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On 2-Killing and Conformal Vector Fields on Riemannian Manifolds

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Abstract

A classical object of differential geometry are Killing vector fields. This notion has been generalized to conformal vector fields and recently to 2-Killing vector fields . In this paper we obtain some relations between 2-Killing vector fields and conformal vector fields on a Riemannian manifold and among other results we show that a 2-Killing conformal vector field on a compact Riemannian manifold must be Killing if the dimension of manifold is greater than two.

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1 Introduction

A classical object of differential geometry are Killing vector fields. These are by definition infinitesimal isometries, i.e. the flow of such a vector field preserves a given metric. More precisely, a smooth vector field ξ on a Riemannian manifold (M, g) is said to be Killing vector field if the Lie derivative of the metric tensor g with respect to ξ is zero, that is $L_\xi\, g = 0$. Killing vector fields play an important role in the geometry as well as the topology of a Riemannian manifold, for instance, it is known that the existence of a nontrivial Killing vector field on a compact Riemannian manifold implies that all its Ricci curvatures can not be negative. Also it is known that on an even dimensional positively curved Riemannian manifold, every Killing vector field must have a zero. In [2] the authors have studied Riemannian manifolds admitting nontrivial Killing vector fields of constant length and obtained interesting results.

Slightly more generally one can consider conformal vector fields, i.e. vector fields with a flow preserving a given conformal class of metrics. More precisely, a smooth vector field ξ on a Riemannian manifold (M, g) is said to be a conformal vector field if there exists a smooth function f on M such that $L_\xi\, g = 2fg$. Conformal vector fields are important objects on a space and have been studied quite extensively on Riemannian manifolds. We call ξ a nontrivial conformal vector field if the potential function f is not a constant. We note that on a compact M, if f is a constant it has to be zero and consequently ξ is Killing. If in addition ξ is a closed vector field, that is the 1-form dual to ξ with respect to g is a closed form, then ξ is called a closed conformal vector field. Riemannian manifolds admitting closed conformal vector fields or conformal gradient vector fields have been investigated by many authors. Also in [1] it is shown that a compact Riemannian manifold with positive constant scalar curvature admitting a nonzero conformal gradient vector field is isometric to a sphere using the classical result of Obata [10]. There are several geometric conditions which force a conformal vector field to be Killing, for instance, a conformal vector field on a compact Kähler manifold of dimension greater than two is a Killing vector field [8]. We recall that Killing forms as a generalization of Killing vector fields were introduced by Yano [13]. Kashiwada [6] introduced conformal Killing forms generalizing conformal vector fields. In fact on a Riemannian manifold, a vector field ξ is dual to a conformal Killing 1-form if and only if it is a conformal vector field. Moreover ξ is dual to a Killing 1-form if and only if it is a Killing vector field. The simplest examples of manifolds with conformal Killing forms are the spaces of constant curvature. See [12] for more details.

Recently the notion of Killing vector fields is generalized to 2-Killing vector fields in [11]. This class of vector fields on a Riemannian manifold (M, g) enlarges the class of Killing vector fields. In fact a smooth vector field ξ is called 2-Killing if $L_\xi\, L_\xi\, g = 0$. In [11] the author has studied the relations between 2-Killing vector fields and monotone vector fields introduced in [9] and obtained interesting results. In this paper, we work in the same direction and use similar ideas. In particular, we are interested in finding conditions under which a 2-Killing vector field on a Riemannian manifold is Killing. Some of our results are comparable to those of [11]. The main results of this paper are Theorems 3.2, 3.3 and Corollaries 4.2 and 4.3.

2 Preliminaries

Let (M, g) be an n-dimensional Riemannian manifold with Lie algebra $\mathcal{X}(M)$ of smooth vector fields and Riemannian connection ∇. Recall that the curvature tensor R is a correspondence that associates to every pair of vector fields X, Y a mapping $R(X,Y) : \mathcal{X}(M) \to \mathcal{X}(M)$ given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The Ricci curvature tensor is defined by

$$Ric(X,Y) = \sum_{i=1}^{n} g(R(e_i, X)Y, e_i)$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M and $X, Y \in \mathcal{X}(M)$.

A smooth vector field ξ is said to be a Killing vector field if $L_\xi\, g = 0$ where L_ξ is the Lie derivative with respect to ξ. This means that for every pair of $X, Y \in \mathcal{X}(M)$ we have

$$(L_\xi\, g)(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0.$$

A smooth vector field ξ is said to be a conformal vector field if $L_\xi\, g = 2fg$ for a smooth function f on M called the potential function. Note that div $\xi = nf$, so if M is compact and f is a constant function on M, then by Stokes' theorem we have $f = 0$, that is ξ is a Killing vector field. Using Kozul's formula we obtain the following for any vector field ξ on M

$$2g(\nabla_X \xi, Y) = (L_\xi\, g)(X,Y) + d\eta(X,Y), \quad X, Y \in \mathcal{X}(M)$$

where η is the 1-form dual to ξ, that is $\eta(X) = g(X, \xi)$. Define a skew symmetric tensor field φ on M by

$$d\eta(X,Y) = 2g(\varphi X, Y), \quad X, Y \in \mathcal{X}(M).$$

Then we get the following (cf. [3])

Lemma 2.1. *Let ξ be a conformal vector field on M with potential function f. Then for any vector field X we have*

$$\nabla_X \xi = fX + \varphi X.$$

Note that if ξ is a conformal *closed* vector field, that is the 1-form η dual to ξ is closed, we get $\varphi = 0$. So by the above lemma we have $\nabla_X \xi = fX$ for any vector field X on M. In particular if ξ is a conformal *gradient* vector field, then ξ is a conformal closed vector field as well and we have $\nabla_X \xi = fX$ for any $X \in \mathcal{X}(M)$. Recall that a vector field X is called a gradient vector field on M if X is the gradient of a smooth function h on M, that is $X = \nabla h$ where ∇h denotes the gradient of h.

A smooth vector field ξ on M is called 2-Killing if $L_\xi L_\xi g = 0$. This is a generalization of the notion of a Killing vector field. In this paper, we consider 2-Killing and conformal vector fields and we are interested in finding conditions under which a 2-Killing vector field is Killing. We need the following fact concerning 2-Killing vector fields (cf. [11]).

Lemma 2.2. *Let ξ be a 2-Killing vector field on a Riemannian manifold (M, g). Then we have*

$$Ric(\xi, \xi) = \operatorname{div}(\nabla_\xi \xi) + \operatorname{Tr}(g(\nabla \xi, \nabla \xi)) = \operatorname{div}(\nabla_\xi \xi) + \left\|\nabla \xi\right\|^2.$$

We use also the notion of harmonic vector fields as following.

In [5] the authors have defined the Laplacian operator acting on smooth vector fields $\Delta : \mathcal{X}(M) \to \mathcal{X}(M)$ by

$$\Delta X = \sum_{i=1}^{n} \nabla^2 X(e_i, e_i) = \sum_{i=1}^{n} (\nabla_{e_i} \nabla_{e_i} X - \nabla_{\nabla_{e_i} e_i} X)$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M. This operator is a self adjoint elliptic operator with respect to the inner product

$$< X, Y > = \int_M g(X, Y)$$

on the set of compactly supported vector fields in $\mathcal{X}(M)$. A vector field is called harmonic if $\Delta X = 0$.

3 Conformal vector fields

Suppose that ξ is a 2-Killing vector field on M. In [11] a necessary and sufficient condition for a vector field to be 2-Killing is given in terms of the curvature tensor R. We can present an easier equivalent condition if ξ is conformal too.

Lemma 3.1. *Suppose that ξ is a conformal vector field with a potential function f on a Riemannian manifold (M, g). Then ξ is a 2-Killing vector field if and only if*

$$\xi \cdot f + 2f^2 = 0 \tag{1}$$

Proof. As ξ is a conformal vector field, for any pair of $V, W \in \mathcal{X}(M)$ we have by definition

$$L_\xi\, g(V,W) = 2fg(V,W) = \xi \cdot g(V,W) - g([\xi,V],W) - g([\xi,W],V) \tag{2}$$

and from equation (2), we obtain

$$\begin{aligned} L_\xi\, L_\xi\, g(V,W) &= L_\xi(2fg)(V,W) \\ &= (\xi \cdot (2fg))(V,W) - 2fg([\xi,V],W) - 2fg([\xi,W],V) \\ &= 2\ (\xi \cdot f)g(V,W) + 2f\xi \cdot g(V,W) - 2fg([\xi,V],W) \\ &- 2fg([\xi,W],V) \\ &= 2\ (\xi \cdot f)g(V,W) + 4f^2 g(V,W) \end{aligned}$$

therefore

$$L_\xi\, L_\xi\, g = (2\ \xi \cdot f + 4f^2)g \tag{3}$$

so by equation (3) we conclude that ξ is a 2-Killing vector field if and only if

$$\xi \cdot f + 2f^2 = 0.$$

□

Note that if ξ is a conformal vector field with a potential function f then div $\xi = nf$.

After describing a relation between 2-Killing vector fields and conformal vector fields, we show that in a *compact* Riemannian manifold, Killing property is equivalent to 2-Killing property for a conformal vector field.

Theorem 3.2. *Suppose that ξ is a conformal vector field with a potential function f on a compact Riemannian manifold (M,g) with dimension $n > 2$. Then the next assertions are equivalent*
i) ξ is a Killing vector field.
ii) ξ is a 2-Killing vector field.

Proof. If ξ is a Killing vector field then clearly ξ is a 2-Killing vector field. Suppose that ξ is a 2-Killing vector field. Then by Lemma 3.1, we have

$$\xi \cdot \operatorname{div}\, \xi + \frac{2}{n}(\operatorname{div}\, \xi)^2 = 0. \tag{4}$$

By a direct calculation we know

$$\operatorname{div}((\operatorname{div}\,\xi)\xi) = (\operatorname{div}\,\xi)^2 + \xi \cdot \operatorname{div}\,\xi \tag{5}$$

so together with equation (4), we have

$$\operatorname{div}((\operatorname{div}\,\xi)\xi) = \frac{n-2}{n}(\operatorname{div}\,\xi)^2 \tag{6}$$

hence by Stokes' theorem, integrating equality (6) on the compact Riemannian manifold M, we obtain

$$\frac{n-2}{n}\int_M (\operatorname{div}\,\xi)^2 = 0. \tag{7}$$

Therefore div $\xi = 0$, so $f = 0$, i.e. ξ is a Killing vector field. $\square$

Recall that by Lemma 2.1, if ξ is a conformal closed vector field on M with a potential function f, then for any vector field X we have

$$\nabla_X \xi = fX.$$

Now we can show the following

Theorem 3.3. *Suppose that ξ is a conformal closed 2-Killing vector field on a Riemannian manifold (M, g) with dimension $n > 1$. If $Ric(\xi, \xi) \leqslant 0$ then ξ is a parallel vector field.*

Proof. Consider a local orthonormal frame $\{e_1, ..., e_n\}$ on M. We directly compute the value of $Ric(X, \xi)$ for an arbitrary vector field X.

$$\begin{aligned} Ric(X,\xi) &= \sum_{i=1}^n g(\nabla_{e_i}\nabla_X\xi - \nabla_X\nabla_{e_i}\xi - \nabla_{\nabla_{e_i}X}\xi + \nabla_{\nabla_X e_i}\xi, e_i) \\ &= \sum_{i=1}^n g(\nabla_{e_i} fX - \nabla_X f e_i - f\nabla_{e_i}X + f\nabla_X e_i, e_i) \\ &= \operatorname{div}(fX) - f\operatorname{div}\,X + \sum_{i=1}^n g(-(X\cdot f)e_i, e_i) \\ &= X\cdot f - n\ X\cdot f = -(n-1)\ X\cdot f. \end{aligned}$$

In particular we have (compare with the formula (30) in [7])

$$Ric(\xi,\xi) = -(n-1)\ \xi\cdot f \tag{8}$$

as ξ is a 2-Killing vector field by Lemma 3.1 and equation (8) we get

$$f^2 = \frac{1}{2(n-1)} Ric(\xi,\xi) \leqslant 0. \tag{9}$$

Therefore $f = 0$ on M and ξ is Killing. So by Lemma 2.1 ξ is a parallel vector field. $\square$

4 Laplacian

As we mentioned before, in [5] the Laplacian operator acting on smooth vector fields on a Riemannian manifold has been defined by

$$\Delta X = \sum_{i=1}^{n} \nabla^2 X(e_i, e_i) = \sum_{i=1}^{n} (\nabla_{e_i}\nabla_{e_i}X - \nabla_{\nabla_{e_i}e_i}X).$$

We shall denote by Δ both the Laplacian operators, the one acting on smooth functions on M as well as that acting on smooth vector fields. Now we use this notion to prove the following lemma (compare with the formula (2.5) in [4]).

Lemma 4.1. *Suppose ξ is a conformal vector field with a potential function f on a Riemannian manifold (M, g). Then*

$$Ric(\xi, \xi) = -(n-2)\ \xi \cdot f - g(\Delta\xi, \xi). \tag{10}$$

Proof. By definition we have

$$Ric(\xi, \xi) = \sum_{i=1}^{n} g(\nabla_{e_i}\nabla_{\xi}\xi - \nabla_{\xi}\nabla_{e_i}\xi + \nabla_{\nabla_{\xi}e_i}\ \xi - \nabla_{\nabla_{e_i}\xi}\ \xi, e_i) \tag{11}$$

where $\{e_1, ..., e_n\}$ is a local orthonormal frame on M. Note that for any fixed $p \in M$, we can take the frame such that $\nabla_{e_i}e_j(p) = 0$ for all $1 \leq i, j \leq n$. As ξ is a conformal vector field, for any pair of $X, Y \in \mathcal{X}(M)$, we have

$$g(\nabla_X\xi, Y) = 2fg(X, Y) - g(\nabla_Y\xi, X). \tag{12}$$

By direct computations and using equation (12), we obtain the following equations

$$\begin{aligned}
\sum_{i=1}^{n} g(\nabla_{e_i}\nabla_{\xi}\xi, e_i) &= 2\ \mathrm{div}(f\xi) - g(\Delta\xi, \xi) - \left\|\nabla\xi\right\|^2 && (13)\\
\sum_{i=1}^{n} g(-\nabla_{\xi}\nabla_{e_i}\xi, e_i) &= -n\ \xi \cdot f - \sum_{i=1}^{n} g(\nabla_{\nabla_{\xi}e_i}\xi, e_i) && (14)\\
\sum_{i=1}^{n} g(-\nabla_{\nabla_{e_i}\xi}\ \xi, e_i) &= \left\|\nabla\xi\right\|^2 - 2f\ \mathrm{div}\ \xi. && (15)
\end{aligned}$$

Now from equations (13), (14) and (15) we immediately get

$$Ric(\xi, \xi) = -(n-2)\ \xi \cdot f - g(\Delta\xi, \xi).$$

□

Now we present some corollaries giving us relations between conformal 2-Killing vector fields and Laplacian operator.

Corollary 4.2. *Suppose that ξ is a harmonic conformal 2-Killing vector field on a Riemannian manifold (M, g) with dimension $n > 2$. If $Ric(\xi, \xi) \leqslant 0$, then ξ is a Killing vector field.*

Proof. Since ξ is a harmonic vector field, so $\Delta\xi = 0$. By using Lemma 4.1 we obtain

$$Ric(\xi, \xi) = -(n-2)\, \xi \cdot f \tag{16}$$

which together with Lemma 3.1 gives

$$Ric(\xi, \xi) = 2(n-2)f^2 \leqslant 0 \tag{17}$$

then $f = 0$ which implies that ξ is a Killing vector field. □

Corollary 4.3. *Suppose that ξ is a 2-Killing vector field on a compact Riemannian manifold (M, g). If ξ is a gradient vector field of a smooth function f on M such that $\nabla\Delta f = \Delta\nabla f$, then ξ is a parallel vector field.*

Proof. The Bochner-Weitzenbock formula gives

$$g(\xi, \nabla\Delta f) = -\left\|\nabla\xi\right\|^2 - Ric(\xi, \xi) + \frac{1}{2}\Delta(\left\|\xi\right\|^2). \tag{18}$$

On the other hand we have

$$\begin{aligned} g(\xi, \Delta\nabla f) &= \sum_{i=1}^{n} g(\nabla^2\xi(e_i, e_i), \xi) \\ &= \sum_{i=1}^{n} g(\nabla^2\xi(e_i, \xi), e_i) \\ &= \sum_{i=1}^{n} \left(g(\nabla_{e_i}\nabla_\xi\xi, e_i) - g(\nabla_{e_i}\xi, \nabla_{e_i}\xi)\right) \end{aligned}$$

hence

$$g(\xi, \Delta\nabla f) = \operatorname{div}(\nabla_\xi\xi) - \left\|\nabla\xi\right\|^2. \tag{19}$$

As ξ is a 2-Killing vector field, by Lemma 2.2 we have

$$Ric(\xi, \xi) = \operatorname{div}(\nabla_\xi\xi) + \left\|\nabla\xi\right\|^2. \tag{20}$$

Now by equations (18) , (19) and (20) we get

$$g(\nabla\Delta f - \Delta\nabla f, \xi) = \frac{1}{2}\Delta(\|\xi\|^2) - 2\,\mathrm{div}(\nabla_\xi \xi) - \|\nabla\xi\|^2 \tag{21}$$

so by Stokes' theorem and the assumption we obtain

$$\int_M \|\nabla\xi\|^2 = 0 \tag{22}$$

hence $\|\nabla\xi\| = 0$ which implies that ξ is a parallel vector field.

□

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