque, *Topics in Number Theory,* Volume I, Addison-Wesley, 1956.

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