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Diophantine Equations. Elementary Methods II

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Abstract

In this article we study some general diophantine equations. Our methods of solution are different and very elementary.

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1 Introduction and Main Results

In a previous article [2] we define derivative solution of a solution and complete system of solutions to an equation. For sake of completeness we establish these definitions here.

Let us consider the equation

$$\sum_{j=1}^{h} k_j x_j^{r_j} = k_{h+1} x_{h+1}^{r_{h+1}} \tag{1}$$

where $h \geq 2$, the coefficients k_j $(j = 1, \ldots, h+1)$ are integers different of zero and the exponents $r_j \geq 2$ $(j = 1, \ldots, h+1)$ are positive integers.

Let us consider a solution

$$(x_1, x_2, \ldots, x_h, x_{h+1}) \tag{2}$$

to equation (1) where the x_j $(j = 1, \ldots, h+1)$ are integers. If we multiply both sides of equation (1) by E^L, where E is an integer different of zero and L

is the least common multiple (lcm)of the exponents r_j $(j = 1, \ldots, h+1)$, then we obtain the solution

$$\left(x_1 E^{\frac{L}{r_1}}, x_2 E^{\frac{L}{r_2}}, \ldots, x_h E^{\frac{L}{r_h}}, x_{h+1} E^{\frac{L}{r_{h+1}}}\right) \tag{3}$$

The solution (3) will be called derivative solution of (1). Note that solution (2) is derivative solution of solution (2) if we put $E = 1$. Clearly, from (3) we can obtain (2) by common factor. If a set of solutions of equation (1) contain at least one derivative solution of each solution of equation (1) we shall call this set of solutions a complete system of solutions to equation (1). Note that from a complete system of solutions we can obtain all solution to the equation by common factor. This method if not very different to consider the set of primitive solutions to, for example, the equation $x^2 + y^2 = z^2$ and to obtain the rest of the solutions by multiplication of the primitive solutions.

If we consider a certain subset S of solutions to equation (1) then a complete system of solutions in relation to S is a subset of S that contain at least a derivative solution of each solution of the set S.

In this note we study the solutions to the diophantine equation

$$\sum_{j=1}^{h} k_j x_j^2 = k_{h+1} x_{h+1}^r \tag{4}$$

where $h \geq 2$, the coefficients k_j $(j = 1, \ldots, h+1)$ are positive integers and the exponent $r \geq 2$ is an arbitrary but fixed positive integer.

A particular case of this general diophantine equation is well-known, namely, the equation

$$x^2 + y^2 = z^r \tag{5}$$

This equation is studied in [1] as part of the dihedral cases.

We also study another general diphantine equations and as a particular case of our general theorems the equation $x^2 + y^4 = z^6$ is studied. This particular equation is studied in [1] as part of the hyperbolic case. Also, as a particular case of our general theorems the equation $x^2 - y^2 = z^r$ is studied. This particular equation is studied in [1] as part of the dihedral cases.

Our methods of solution are different and very elementary.

In [2] is proved the following general theorem.

Theorem 1.1 *Let us consider the diophantine equation*

$$\sum_{j=1}^{h} k_j x_j^{r_j} = k_{h+1} x_{h+1}^{\frac{M+1}{d}}$$

where $h \geq 2$*, the coefficients* k_j $(j = 1, \ldots, h)$ *and* k_{h+1} *are integers differents of zero, each integer exponent* $r_j \geq 2$ $(j = 1, \ldots, h)$ *divides the positive integer*

M and the positive integer d $(0 < d < M+1)$ *divides* $M+1$. *Let us consider the solutions to the equation* $(x_1, \ldots, x_h, x_{h+1})$, *where* $x_{h+1} \neq 0$. *Then a complete system of solutions to the equation is*

$$x_j = k_{h+1}^{\frac{M^2}{r_j}} A^{\frac{M}{r_j}} b_j \qquad (j = 1, 2, \ldots, h), \qquad x_{h+1} = \left(k_{h+1}^{M-1} A\right)^d$$

where $A = \sum_{i=1}^{h} k_i b_i^{r_i}$ *and the* b_j *are arbitrary integers such that* $A \neq 0$.

The case r odd in equation (4) is a particular case of Theorem 1.1 when $r_1 = r_2 = \cdots = r_h = 2$, $M = r - 1$ $(r \geq 3)$ and $d = 1$.

Consequently we have the following theorem.

Theorem 1.2 *Let us consider the diophantine equation*

$$\sum_{j=1}^{h} k_j x_j^2 = k_{h+1} x_{h+1}^r$$

where $h \geq 2$, *the coefficients* k_j $(j = 1, \ldots, h+1)$ *are positive integers and the exponent* $r \geq 3$ *is an arbitrary but fixed odd positive integer. Let us consider the solutions to the equation* $(x_1, \ldots, x_h, x_{h+1})$ *where* $x_{h+1} \neq 0$. *Then a complete system of solutions to the equation is*

$$x_j = k_{h+1}^{\frac{(r-1)^2}{2}} A^{\frac{r-1}{2}} b_j \qquad (j = 1, 2, \ldots, h) \qquad x_{h+1} = k_{h+1}^{r-2} A$$

where $A = \sum_{i=1}^{h} k_i b_i^2$ *and the* b_j *are arbitrary integers such that* $A \neq 0$.

Corollary 1.3 *Let us consider equation (5). Then, a complete system of solutions to equation (5) when* $r \geq 3$ *is odd is*

$$x = \left(a^2 + b^2\right)^{\frac{r-1}{2}} a \qquad y = \left(a^2 + b^2\right)^{\frac{r-1}{2}} b \qquad z = a^2 + b^2$$

where a and b are arbitrary integers.

In the following two theorems we examine equation (5) when r is even.

Theorem 1.4 *Let us consider the diophantine equation*

$$x^2 + y^2 = z^2$$

where $xyz \neq 0$. *Then, a complete system of solutions to the equation is*

$$x = a^2 - b^2, \qquad y = 2ab, \qquad z = -a^2 - b^2$$

where a and b are arbitrary integers such that $xyz \neq 0$.

Proof. See [2]. The theorem is proved.

Theorem 1.5 *Let s an arbitrary but fixed positive integer. Let us consider the diophantine equation*

$$x^2 + y^2 = z^{2s+2} \tag{6}$$

where $xyz \neq 0$. Then, a complete system of solutions to the equation is

$$x = (-a^2 - b^2)^s (a^2 - b^2), \qquad y = (-a^2 - b^2)^s 2ab, \qquad z = -a^2 - b^2 \tag{7}$$

where a and b are arbitrary integers such that $xyz \neq 0$.

Proof. We have the identity

$$\left((-a^2 - b^2)^s (a^2 - b^2)\right)^2 + \left((-a^2 - b^2)^s 2ab\right)^2 = \left(-a^2 - b^2\right)^{2s+2} \tag{8}$$

where a and b are arbitrary integers.

Consequently equation (6) has infinite solutions (x, y, z) such that $xyz \neq 0$.

Let us consider a solution (x, y, z) such that $xyz \neq 0$. We can write is solution in the form $(x, y, z) = (C^s a_1, C^s a_2, C)$ where a_1 and a_2 are rational numbers. Therefore we have

$$(C^s a_1)^2 + (C^s a_2)^2 = C^{2s+2} \tag{9}$$

We can write $a_1 = \frac{b_1}{d}$ and $a_2 = \frac{b_2}{d}$, where b_1, b_2 and d are integers. Hence (9) becomes

$$\left(C^s \frac{b_1}{d}\right)^2 + \left(C^s \frac{b_2}{d}\right)^2 = C^{2s+2} \tag{10}$$

If we multiply both sides of (10) by d^{2s+2} then we obtain

$$\left((Cd)^s\, b_1\right)^2 + \left((Cd)^s\, b_2\right)^2 = (Cd)^{2s+2} \tag{11}$$

Equation (11) gives

$$(b_1)^2 + (b_2)^2 = (Cd)^2 \tag{12}$$

By Theorem 1.4 there exists h such that if we multiply both sides of (12) by h^2 we obtain

$$(hb_1)^2 + (hb_2)^2 = (Cdh)^2$$

That is, we obtain

$$(hb_1 = a^2 - b^2)^2 + (hb_2 = 2ab)^2 = (Cdh = -a^2 - b^2)^2 \tag{13}$$

Now, if we multiply both sides of (11) by h^{2s+2} then we obtain the following derivative solution of the solution (x, y, z)

$$((Cdh)^s(hb_1))^2 + ((Cdh)^s(hb_2))^2 = (Cdh)^{2s+2} \tag{14}$$

This derivative solution can be written in the form (see (13) and (14))

$$\left((-a^2-b^2)^s(a^2-b^2)\right)^2 + \left((-a^2-b^2)^s 2ab\right)^2 = \left(-a^2-b^2\right)^{2s+2}$$

Compare with (8). The theorem is proved.

In the following two theorems we complete the study of equation (4) when r is even.

Theorem 1.6 *Let us consider the diophantine equation*

$$\sum_{j=1}^{h} k_j x_j^2 = k_{h+1} x_{h+1}^2$$

where $h \geq 2$ and the coefficients k_j $(j = 1, \ldots, h)$ and k_{h+1} are positive integers. Suppose that this equation has a solution

$$(x_1, x_2, \ldots, x_h, x_{h+1}) = (b_1, b_2, \ldots, b_h, b_{h+1})$$

different of the trivial solution $(0, 0, \ldots, 0, 0)$ and besides $\gcd(b_1, b_2, \ldots, b_h, b_{h+1}) = 1$. Then a complete system of solutions is

$$x_j = -b_j \sum_{i=1}^{h} k_i c_i^2 + 2c_j \sum_{i=1}^{h} k_i b_i c_i \qquad (j = 1, 2, \ldots, h) \qquad x_{h+1} = -b_{h+1} \sum_{i=1}^{h} k_i c_i^2$$

where the c_i $(i = 1, \ldots, h)$ are arbitrary integers.

Proof. See [2]. The theorem is proved.

Theorem 1.7 *Let s an arbitrary but fixed positive integer. Let us consider the diophantine equation*

$$\sum_{j=1}^{h} k_j x_j^2 = k_{h+1} x_{h+1}^{2s+2}$$

Suppose that is diophantine equation has a solution different of the trivial.

Then a complete system of solutions is

$$x_j = \left(-b_{h+1} \sum_{i=1}^{h} k_i c_i^2\right)^s \left(-b_j \sum_{i=1}^{h} k_i c_i^2 + 2c_j \sum_{i=1}^{h} k_i b_i c_i\right) \qquad (j = 1, 2, \ldots, h)$$

$$x_{h+1} = -b_{h+1} \sum_{i=1}^{h} k_i c_i^2$$

where the c_i $(i = 1, \ldots, h)$ are arbitrary integers.

Proof. The proof is the same as the proof of Theorem 1.5 using now Theorem 1.6. Note that we have the identity

$$\sum_{j=1}^{h} k_j \left(A_{h+1}^{s} A_j\right)^2 = k_{h+1} \left(A_{h+1}\right)^{2s+2}$$

where

$$A_j = -b_j \sum_{i=1}^{h} k_i c_i^2 + 2c_j \sum_{i=1}^{h} k_i b_i c_i \qquad (j = 1, 2, \ldots, h)$$

and

$$A_{h+1} = -b_{h+1} \sum_{i=1}^{h} k_i c_i^2$$

The theorem is proved.

Lemma 1.8 *Let s be an arbitrary but fixed positive integer. Let us consider the diophantine equation*

$$x_1^2 + \sum_{j=2}^{h} k_j x_j^2 = x_{h+1}^2 \tag{15}$$

where $h \geq 2$, the coefficients k_j $(j = 2, \ldots, h)$ are positive integers and some x_j $(j = 2, \ldots, h)$ is different of zero. Then a complete system of solutions to the equation is

$$x_1 = \left(a_1^2 - \sum_{j=2}^{h} k_j a_j^2\right) (2a_1)^{s-1}, \quad x_j = (2a_1)^s a_j \quad (j = 2, \ldots, h) \tag{16}$$

$$x_{h+1} = \left(-a_1^2 - \sum_{j=2}^{h} k_j a_j^2\right) (2a_1)^{s-1} \tag{17}$$

where the a_j $(j = 1, \ldots, h)$ are arbitrary integers such that some x_j $(j = 2, \ldots, h)$ is different of zero.

Proof. The equation has solutions with is property, since we have the identity (see (16) and (17))

$$\begin{aligned} &\left(\left(a_1^2 - \sum_{j=2}^{h} k_j a_j^2\right) (2a_1)^{s-1}\right)^2 + \sum_{j=2}^{h} k_j \left((2a_1)^s a_j\right)^2 \\ = &\left(\left(-a_1^2 - \sum_{j=2}^{h} k_j a_j^2\right) (2a_1)^{s-1}\right)^2 \end{aligned} \tag{18}$$

Let us consider then a solution $(x_1, \ldots, x_h, x_{h+1})$ with is property. We can write

$$(x_1, x_2, \ldots, x_h, x_{h+1}) = (C + a_1, a_2, \ldots, a_h, C) \tag{19}$$

Note that $C \neq 0$ and $a_1 \neq 0$, since in contrary case the property is not fulfilled. Consequently (see (15) and (19))

$$(C + a_1)^2 + \sum_{j=2}^{h} k_j a_j^2 = C^2 \tag{20}$$

Therefore

$$2Ca_1 + a_1^2 + \sum_{j=2}^{h} k_j a_j^2 = 0$$

That is

$$C = -\frac{a_1^2 + \sum_{j=2}^{h} k_j a_j^2}{2a_1} \tag{21}$$

Substituting (21) into (20) we obtain

$$\left(-\frac{a_1^2 + \sum_{j=2}^{h} k_j a_j^2}{2a_1} + a_1\right)^2 + \sum_{j=2}^{h} k_j a_j^2 = \left(-\frac{a_1^2 + \sum_{j=2}^{h} k_j a_j^2}{2a_1}\right)^2 \tag{22}$$

If we now multiply both sides of equation (22) by $(2a_1)^{2s}$ then we obtain the following derivative solution

$$\left(\left(a_1^2 - \sum_{j=2}^{h} k_j a_j^2\right)(2a_1)^{s-1}\right)^2 + \sum_{j=2}^{h} k_j \left((2a_1)^s a_j\right)^2$$
$$= \left(\left(-a_1^2 - \sum_{j=2}^{h} k_j a_j^2\right)(2a_1)^{s-1}\right)^2$$

of the solution $(x_1, \ldots, x_h, x_{h+1})$. Compare with (18). The lemma is proved.

Theorem 1.9 *Let us consider the diophantine equation*

$$x_1^2 + \sum_{j=2}^{h} k_j x_j^{\frac{2s}{s_j}} = x_{h+1}^{2s+2} \tag{23}$$

where the coefficients k_j $(j = 2, \ldots, h)$ are positive integers, s is a positive integer, the s_j $(j = 2, \ldots, h)$ are divisors of s and some x_j $(j = 2, \ldots, h)$ is different of zero. Then a complete system of solutions to the equation is

$$x_1 = A^s B \quad x_j = A^{s_j}(2t_1)^{s_j} t_j \quad (j = 2, \ldots, h) \quad x_{h+1} = A \tag{24}$$

where

$$A = -\left(t_1^2 + \sum_{j=2}^{h} k_j t_j^{\frac{2s}{s_j}}\right)(2t_1)^{s-1} \tag{25}$$

$$B = \left(t_1^2 - \sum_{j=2}^{h} k_j t_j^{\frac{2s}{s_j}}\right)(2t_1)^{s-1} \tag{26}$$

and the t_j $(j = 1, \ldots, h)$ are arbitrary integers such that some x_j is different of zero. That is, $t_1 \neq 0$ and some t_j $(j = 2, \ldots, h)$ is different of zero.

Proof. We have the identity

$$(A^s B)^2 + \sum_{j=2}^{h} k_j \left(A^{s_j}(2t_1)^{s_j} t_j\right)^{\frac{2s}{s_j}} = A^{2s+2} \tag{27}$$

consequently there exist solutions such that some x_j $(j = 2, \ldots, h)$ is different of zero. Let us consider a solution

$$(x_1, x_2, \ldots, x_h, x_{h+1}) \tag{28}$$

to the equation with is property. This solution can be written in the form

$$(C^s u_1, C^{s_1} u_2, \ldots, C^{s_h} u_h, C) \tag{29}$$

where the u_j $(j = 1, \ldots, h)$ are rational numbers.

We can write $u_j = \frac{n_j}{d}$ $(j = 1, \ldots, h)$ where d and the n_j $(j = 1, \ldots, h)$ are integers. Therefore we have (see (23) and (29))

$$\left(C^s \frac{n_1}{d}\right)^2 + \sum_{j=2}^{h} k_j \left(C^{s_j} \frac{n_j}{d}\right)^{\frac{2s}{s_j}} = C^{2s+2} \tag{30}$$

If we multiply both sides of equation (30) by $d^{2(2s)(2s+2)}$ then we obtain

$$\left(\left(d^{2(2s)}C\right)^s d^{4s-1} n_1\right)^2 + \sum_{j=2}^{h} k_j \left(\left(d^{2(2s)}C\right)^{s_j} d^{4s_j-1} n_j\right)^{\frac{2s}{s_j}} = \left(d^{2(2s)}C\right)^{2s+2} \tag{31}$$

Equation (31) gives

$$\left(d^{4s-1} n_1\right)^2 + \sum_{j=2}^{h} k_j \left(d^{4s_j-1} n_j\right)^{\frac{2s}{s_j}} = \left(d^{2(2s)}C\right)^2$$

That is

$$\left(d^{4s-1} n_1\right)^2 + \sum_{j=2}^{h} k_j \left(\left(d^{4s_j-1} n_j\right)^{\frac{s}{s_j}}\right)^2 = \left(d^{2(2s)}C\right)^2 \tag{32}$$

By Lemma 1.8 here exists h such that if we multiply both sides of (32) by h^{2s} we obtain

$$\left(h^s d^{4s-1} n_1\right)^2 + \sum_{j=2}^{h} k_j \left(\left(h^{s_j} d^{4s_j-1} n_j\right)^{\frac{s}{s_j}}\right)^2 = \left(h^{s_j} d^{2(2s)} C\right)^2 \tag{33}$$

where

$$h^s d^{4s-1} n_1 = \left(a_1^2 - \sum_{j=2}^{h} k_j a_j^2\right)(2a_1)^{s-1} \tag{34}$$

$$\left(h^{s_j} d^{4s_j-1} n_j\right)^{\frac{s}{s_j}} = (2a_1)^s a_j \qquad (j = 2, \ldots, h) \tag{35}$$

$$h^s d^{2(2s)} C = \left(-a_1^2 - \sum_{j=2}^{h} k_j a_j^2\right)(2a_1)^{s-1} \tag{36}$$

Equation (35) gives

$$\left(h^{s_j} d^{4s_j-1} n_j\right)^{\frac{s}{s_j}} = \left((2a_1)^{s_j}\right)^{\frac{s}{s_j}} a_j \qquad (j = 2, \ldots, h) \tag{37}$$

Therefore

$$a_j = t_j^{\frac{s}{s_j}} \qquad (j = 2, \ldots, h) \tag{38}$$

where t_j is the integer $\frac{h^{s_j} d^{4s_j-1} n_j}{(2a_1)^{s_j}}$ $(j = 2, \ldots, h)$ and consequently

$$h^{s_j} d^{4s_j-1} n_j = (2a_1)^{s_j} t_j \qquad (j = 2, \ldots, h) \tag{39}$$

If we multiply both sides of equation (31) by $h^{2s(s+1)}$ then we obtain the following derivative solution of solution (28)

$$\begin{aligned} &\left(\left(h^s d^{2(2s)} C\right)^s \left(h^s d^{4s-1} n_1\right)\right)^2 + \sum_{j=2}^{h} k_j \left(\left(h^s d^{2(2s)} C\right)^{s_j} \left(h^{s_j} d^{4s_j-1} n_j\right)\right)^{\frac{2s}{s_j}} \\ = &\left(h^s d^{2(2s)} C\right)^{2s+2} \end{aligned} \tag{40}$$

Substituting equations (34), (36) and (39) into equation (40) we obtain equation (27). Note that we have written $a_1 = t_1$. The theorem is proved.

Theorem 1.10 *Let us consider the diophantine equation*

$$\sum_{j=1}^{h} k_j x_j^{r_j} + \sum_{j=h+1}^{t} k_j x_j^{s_j} = k_{t+1} x_{t+1}^{M+1} \tag{41}$$

where the k_j $(j = 1, \ldots, t{+}1)$ *are integers different of zero, there exist a positive integer* $M \geq 2$ *such that the exponents* $r_j \geq 2$ *are divisors of* M $(j = 1, \ldots, h)$ *and the exponents* $s_j \geq 2$ $(j = h{+}1, \ldots, t)$ *are divisors of* $M{+}1$*. Let us consider the solutions to the equation* $(x_1, \ldots, x_h, x_{h+1}, \ldots, x_t, x_{t+1})$ *such that* $x_{t+1} \neq 0$ *and* $\sum_{j=h+1}^{t} k_j x_j^{s_j} - k_{t+1} x_{t+1}^{M+1} \neq 0$*. Then a complete system of solutions to the equation is*

$$x_j = A^{\frac{M}{r_j}} B^{\frac{M^2}{r_j}} m^{\frac{M(M+1)}{r_j} - 1} v_j \qquad (j = 1, \ldots, h) \tag{42}$$

$$x_j = A^{\frac{M+1}{s_j}} B^{\frac{M^2-1}{s_j}} m^{\frac{M(M+1)}{s_j} - 1} v_j \qquad (j = h+1, \ldots, t) \tag{43}$$

$$x_{t+1} = m^M A B^{M-1} \tag{44}$$

where

$$A = -\sum_{j=1}^{h} k_j \left(m^{\frac{M(M+1)}{r_j} - 1} v_j \right)^{r_j} \tag{45}$$

$$B = -k_{t+1} m^{M(M+1)} + \sum_{j=h+1}^{t} k_j \left(m^{\frac{M(M+1)}{s_j} - 1} v_j \right)^{s_j} \tag{46}$$

the integers v_j $(j = 1, \ldots, t)$ *and* m *are arbitrary and such that* $A \neq 0$ *and* $B \neq 0$*.*

Proof. Note that we have the identity

$$\begin{aligned} & \sum_{j=1}^{h} k_j \left(A^{\frac{M}{r_j}} B^{\frac{M^2}{r_j}} m^{\frac{M(M+1)}{r_j} - 1} v_j \right)^{r_j} + \sum_{j=h+1}^{t} k_j \left(A^{\frac{M+1}{s_j}} B^{\frac{M^2-1}{s_j}} m^{\frac{M(M+1)}{s_j} - 1} v_j \right)^{s_j} \\ = \; & k_{t+1} \left(m^M A B^{M-1} \right)^{M+1} \end{aligned} \tag{47}$$

Therefore there exist solutions to the equation with the properties of the theorem.

Let us consider a solution

$$(x_1, \ldots, x_h, x_{h+1}, \ldots, x_t, x_{t+1}) \tag{48}$$

to the equation with the properties of the theorem. This solution can be written in the form

$$\left(C^{\frac{M}{r_1}} u_1, \ldots, C^{\frac{M}{r_h}} u_h, C^{\frac{M+1}{s_{h+1}}} u_{h+1}, \ldots, C^{\frac{M+1}{s_t}} u_t, C \right) \tag{49}$$

where the u_j $(j = 1, \ldots, t)$ are certain rational numbers. Hence we have (see (41) and (49))

$$\sum_{j=1}^{h} k_j \left(C^{\frac{M}{r_j}} u_j\right)^{r_j} + \sum_{j=h+1}^{t} k_j \left(C^{\frac{M+1}{s_j}} u_j\right)^{s_j} = k_{t+1} C^{M+1} \tag{50}$$

We can write $u_j = \frac{v_j}{m}$ where m and the v_j $(j = 1, \ldots, t)$ are integers. Therefore (50) becomes

$$\sum_{j=1}^{h} k_j \left(C^{\frac{M}{r_j}} \frac{v_j}{m}\right)^{r_j} + \sum_{j=h+1}^{t} k_j \left(C^{\frac{M+1}{s_j}} \frac{v_j}{m}\right)^{s_j} = k_{t+1} C^{M+1} \tag{51}$$

If we multiply both sides of (51) by $m^{M(M+1)}$ then we obtain

$$\begin{aligned} &\sum_{j=1}^{h} k_j \left(C^{\frac{M}{r_j}} m^{\frac{M(M+1)}{r_j}-1} v_j\right)^{r_j} + \sum_{j=h+1}^{t} k_j \left(C^{\frac{M+1}{s_j}} m^{\frac{M(M+1)}{s_j}-1} v_j\right)^{s_j} \\ = \; & k_{t+1} \left(m^M C\right)^{M+1} \end{aligned} \tag{52}$$

Consequently we have

$$C = \frac{-\sum_{j=1}^{h} k_j \left(m^{\frac{M(M+1)}{r_j}-1} v_j\right)^{r_j}}{-k_{t+1} m^{M(M+1)} + \sum_{j=h+1}^{t} k_j \left(m^{\frac{M(M+1)}{s_j}-1} v_j\right)^{s_j}} = \frac{A}{B} \tag{53}$$

Substituting (53) into (52) and multiply both sides by $B^{M(M+1)}$ we obtain the following derivative solution of solution (48) (Compare with (47)).

$$\begin{aligned} &\sum_{j=1}^{h} k_j \left(A^{\frac{M}{r_j}} B^{\frac{M^2}{r_j}} m^{\frac{M(M+1)}{r_j}-1} v_j\right)^{r_j} + \sum_{j=h+1}^{t} k_j \left(A^{\frac{M+1}{s_j}} B^{\frac{M^2-1}{s_j}} m^{\frac{M(M+1)}{s_j}-1} v_j\right)^{s_j} \\ = \; & k_{t+1} \left(m^M A B^{M-1}\right)^{M+1} \end{aligned}$$

The theorem is proved.

Theorem 1.11 *Let us consider the diophantine equation*

$$\sum_{j=1}^{h} k_j x_j^{r_j} + \sum_{j=h+1}^{t} k_j x_j^{s_j} = k_{t+1} x_{t+1}^{M}$$

where the k_j $(j = 1, \ldots, t+1)$ *are integers different of zero, there exist a positive integer* $M \geq 2$ *such that the exponents* $r_j \geq 2$ *are divisors of* $M+1$ $(j = 1, \ldots, h)$ *and the exponents* $s_j \geq 2$ $(j = h+1, \ldots, t)$ *are divisors of* M*. Let us consider the solutions to the equation* $(x_1, \ldots, x_h, x_{h+1}, \ldots, x_t, x_{t+1})$

such that $x_{t+1} \neq 0$ *and* $\sum_{j=h+1}^{t} k_j x_j^{s_j} - k_{t+1} x_{t+1}^{M+1} \neq 0$. *Then a complete system of solutions to the equation is*

$$x_j = A^{\frac{M+1}{r_j}} B^{\frac{M^2-1}{r_j}} m^{\frac{M(M+1)}{r_j}-1} v_j \qquad (j = 1, \dots, h)$$

$$x_j = A^{\frac{M}{s_j}} B^{\frac{M^2}{s_j}} m^{\frac{M(M+1)}{s_j}-1} v_j \qquad (j = h+1, \dots, t)$$

$$x_{t+1} = m^{M+1} A B^M$$

where

$$A = k_{t+1} m^{M(M+1)} - \sum_{j=h+1}^{t} k_j \left(m^{\frac{M(M+1)}{s_j}-1} v_j \right)^{s_j}$$

$$B = \sum_{j=1}^{h} k_j \left(m^{\frac{M(M+1)}{r_j}-1} v_j \right)^{r_j}$$

the integers v_j $(j = 1, \dots, t)$ *and* m *are arbitrary and such that* $A \neq 0$ *and* $B \neq 0$.

Proof. The proof is the same as the proof of Theorem 1.10. The theorem is proved.

Theorem 1.12 *Let us consider the equation*

$$x_1^2 - x_2^2 + \sum_{j=3}^{h} k_j x_j^{r_j} = 0 \tag{54}$$

where the coefficients k_j $(j = 3, \dots, h)$ *are integers different of zero and the exponents* $r_j \geq 2$ $(j = 3, \dots, h)$ *are positive integers. Let us consider the solutions* $(x_1, x_2, x_3, \dots, x_h)$ *such that* $x_1 \neq x_2$. *Then, a complete system of solutions is*

$$x_1 = \left(-b_1^2 - \sum_{j=3}^{h} k_j a_j^{r_j} \right) (2b_1)^{\frac{L}{2}-1} + (2b_1)^{\frac{L}{2}} b_1 = A \tag{55}$$

$$x_2 = \left(-b_1^2 - \sum_{j=3}^{h} k_j a_j^{r_j} \right) (2b_1)^{\frac{L}{2}-1} = B \tag{56}$$

$$x_j = a_j (2b_1)^{\frac{L}{r_j}} \qquad (j = 3, \dots, h) \tag{57}$$

where $b_1 \neq 0$ *and* a_j $(j = 3, \dots, h)$ *are arbitrary integers and* L *is a fixed positive integer multiple of the least common multiple of the exponents* 2 *and* r_j $(j = 3, \dots, h)$.

Proof. We have the identity (see (55), (56) and (57))

$$A^2 - B^2 + \sum_{j=3}^{h} k_j \left(a_j (2b_1)^{\frac{L}{r_j}} \right)^{r_j} = 0 \tag{58}$$

Hence there exist solutions to equation (54) such that $x_1 \neq x_2$.

Let us consider a solution $(x_1, x_2, x_3, \ldots, x_h)$ such that $x_1 \neq x_2$. This solution can be written in the form

$$(x_1, x_2, x_3, \ldots, x_h) = (C + b_1, C, a_3, \ldots, a_h) \tag{59}$$

Consequently (see (54) and (59)) we have

$$(C + b_1)^2 - C^2 + \sum_{j=3}^{h} k_j a_j^{r_j} = 0 \tag{60}$$

Equation (60) gives

$$C = \frac{-b_1^2 - \sum_{j=3}^{h} k_j a_j^{r_j}}{2b_1} \tag{61}$$

Substituting (61) into (60) and multiply both sides by $(2b_1)^L$ we obtain the following derivative solution

$$A^2 - B^2 + \sum_{j=3}^{h} k_j \left(a_j (2b_1)^{\frac{L}{r_j}} \right)^{r_j} = 0$$

of solution (59). Compare with (58). The theorem is proved.

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