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Bishop 2-Type Frame for Non-null Curves

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Abstract

In this paper, we study Bishop 2-type frame for non-null curves in three and four dimensional semi Euclidean space. Later we obtain $\{Z_1, Z_2, Z_3, V_3\}$ Bishop 2-type frame for spacelike curves from Frenet 4-frame $\{T, V_1, V_2, V_3\}$ with aid semi-Euclidean rotation matrix in the semi-Euclidean 4-space.

Keywords: Bishop 2-type, semi-Euclidean space

1 Introduction

As First Bishop has studied the parallel frame in 1975 [1]. Recently, Parallel frame has been studied by some authors [2-5]. New version of Bishop frame using a common vector field as binormal vector field of a regular curve and an application to spherical images was introduced by Yılmaz and Turgut [6]. Yılmaz, Ünlütürk and Mağden have studied characterizations of some special curves of timelike curves according to the Bishop frame of type-2 in Minkowski 3-space [7]. Gürbüz has studied Bishop 2-type frame for inelastic curves in Minkowski 3-space [8]. Later Gürbüz and Işık have investigated nonlinear heat equation according to Bishop 2-type frame in Minkowski 3-space [9].

In section 2, we give some preliminaries. In section 3, we study Bishop 2-type frame for non-null curves in three and four dimensional semi Euclidean space and we obtain $\{Z_1, Z_2, Z_3, V_3\}$ Bishop 2-type frame from Frenet 4-frame $\{T, V_1, V_2, V_3\}$ for three case of spacelike curves with aid semi-Euclidean rotation matrix in the semi-Euclidean 4-space.

2 Preliminaries

The semi-Euclidean n-space E_1^n is given the following metric [10]: $\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2 + ... + dx_n^2$

Frenet-Serret frame $\{T, N, B\}$ derivative formulas are given as following in the semi-Euclidean 3-space E_1^3 :

$$T_s = \epsilon_2 \kappa N, \quad N_s = -\epsilon_1 \kappa T + \epsilon_3 \tau B, \; B_s = -\epsilon_2 \tau N. \tag{1}$$

Here κ, τ Frenet curvatures with $\langle T, T\rangle = \epsilon_1$, $\langle N, N\rangle = \epsilon_2$, $\langle B, B\rangle = \epsilon_3$. Frenet-Serret frame $\{T, V_1, V_2, V_3\}$ derivative formulas are given as following in E_1^4 :

$$T_s = \epsilon_2 \kappa V_1, \; V_{1s} = -\epsilon_1 \kappa T + \epsilon_3 \tau V_2, \; V_{2s} = -\epsilon_2 \tau V_1 + \epsilon_4 \rho V_3, \; V_{3s} = -\epsilon_3 \rho V_2 \tag{2}$$

where s is arc length, κ, τ, ρ and Frenet curvatures with

$$\langle T, T\rangle = \epsilon_1, \langle V_1, V_1\rangle = \epsilon_2, \langle V_2, V_2\rangle = \epsilon_3, \langle V_3, V_3\rangle = \epsilon_4$$

For an arbitrary vector $x = (x_1, x_2, x_3, ..., x_n)$ in E_1^n, if $\langle x, x\rangle > 0$, x is spacelike, if $\langle x, x\rangle < 0$, x is timelike. Spacelike and timelike vectors are called non-null vectors. The norm of the vector $x = |\langle x, x\rangle|^{1/2}$[10].

3 Bishop 2-type frame in semi Euclidean space

Bishop 2-type frame for spacelike curves with timelike normal:

Theorem 3.1. Let $\{T, N, B\}$ be Frenet frame for a spacelike curve with timelike normal $\langle T, T\rangle = 1$, $\langle N, N\rangle = -1$, $\langle B, B\rangle = 1$ and let $\{Z_1, Z_2, B\}$ be Bishop 2-type frame with $\langle Z_1, Z_1\rangle = 1, \langle Z_2, Z_2\rangle = -1, \langle B, B\rangle = 1$. Bishop 2-type frame derivative formulas are given by as following in E_1^3 :

$$Z_{1s} = -\delta_1 B, \quad Z_{2s} = \delta_2 B, \; B_s = \delta_1 Z_1 + \delta_2 Z_2$$

where δ_1, δ_2 are curvatures according to Bishop 2-type frame in the semi Euclidean 3-space. Connection between Frenet frame and Bishop 2-type frame is expressed as following:

$$T = \cosh\theta Z_1 + \sinh\theta Z_2, N = \sinh\theta Z_1 + \cosh\theta Z_2, \; B = B$$

First and second Bishop 2-type curvatures are $\delta_1 = -\langle Z_{1s}, B\rangle = \tau \sinh\theta$, $\delta_2 = \langle Z_{2s}, B\rangle = \tau \cosh\theta$. Also $\theta_s = -\kappa$, $\tau = \left|\delta_1^2 - \delta_2^2\right|^{1/2}$.

Proof. The tangent vector T can be written by

$$T = \cosh\theta Z_1 + \sinh\theta Z_2 \tag{3}$$

Taking derivative of (3), substituting $Z_{1s} = -\delta_1 B$, $Z_{2s} = \delta_2 B$ we obtain

$$N = \sinh\theta Z_1 + \cosh\theta Z_2, \quad \theta = \arg\tanh\frac{\delta_1}{\delta_2}, \theta_s = -\kappa$$

From derivative of binormal,

$$B_s = \delta_1 Z_1 + \delta_2 Z_2 = \tau N \tag{4}$$

Taking norm of (4), we have $\tau = \left|\delta_1^2 - \delta_2^2\right|^{1/2}$.

Bishop 2-type frame for timelike curves:

Theorem 3.2. Let $\{T, N, B\}$ be Frenet frame with $\langle T, T\rangle = -1$, $\langle N, N\rangle = 1$, $\langle B, B\rangle = 1$ and let $\{Z_1, Z_2, B\}$ be Bishop 2-type frame with $\langle Z_1, Z_1\rangle = -1, \langle Z_2, Z_2\rangle = 1, \langle B, B\rangle = 1$. Bishop 2-type frame derivative formulas are given as following:

$$\begin{bmatrix} Z_{1s} \\ Z_{2s} \\ B_s \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\delta_1 \\ 0 & 0 & \delta_2 \\ -\delta_1 & -\delta_2 & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ B \end{bmatrix} \tag{5}$$

where δ_1, δ_2 are curvatures according to Bishop 2-type frame in the semi Euclidean 3-space. Connection between Frenet frame and Bishop 2-type frame is expressed as following:

$$\begin{aligned} T &= \cosh\theta Z_1 + \sinh\theta Z_2; \\ N &= \sinh\theta Z_1 + \cosh\theta Z_2;\ B = B \end{aligned} \tag{6}$$

First and second Bishop 2-type curvatures are $\delta_1 = \tau\sinh\theta$, $\delta_2 = \tau\cosh\theta$. Here $\theta_s = \kappa$. Proof is obtained as similar with Theorem 3.1.

Bishop 2-type frame for spacelike curves with timelike binormal

Theorem 3.3. Bishop 2-type frame derivative formulas are given as following:

$$Z_{1s} = k_1 B, \quad Z_{2s} = -k_2 B, \quad B_s = k_1 Z_1 - k_2 Z_2$$

Connection between Frenet frame and Bishop 2-type frame is expressed as following:

$$T = \cos\theta Z_1 + \sin\theta Z_2, N = -\sin\theta Z_1 + \cos\theta Z_2, B = B$$

First and second Bishop 2-type curvatures are $\delta_1 = \tau \sin\theta, \delta_2 = \tau\cos\theta$. Also $\theta_s = \kappa$.

Proof. The tangent vector T can be written by

$$T = \cos\theta Z_1 + \sin\theta Z_2 \tag{7}$$

Taking derivative of (7) and substituting

$$Z_{1s} = \delta_1 B, Z_{2s} = -\delta_2 B$$

we obtain $N = \sin\theta Z_1 + \cos\theta Z_2, \quad \theta = \arctan\frac{\delta_1}{\delta_2}$, $\theta_s = \kappa$. From derivative of binormal we obtain $B_s = \delta_1 Z_1 - \delta_2 Z_2 = \tau N$.

Bishop 2-type frame derivative formulas with aid Theorem 3.1, Theorem 3.2 and Theorem 3.3 are written as following:

$$\begin{bmatrix} Z_{1s} \\ Z_{2s} \\ B_s \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\varepsilon_3 k_1 \\ 0 & 0 & \varepsilon_3 k_2 \\ \varepsilon_1 k_1 & -\varepsilon_2 k_2 & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ B \end{bmatrix} \tag{8}$$

where $\langle Z_1, Z_1\rangle = \varepsilon_1$, $\langle Z_2, Z_2\rangle = \varepsilon_2, \langle B, B\rangle = \varepsilon_3 = \epsilon_3$

Bishop 2- type frame derivative formulas in semi Euclidean 4-space are expressed as following:

$$\begin{bmatrix} Z_{1s} \\ Z_{2s} \\ Z_{3s} \\ V_{3s} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -\varepsilon_4\delta_1 \\ 0 & 0 & 0 & \varepsilon_4\delta_2 \\ 0 & 0 & 0 & -\varepsilon_4\delta_3 \\ \varepsilon_1\delta_1 & -\varepsilon_2\delta_2 & \varepsilon_3\delta_3 & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ V_3 \end{bmatrix} \tag{9}$$

where $\delta_1, \delta_2, \delta_3$ curvatures of Bishop 2 type frame in E_1^4.

$$\langle Z_1, Z_1\rangle = \varepsilon_1, \langle Z_2, Z_2\rangle = \varepsilon_2, \langle Z_3, Z_3\rangle = \varepsilon_3, \langle V_3, V_3\rangle = \varepsilon_4 = \epsilon_4$$

As result, we can give Bishop 2-type frame derivative formulas in n-dimensional semi Euclidean space E_1^n as following:

$$\begin{bmatrix} Z_{1s} \\ Z_{2s} \\ Z_{3s} \\ . \\ . \\ Z_{(n-1)s} \\ B_s \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & . & . & 0 & -\varepsilon_n\delta_1 \\ 0 & 0 & 0 & . & . & 0 & \varepsilon_n\delta_2 \\ 0 & 0 & 0 & . & . & 0 & -\varepsilon_n\delta_3 \\ 0 & 0 & 0 & . & . & 0 & . \\ 0 & 0 & 0 & . & . & 0 & . \\ 0 & 0 & 0 & . & . & 0 & (-1)^{n-1}\varepsilon_n\delta_{n-1} \\ \varepsilon_1\delta_1 & -\varepsilon_2\delta_2 & \varepsilon_3\delta_3 & . & . & (-1)^{n-1}\varepsilon_{n-1}\delta_{n-1} & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ . \\ . \\ Z_{(n-1)} \\ B \end{bmatrix} \tag{10}$$

where $\langle Z_1, Z_1\rangle = \varepsilon_1, \langle Z_2, Z_2\rangle = \varepsilon_2, \langle Z_3, Z_3\rangle = \varepsilon_3, ..., \langle Z_{n-1}, Z_{n-1}\rangle = \varepsilon_{n-1}, \langle B, B\rangle = \varepsilon_n$ and $\delta_i = (-1)^i \langle Z_{is}, B\rangle$, $i = 1, 2, ...n-1$ are Bishop 2-type curvatures in n-dimensional semi Euclidean space.

Theorem 3.4. $\{Z_1, Z_2, Z_3, V_3\}$ Bishop 2-type satisfying the first case

$$\begin{aligned} \langle Z_1, Z_1\rangle &= 1, \ \langle Z_2, Z_2\rangle = -1, \ \langle Z_3, Z_3\rangle = 1, \langle T, T\rangle = 1, \\ \langle V_1, V_1\rangle &= -1, \ \langle V_2, V_2\rangle = 1, \ \langle V_3, V_3\rangle = 1 \end{aligned} \tag{11}$$

is obtained from Frenet 4-frame $\{T, V_1, V_2, V_3\}$ with aid semi-Euclidean rotation matrix SR_1 as following:

$$\begin{aligned} Z_1 &= \cos\gamma\cosh\xi T + \cos\gamma\sinh\xi V_1 - \sin\gamma V_2 \qquad (12) \\ Z_2 &= (\sinh\theta\sin\gamma\cosh\xi + \cosh\theta\sinh\xi)T + (\sinh\theta\sin\gamma\sinh\xi + \cosh\theta\cosh\xi)V_1 \\ &\quad + \sinh\theta\cos\gamma V_2 \\ Z_3 &= (\cosh\theta\sin\gamma\cosh\xi + \sinh\theta\sinh\xi)T + (\cosh\theta\sin\gamma\sinh\xi + \sinh\theta\cosh\xi)V_1 \\ &\quad + \cosh\theta\cos\gamma V_2; \ \ V_3 = V_3 \end{aligned}$$

where respectively θ, γ, ξ are angles between V_1 and V_2, T and V_2, T and V_1.

Bishop 2- type frame derivative formulas considering (9) are expressed as following:

$$\begin{bmatrix} Z_{1s} \\ Z_{2s} \\ Z_{3s} \\ V_{3s} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -\delta_1 \\ 0 & 0 & 0 & \delta_2 \\ 0 & 0 & 0 & -\delta_3 \\ \delta_1 & \delta_2 & \delta_3 & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ V_3 \end{bmatrix} \tag{13}$$

where $\delta_1, \delta_2, \delta_3$ curvatures of Bishop 2 type frame in E_1^4. Respectively $\delta_1, \delta_2, \delta_3, \kappa, \tau$ are obtained as following:

$$\begin{aligned} \delta_1 &= \rho\sin\gamma, \qquad \delta_2 = \rho\sinh\theta\cos\gamma, \qquad \delta_3 = -\rho\cosh\theta\cos\gamma, \\ \kappa &= -\gamma_s\tan\gamma\coth\xi + \xi_s, \ \ \tau = \frac{\gamma_s}{\sinh\xi} \end{aligned}$$

Proof. The semi-Euclidean rotation matrix SR_1 according to first case is obtained with aid three rotations in E_1^4. Respectively first, second, third semi-Euclidean rotations are between V_1 and V_2, T and V_2, T and V_1.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta & 0 \\ 0 & \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\gamma & 0 & -\sin\gamma & 0 \\ 0 & 1 & 0 & 0 \\ \sin\gamma & 0 & \cos\gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh\xi & \sinh\xi & 0 & 0 \\ \sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}$$

Bishop 2-type frame (12) is obtained from the Frenet 4-frame $\{T_1, V_1, V_2, V_3\}$ with aid SR_1 in the semi-Euclidean 4-space.

Using derivative of Z_1 in (12) and $Z_{1s} = -\delta_1 V_3$ in (13), it can be written

$$\begin{gathered} (-\gamma_s \sin\gamma \cosh\xi + \xi_s \cos\gamma \sinh\xi - \kappa \sinh\xi \cos\gamma)T \\ -(\kappa \cosh\xi \cos\gamma + \gamma_s \sin\gamma \sinh\xi - \xi_s \cos\gamma \cosh\xi + \sin\gamma\tau)V_1 \\ +(\cos\gamma \sinh\xi\tau - \gamma_s \cos\gamma)V_2 - \rho \sin\gamma V_3 = -\delta_1 V_3 \end{gathered} \tag{15}$$

From (15), it is obtained

$$\cos\gamma \sinh\xi\tau - \gamma_s \cos\gamma = 0, \tag{16}$$

$$-(\kappa \cosh\xi \cos\gamma + \gamma_s \sin\gamma \sinh\xi - \xi_s \cos\gamma \cosh\xi + \sin\gamma\tau) = 0 \tag{17}$$

$$\delta_1 = \rho \sin\gamma \tag{18}$$

From (16), (17) and (18), we have

$$\tau = \frac{\gamma_s}{\sinh\xi},\ \delta_1 = \rho \sin\gamma, \kappa = -\gamma_s \tan\gamma \coth\xi + \xi_s \tag{19}$$

As similar, we have

$$\begin{aligned} &((\sinh\theta \sin\gamma \cosh\xi + \cosh\theta \sinh\xi)T + (\sinh\theta \sin\gamma \sinh\xi + \cosh\theta \cosh\xi)V_1 \\ &+ \sinh\theta \cos\gamma V_2)_s = \delta_2 V_3, \end{aligned} \tag{20}$$

$$\begin{aligned} &((\cosh\theta \sin\gamma \cosh\xi + \sinh\theta \sinh\xi)T + (\cosh\theta \sin\gamma \sinh\xi + \sinh\theta \cosh\xi)V_1 \\ &+ \cosh\theta \cos\gamma V_2)_s = -\delta_3 V_3, \end{aligned} \tag{21}$$

From (20) and (21) δ_2, δ_3 are obtained as following: $\delta_2 = \rho \sinh\theta \cos\gamma$, $\delta_3 = -\rho \cosh\theta \cos\gamma$

Result 3.1. The Frenet 4 -frame satisfying (13) is given with aid inverse of semi-Euclidean rotation matrix $(SR_1)^{-1}$ as following:

$$\begin{aligned} T &= \cos\gamma\cosh\xi Z_1 - (\sinh\theta\sin\gamma\cosh\xi + \cosh\theta\sinh\xi)Z_2 \qquad (22)\\ &\quad + (\cosh\theta\sin\gamma\cosh\xi + \sinh\theta\sinh\xi)Z_3 \\ V_1 &= -\cos\gamma\sinh\xi Z_1 + (\sinh\theta\sin\gamma\sinh\xi + \cosh\theta\cosh\xi)Z_2 \\ &\quad -(\cosh\theta\sin\gamma\sinh\xi + \sinh\theta\cosh\xi)Z_3 \\ V_2 &= -\sin\gamma Z_1 - \sinh\theta\cos\gamma Z_2 + \cosh\theta\cos\gamma Z_3;\ V_3 = V_3 \end{aligned}$$

Proof. Frenet frame (22) is obtained easily from $(SR_1)^{-1} = \zeta_1 (SR_1)^T \zeta_1$. Here

$$\zeta_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = diag(1, -1, 1, 1)$$

$(SR_1)^T$ is semi-Euclidean tranpoze rotation matrix of SR_1 and $(SR_1)^{-1}$ is semi-Euclidean inverse rotation matrix of SR_1.

Theorem 3.5. Let α be spacelike curve in semi-Euclidean 4-space. Bishop 2-type $\{Z_1, Z_2, Z_3, V_3\}$ satisfying the second case:$\langle T, T\rangle = 1, \langle V_1, V_1\rangle = 1, \langle V_2, V_2\rangle = -1, \langle V_3, V_3\rangle = 1, \langle Z_1, Z_1\rangle = 1, \langle Z_2, Z_2\rangle = 1, \langle Z_3, Z_3\rangle = -1, \langle V_3, V_3\rangle = 1$ is obtained from Frenet 4-frame $\{T, V_1, V_2, V_3\}$ with aid semi-Euclidean rotation matrix SR_2 as following:

$$\begin{aligned} Z_1 &= \cosh\gamma\cos\xi T - \cosh\gamma\sin\xi V_1 + \sinh\gamma V_2 \qquad (23)\\ Z_2 &= (-\sin\theta\sinh\gamma\cos\xi + \cos\theta\sin\xi)T + (\sin\theta\sinh\gamma\sin\xi + \cos\theta\cos\xi)V_1 \\ &\quad - \sin\theta\cosh\gamma V_2 \\ Z_3 &= (\cos\theta\sinh\gamma\cos\xi + \sin\theta\sin\xi)T - (\cos\theta\sinh\gamma\sin\xi - \sin\theta\cos\xi)V_1 + \cos\theta\cosh\gamma V_2;\ V_3 = V_3 \end{aligned}$$

Bishop 2-type frame derivative formulas for second case are defined as following:

$$\begin{bmatrix} Z_{1s} \\ Z_{2s} \\ Z_{3s} \\ V_{3s} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -\delta_1 \\ 0 & 0 & 0 & \delta_2 \\ 0 & 0 & 0 & -\delta_3 \\ \delta_1 & -\delta_2 & -\delta_3 & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ V_3 \end{bmatrix} \qquad (24)$$

$\delta_1, \delta_2, \delta_3, \kappa, \tau$ are obtained as following:

$$\begin{aligned} \delta_1 &= -\rho\sinh\gamma, \quad \delta_2 = -\rho\sin\theta\cosh\gamma, \qquad \delta_3 = -\rho\cos\theta\cosh\gamma, \\ \kappa &= \xi_s - \gamma_s\cot\xi\tanh\gamma, \tau = -\frac{\gamma_s}{\sin\xi} \end{aligned}$$

Proof.

$$SR_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh\gamma & 0 & \sinh\gamma & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\gamma & 0 & \cosh\gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\xi & -\sin\xi & 0 & 0 \\ \sin\xi & \cos\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{25}$$

Frenet frame (23) is obtained from (25). Considering derivative of Z_1 in (23) and $Z_{1s} = -\delta_1 V_3$ in (24),

$$\begin{aligned} &(\gamma_s \sinh\gamma\cos\xi - \xi_s\cosh\gamma\sin\xi + \kappa\sin\xi\cosh\gamma)T \\ &+(\kappa\cos\xi\cosh\gamma - \gamma_s\sinh\gamma\sin\xi - \xi_s\cosh\gamma\cos\xi - \tau\sinh\gamma)V_1 \\ &+(\tau\cosh\gamma\sin\xi + \gamma_s\cosh\gamma)V_2 + \rho\sinh\gamma V_3 = -\delta_1 V_3 \end{aligned} \tag{26}$$

From (26),

$$\kappa\cos\xi\cosh\gamma - \gamma_s\sinh\gamma\sin\xi - \xi_s\cosh\gamma\cos\xi - \tau\sinh\gamma = 0 \tag{27}$$

$$\tau\cosh\gamma\sin\xi + \gamma_s\cosh\gamma = 0 \tag{28}$$

$$\rho\sinh\gamma = -\delta_1 \tag{29}$$

With aid (28), we have $\tau = -\frac{\gamma_s}{\sin\xi}$. From (27), (29) we obtain $\kappa = \xi_s - \gamma_s\cot\xi\tanh\gamma$, $\delta_1 = -\rho\sinh\gamma$. As similar

$$\begin{aligned} &((-\sin\theta\sinh\gamma\cos\xi + \cos\theta\sin\xi)T + (\sin\theta\sinh\gamma\sin\xi + \cos\theta\cos\xi)V_1 \\ &-\sin\theta\cosh\gamma V_2)_s = \delta_2 V_3, \end{aligned} \tag{30}$$

$$\begin{aligned} ((\cos\theta\sinh\gamma\cos\xi + \sin\theta\sin\xi)T - (\cos\theta\sinh\gamma\sin\xi - \sin\theta\cos\xi)V_1 \\ + \cos\theta\cosh\gamma V_2)_s = -\delta_3 V_3 \end{aligned} \tag{31}$$

From (30), (31) we obtain $\delta_2 = -\rho\sin\theta\cosh\gamma$, $\delta_3 = -\rho\cos\theta\cosh\gamma$.

Result 3.2. The Frenet 4 -frame (32) is given with aid inverse of semi-Euclidean rotation matrix $(SR_2)^{-1}$ as following:

$$\begin{aligned} T &= \cosh\gamma\cos\xi Z_1 - (\sin\theta\sinh\gamma\cos\xi + \cos\theta\sin\xi)Z_2 \\ &\quad -(\cos\theta\sinh\gamma\cos\xi + \sin\theta\sin\xi)Z_3 \\ V_1 &= -\cosh\gamma\sin\xi Z_1 + (\sin\theta\sinh\gamma\sin\xi + \cos\theta\cos\xi)Z_2 \\ &\quad +(\cos\theta\sinh\gamma\sin\xi - \sin\theta\cos\xi)Z_3 \\ V_2 &= -\sinh\gamma Z_1 + \sin\theta\cosh\gamma Z_2 + \cos\theta\cosh\gamma Z_3; \quad V_3 = V_3 \end{aligned} \tag{32}$$

Proof. Bishop 2-type frame (32) is obtained from $(SR_2)^{-1} = \zeta_2 (SR_2)^T \zeta_2$,where $\zeta_2 = diag(1, 1, -1, 1)$.

Theorem 3.6. Bishop 2-type $\{Z_1, Z_2, Z_3, V_3\}$ is obtained from Frenet 4-frame $\{T, V_1, V_2, V_3\}$ according to third case

$$\begin{aligned} \langle Z_1, Z_1 \rangle &= 1, \langle Z_2, Z_2 \rangle = 1, \langle Z_3, Z_3 \rangle = 1; \langle T, T \rangle = 1, \\ \langle V_1, V_1 \rangle &= 1, \langle V_2, V_2 \rangle = 1, \langle V_3, V_3 \rangle = -1 \end{aligned}$$

using semi-Euclidean rotation SR_3 as following:

$$\begin{aligned} Z_1 &= \cos\gamma \cos\xi T - \cos\gamma \sin\xi V_1 - \sin\gamma V_2 \quad (33) \\ Z_2 &= (-\sin\theta \sin\gamma \cos\xi + \cos\theta \sin\xi) T \\ &\quad + (\sin\theta \sin\gamma \sin\xi + \cos\theta \cos\xi) V_1 - \sin\theta \cos\gamma V_2 \quad (34) \\ Z_3 &= (\cos\theta \sin\gamma \cos\xi + \sin\theta \sin\xi) T - (\cos\theta \sin\gamma \sin\xi - \sin\theta \cos\xi) V_1 + \cos\theta \cos\gamma V_2; \ V_3 = V_3 \end{aligned}$$

Bishop 2-type frame derivative formulas satisfying third case are expressed as following:

$$\begin{bmatrix} Z_{1s} \\ Z_{2s} \\ Z_{3s} \\ V_{3s} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \delta_1 \\ 0 & 0 & 0 & -\delta_2 \\ 0 & 0 & 0 & \delta_3 \\ \delta_1 & -\delta_2 & \delta_3 & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ V_3 \end{bmatrix} \quad (35)$$

$\delta_1, \delta_2, \delta_3, \kappa, \tau$ are obtained as following:

$$\begin{aligned} \delta_1 &= -\rho \sin\gamma, \quad \delta_2 = -\rho \sin\theta \cos\gamma, \quad \delta_3 = \rho \cos\theta \cos\gamma, \\ \kappa &= \gamma_s \cot\xi \tan\gamma + \xi_s, \quad \tau = -\frac{\gamma_s}{\sin\xi} \end{aligned}$$

Proof.

$$SR_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\gamma & 0 & -\sin\gamma & 0 \\ 0 & 1 & 0 & 0 \\ \sin\gamma & 0 & \cos\gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\xi & -\sin\xi & 0 & 0 \\ \sin\xi & \cos\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (36)$$

Bishop 2-type frame (33) is obtained from SR_3. From

$$\begin{aligned}&(-\gamma_s \sin\gamma\cos\xi - \xi_s \cos\gamma\sin\xi + \kappa\sin\xi\cos\gamma)T\\&+(\kappa\cos\xi\cos\gamma + \gamma_s\sin\gamma\sin\xi - \xi_s\cos\gamma\cos\xi + \tau\sin\gamma)V_1\\&-(\tau\cos\gamma\sin\xi + \gamma_s\cos\gamma)V_2 + \rho\sin\gamma V_3 = \delta_1 V_3\end{aligned}$$

we obtain

$$\delta_1 = \rho\sin\gamma, \quad \kappa = \gamma_s\cot\xi\tan\gamma + \xi_s, \quad \tau = \frac{\gamma_s}{\sin\xi}$$

As similar, we have

$$\begin{aligned}&((-\sin\theta\sin\gamma\cos\xi + \cos\theta\sin\xi)T + (\sin\theta\sin\gamma\sin\xi + \cos\theta\cos\xi)V_1\\&-\sin\theta\cos\gamma V_2)_s = -\delta_2 V_2,\end{aligned} \tag{37}$$

$$\begin{aligned}&((\cos\theta\sin\gamma\cos\xi + \sin\theta\sin\xi)T - (\cos\theta\sin\gamma\sin\xi - \sin\theta\cos\xi)V_1\\&+\cos\theta\cos\gamma V_2)_s = \delta_3 V_3,\end{aligned} \tag{38}$$

From (37), (38), we obtain $\delta_2 = -\rho\sin\theta\cos\gamma$, $\delta_3 = \rho\cos\theta\cos\gamma$.

Result 3.3. The Frenet 4 -frame is given with aid inverse of semi-Euclidean rotation matrix $(SR_3)^{-1}$ as following:

$$\begin{aligned}T &= \cos\gamma\cos\xi Z_1 - (\sin\theta\sin\gamma\cos\xi + \cos\theta\cos\xi)Z_2\\&\quad+(\cos\theta\sin\gamma\cos\xi + \sin\theta\sin\xi)Z_3\\V_1 &= -\cos\gamma\sin\xi Z_1 + (\sin\theta\sin\gamma\sin\xi + \cos\theta\cos\xi)Z_2\\&\quad-(\cos\theta\sin\gamma\sin\xi - \sin\theta\cos\xi)Z_3\\V_2 &= -\sin\gamma Z_1 - \sin\theta\cos\gamma Z_2 + \cos\theta\cos\gamma Z_3 \quad ; V_3 = V_3\end{aligned} \tag{39}$$

Proof. Frenet 4-frame for third case is obtained from $(SR_3)^{-1} = \zeta_3(SR_3)^T\zeta_3$.

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