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Plancherel Measures on Siegel Type Nilpotent Lie Groups

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Abstract

In this article, we use a kind of coupling inner products to get the Plancherel measure.

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1 Introduction

Plancherel measure plays an important role in Harmonic analysis. This measure is defined on the set of irreducible unitary representations of a nilpotent Lie group, that describes how the regular representation breaks up into irreducible unitary representations; see [5].

In this article, we take advantage of Jordan triple systems and orbits theoretic methods to obtain the Plancherel measure. More precisely, for any $\ell \in \mathcal{O}_\varepsilon^*$, we will define a kind of coupling inner product $\gamma : U \times \mathcal{O}_\varepsilon^* \to \mathbb{C}$, where $U = X + iX$.

This paper is organized as follows. In Section 2, we recall some preliminaries of the Jordan characterization of the symmetric Siegel domains which play a crucial and deep role in our study. Section 3, we consider the Plancherel measure (see Theorem 3.6 below).

2 Boundary components of Siegel domains

We start this section by recalling some notations and conventions which we will follow. The Hermitian Jordan triple is on the operator-valued inner product in the general irreducible domains. Certainly, it is also valid on the symmetric Siegel domains. Now we state the definition of the positive Hermitian Jordan triple system (see [6, 1]).

Definition 2.1. Let U and V ($V \neq \{0\}$) be two finite dimensional complex vector spaces, and let $Z = U \times V$ and L be a sesqui-linear map, that is $Z \times Z \to L(Z)$. Then, for all $x, y, z, w \in Z$ and $t \in \mathbb{C}$, (L, Z) is called a positive Hermitian Jordan triple system if

(i) $\{xyz\} := L(x, y)(z)$ is symmetric bilinear in the outer variables x, z and conjugate linear in the inner variable y;

(ii) $[L(x, y), L(z, w)] = L(\{xyz\}, w) - L(z, \{wxy\})$, where $[\cdot, \cdot]$ denotes the commutator of operators;

(iii) $\{xxx\} = tx$ implies $t = |t| > 0$ or $x = 0$.

In Definition 2.1(ii), $[\cdot, \cdot]$ is so-called the Jordan triple identity. It implies that the linear span of all operators $L(x, y)$ for all $x, y \in Z$ is a Lie subalgebra of $\mathcal{L}(Z)$. Let $(x|y) := \mathrm{tr}(L(x, y))$ be a trace form of the positive-definite inner product on Z that is invariant under the automorphism group, meaning that $L(gx, gy) = gL(x, y)g^{-1}$ for all $g \in \mathrm{Aut}(Z)$, where

$$\mathrm{Aut}(Z) := \{g \in GL(E) : \ g\{xyz\} = \{(gx)(gy)(gz)\} \text{ for all } x, y, z \in Z\}.$$

In particular, $L^*(x, y) = L(y, x)$ is the corresponding adjoint of $L(x, y)$, thus it justifies the name of Hermitian Jordan triple system (see [6, 4]).

For any $x, y, z \in Z$, in Definition 2.1(i), $\{xyz\} \in Z$ is so-called the Jordan triple product. An element c of Z is called a triple idempotent if $\{ccc\} = c$. In this case there exists a Peirce decomposition

$$Z = Z_1(c) \bigoplus Z_{1/2}(c) \bigoplus Z_0(c),$$

where $Z_s(c) := \{z \in Z : \ \{ccz\} = sz\}$ for $s = 0, 1/2, 1$ (see [10]).

Let U be as in Definition 2.1 and let Ω denote an open non-void regular convex cone in the self-adjoint part $X := \{x \in U : x^* = x\}$ of U. Given a point $e \in \Omega$. Notice that X is a simple Eucidean Jordan algebra with the identity e. Suppose Z is a Jordan algebra under the product

$$z \circ w := \{zew\} \tag{2.1}$$

and $U = Z_1(e)$ is a subalgebra with unit element e and $V = Z_{1/2}(e)$. The involution on U is given by $u^* = \{eue\}$. The base point $e \in \Omega$ is a triple idempotent of Z with $Z_0(e) = \{0\}$, and $\mathcal{D}$ has the following Jordan theoretic description (see [10]).

Definition 2.2. Let $\Phi:\ V\times V\to U$ be an Ω-Hermitian mapping and

$$\Phi(v,\,b)=2\{vbe\}. \tag{2.2}$$

The symmetric Siegel domains of second kind is defined by

$$\mathcal{D}=\mathcal{D}(\Omega,\,\Phi):=\{(u,\,v)\in U\times V:\ 2\mathrm{Re}u-\Phi(v,\,v)\in\Omega\},$$

where $\mathrm{Re}u:=(u+u^*)/2$ is the real part of $u\in U$ with respect to an involution $u\mapsto u^*$ of U. The boundary of $\mathcal{D}=\mathcal{D}(\Omega,\,\Phi)$ is

$$\partial D=\{(u,\,v)\in U\times V:\ 2\mathrm{Re}u-\Phi(v,\,v)\in\partial\Omega\},$$

where $\partial\Omega$ denotes the boundary of Ω in X.

Remark 2.3. (i) The Ω-Hermitian mapping Φ on V satisfies $\Phi(b,\,v)^*=\Phi(v,\,b)$ for all $b,\,v\in V$ and $\Phi(v,\,v)\in\bar{\Omega}\setminus\{0\}$ whenever $v\neq 0$ (see [8]).

(ii) For any $x,\,y\in Z$, we define the bilinear map by $(x,\,y)\mapsto x\circ y=\{xey\}$. Then $x,\,y$ are called orthogonal if $x\circ y=0$ holds, where the symbol $\circ$ is as in (2.1). The triple idempotent $c\neq 0$ is called minimal if it is not the sum of two orthogonal nonzero idempotents.

The frame $\{e_1,e_2,\cdots,e_r\}$ gives the Peirce decompositions

$$U=X+iX=\sum_{1\le i\le j\le r}X_{ij}^{\mathbb{C}}\quad\text{and}\quad V=\sum_{1\le j\le r}V_j, \tag{2.3}$$

where

$$V_j:=\{v\in V:\ \{e_ke_kv\}=\delta_{jk}v/2\text{ for any }1\le k\le r\}\,. \tag{2.4}$$

Let $a=\mathrm{dim}X_{ij}$ and $b=\mathrm{dim}V_j$, where a and b are independent of j and $i<j$, respectively, and the choice of $e_1,\,\cdots,\,e_r$. The number a and b are so-called the Peirce multiplicities; see, for example, [6].

The holomorphic automorphism group $G=\mathrm{Aut}(\mathcal{D})$ of $\mathcal{D}$ is a semi-simple Lie group. The Lie algebra of group G is defined by $\mathfrak{g}=\mathrm{aut}(\mathcal{D})$, it consists of all completely integrable holomorphic vector fields $F(z)\frac{\partial}{\partial z}$ on $\mathcal{D}$, with Poisson bracket

$$\left[F(z)\frac{\partial}{\partial z},\,G(z)\frac{\partial}{\partial z}\right]=\left[F'(z)\cdot G(z)-G'(z)\cdot F(z)\right]\frac{\partial}{\partial z}.$$

Let $\mathfrak{l}$ consist of all Jordan triple derivations M of Z vanishing at e and

$$\mathfrak{l}:=\{eez\}\frac{\partial}{\partial z}=u\frac{\partial}{\partial u}+\frac{v}{2}\frac{\partial}{\partial v}\in\mathfrak{g}.$$

Let $\mathfrak{g}^\mu := \{A \in \mathfrak{g} : [\mathfrak{l}, A] = \mu A\}$, where $\mu = -1, -1/2, 0, 1/2, 1$. Similar to the proof of [9, p. 579], we have canonical gradation $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^{1/2} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{-1/2} \oplus \mathfrak{g}^{-1}$ and hence $[\mathfrak{g}^\nu, \mathfrak{g}^\mu] \subset \mathfrak{g}^{\nu+\mu}$ for $\nu, \mu = -1, -1/2, 0, 1/2, 1$, where $\mathfrak{g}^\mu := \{0\}$ if $\mu > 1$ and $\mu < -1$. With the help from Jordan triple derivation of Z, we obtain an important lemma just from [9, Propositions 2.46 and 2.66], the details are omitted.

Lemma 2.4. (i) *Let $\mathfrak{a}$ be a commutative subalgebra of $\mathfrak{g}$ and $\mathfrak{m} := \{B \in \mathfrak{g} : [A, B] = 0$ for all $A \in \mathfrak{a}\}$. Then we have the following decomposition*

$$\mathfrak{g} = \sum_{i\le j} \mathfrak{g}_{ij}^1 \oplus \sum_i \mathfrak{g}_i^{1/2} \oplus \sum_i \mathfrak{g}_{ij}^0 \oplus \sum_i \mathfrak{g}_i^{-1/2} \oplus \sum_{i\le j} \mathfrak{g}_{ij}^{-1} \oplus \mathfrak{m} \tag{2.5}$$

with

$$\begin{aligned}
\mathfrak{g}_{ij}^1 &= \left\{ i\xi \frac{\partial}{\partial z} : \ \xi \in X_{ij} \right\}, \\
\mathfrak{g}_j^{1/2} &= \left\{ (\beta + 2\{e\beta z\}) \frac{\partial}{\partial z} : \ \beta \in V_j \right\}, \\
\mathfrak{g}_{ij}^0 &= \left\{ L(x, e_i) \frac{\partial}{\partial z} : \ x \in X_{ij} \right\}, \\
\mathfrak{g}_j^{-1/2} &= \left\{ (2\{\beta e z\} + \{z\beta z\}) \frac{\partial}{\partial z} : \ \beta \in V_j \right\} \text{ and} \\
\mathfrak{g}_{ij}^{-1} &= \left\{ i\{z\xi z\} \frac{\partial}{\partial z} : \ \xi \in X_{ij} \right\};
\end{aligned}$$

(ii) *Suppose $r := \dim\mathfrak{a}$ and let $\alpha_1, \cdots, \alpha_r$ be the basis of $\mathfrak{a}^*$ dual to $e_1, \cdots, e_r$. The splitting* (2.5) *is the root decomposition of $\mathfrak{g}$ relative to $\mathfrak{a}$. Then $\mathfrak{g}_{ij}^{\pm 1}$ is the root space for $\pm\frac{1}{2}(\alpha_i + \alpha_j)$ $(i \le j)$; $\mathfrak{g}_j^{\pm 1/2}$ is the root space for $\pm\frac{1}{2}\alpha_j$ $(1 \le j \le r)$ and $\mathfrak{g}_{ij}^0$ is the root space for $\frac{1}{2}(\alpha_j - \alpha_i)$ $(i \ne j)$.*

Choose an ordering of the roots such that

$$\mathfrak{n}^\pm = \sum_{i\le j} \mathfrak{g}_{ij}^{\pm 1} \oplus \sum_j \mathfrak{g}_j^{\pm 1/2} \oplus \sum_{i<j} \mathfrak{g}_{ij}^0,$$

which is the nilpotent part of the Iwasawa decomposition $\mathfrak{g} = \mathfrak{n}^\pm \oplus \mathfrak{a} \oplus \mathfrak{k}$. Then the Weyl chamber is still given by $\mathfrak{a}_+ = \{\sum_{j=1}^r t_j e_j : \ 0 < t_1 < \cdots < t_r\}$. Let K be the stabilizer at $(0, e) \in \mathcal{D}$ in G. Then K is a maximal compact subgroup of G, denotes it by $K := \exp(\mathfrak{k})$. The corresponding Iwasawa decomposition of G is $G = N^\pm AK$ with $A = \exp(\mathfrak{a})$ and $N^\pm := \exp(\mathfrak{n}^\pm)$, where the subset N is called the distinguished boundary of $\mathcal{D}$ (see [6, 10]).

Definition 2.5. The Šilov boundary of $\mathcal{D}$ is given by

$$\Sigma := \{(u,\, v) \in U \times V :\ 2\mathrm{Re}u = \Phi(v, v)\},$$

which can be identified as the Siegel-type nilpotent group $N = X \times V$. The Campbell-Hausdorff formula of N is defined by, for any $(\zeta,\, z)$, $(\zeta',\, z') \in N$,

$$(\zeta,\, z)(\zeta',\, z') := (\zeta + \zeta' + \mathrm{Im}\Phi(z',\, z),\, z + z').$$

Let $N = X \times V$. The dimension of X and V are

$$n_1 := \dim_{\mathbb{R}} X = r + \frac{1}{2}ar(r-1) \quad \text{and} \quad n_2 := \dim V = 2rb, \tag{2.6}$$

where a and b denote the Peirce multiplicities. Σ is a real analytic manifold isomorphic to N via the mapping

$$N \ni (\zeta,\, z) \mapsto (\zeta + i\Phi(z,\, z)/2,\, z) \in \Sigma.$$

For any pair $(\zeta,\, z) \in N$, the affine transformation

$$(\zeta,\, z)(u,\, v) := (u + \zeta + i\Phi(v,\, z) + i\Phi(z,\, z)/2,\, v + z) \tag{2.7}$$

leaves $\mathcal{D}$ invariant. It is clear that an element $(\zeta,\, z) := \exp(i\zeta + z + L(2e,\, z))$ for $\zeta \in X$ and $z \in V$ is the action on $\mathcal{D}$ by the formula (2.7) under the definition of Φ as in (2.2) (see [6, Lemma 10.7(3)] for more details).

3 Plancherel measures

Let the Lie algebra $\mathfrak{s}(0)$ leave both finite dimensional vector spaces X and V invariant, where X, V $(V \neq \{0\})$ are as in (2.3) and

$$\mathfrak{s}(0) := \mathfrak{r} \oplus \sum_{i<j} \{L(x,\, e_i) :\ x \in X_{ij}\} \text{ and } \mathfrak{r} := \sum_{1 \le j \le r} \mathbb{R}(L(e_j,\, e_j)).$$

The following lemma is just [7, Lemma 11.1].

Lemma 3.1. *Let $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{a}$ and $\mathfrak{g}^0 = \mathfrak{a} \oplus \sum_{i<j} \mathfrak{g}^0_{ij}$. We say that $p :\ \mathfrak{g} \to \mathfrak{s} = \mathfrak{s}(0) \oplus X \oplus V$ is Lie algebra isomorphism if the map p satisfies, for $t \in \mathfrak{g}^0$, $u \in V$ and $a \in X$,*

$$t(z)\frac{\partial}{\partial z} \mapsto t, \quad (u + 2\{euz\})\frac{\partial}{\partial z} \mapsto u \quad \text{and} \quad ia\frac{\partial}{\partial z} \mapsto a.$$

We recall the definition of normal j-algebra. Suppose

$$J(L(e_j,\, e_j)) = -e_j,\ J(L(x,\, e_i)) = -\frac{1}{2}x,\ Ju = -iu,\ Jx = 2L(x,\ e_i)$$
and $Je_j = L(e_j,\, e_j)$

is a linear operator on $\mathfrak{s}$, where $x \in V_{ij}$, $u \in V$ and $1 \leq j \leq r$. The statement $J^2 = -I$ is evident. We say that J is the complex structure on V if J satisfies $J^2 = -I$. Suppose $e^* \in X^*$ is the trace form of the Euclidean Jordan algebra X, namely, $\langle x,\, e^* \rangle = \mathrm{tr} x$ for $x \in X$. By an element e^* of X^*, we extend e^* to $\mathfrak{s}$ by zero extension such that the triple $(\mathfrak{s},\, J,\, e^*)$ is a normal j-algebra ([7, Proposition 11.2]).

Remark 3.2. As a remark of the normal j-algebra, we refer these works [3, 7] by the actions of $\exp \mathfrak{s}(0)$ and $\exp \mathfrak{g}^0$ which the split solvable Lie group $S := \exp \mathfrak{s}$ can be identified with the subgroup $S = N^+A$.

Suppose that X is a finite dimensional vector space. X can be provided with an inner product $(\cdot|\cdot)$, so that $(X,\, (\cdot|\cdot))$ is a Hilbert space. As usual, we denote by X^* the space dual to X, the space of all linear forms on X, namely, $X \to X^*$ is conjugated linear. Now we consider the coadjoint action of $S(0) = \exp(\mathfrak{s}(0))$ on the dual vector X^* of X. We identity X^* with X by the inner product $(\cdot|\cdot)$. For $t \in S(0)$ and $\xi \in X^*$, we denote by $x \cdot \xi$ the non-degenerate linear form $\xi \circ \mathrm{Ad}(t)^{-1} \in X^*$. Let $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_r)$ be an element of $\{-1,\, 1\}^r$, and $e_{\varepsilon_j} = \varepsilon_j e_j$ for $1 \leq j \leq r$, where e_j is an orthogonal system of frame. The linear form e^*_ε on vector space X induces an isomorphism $X \to X^*$ such that

$$e^*_\varepsilon(x) = (x|e_\varepsilon) = \sum_{j=1}^{r} \varepsilon_j x_j,$$

where $x_j \in \mathbb{R}$ and

$$x = \sum_{1\leq j\leq r} x_j e_j + \sum_{1\leq i<j\leq r} X_{ij} \text{ for } X_{ij} \in \mathfrak{g}^1_{ij}.$$

Motivated by [2], we have the following

Definition 3.3. The family $\{\mathcal{O}^*_\varepsilon :\ \varepsilon \in \{-1,\, 1\}^r\}$ is a $S(0)$-orbit $S(0)\cdot e^*_\varepsilon \subset X^*$ if it satisfies the properties:

(i) Each of the orbits $\mathcal{O}^*_\varepsilon$ is open in X^*;

(ii) The unit $\mathcal{O}^* := \bigsqcup_{\varepsilon\in\{-1,\,1\}^r} \mathcal{O}^*_\varepsilon$ is dense in X^*;

(iii) The group $S(0)$ acts on $\mathcal{O}^*_\varepsilon$ simply transitively.

Remark 3.4. The statement is obvious if the element $e^*_{(1,\cdots,1)} = e^*|_X$. Then orbit $\mathcal{O}^*_{(1,\cdots,1)}$ coincides with the dual cone of $\Omega \subset X$,

$$\Omega^* = \{\xi \in X :\ (x|\xi) > 0 \text{ for all } x \in \bar{\Omega}\backslash\{0\}\}.$$

Our main interest in this paper is $\Omega^* = \Omega$ and $\mathcal{O}^*_\varepsilon \in \partial\Omega$ for $\varepsilon \notin (1, \ldots, 1)$.

Definition 3.5. For any $\ell \in \mathcal{O}^*_\varepsilon$, in Definition 3.3(iii), there exist a unique $t \in S(0)$ and $\varepsilon \in \{-1, 1\}^r$ for which $\ell = t \cdot e^*_\varepsilon$. Then, for any $x, y \in X$, we define a kind of coupling inner product $\gamma :\ \mathcal{O}^* \times U \to \mathbb{C}$ by

$$\gamma(t \cdot e^*_\varepsilon(x+iy)) := t \cdot e^*_{(1,\cdots,1)}(x) + it \cdot e^*_\varepsilon(y),$$

where $t \cdot e^*_\varepsilon(x+iy) := \langle t \cdot e^*_\varepsilon,\, x+iy\rangle$.

Throughout the whole paper, we define the determinant $\det x$ for $x \in X$ by $\Delta(x)$. The application of the coupling inner product γ formally affects a Plancherel measure of the Siegel-type nilpotent Lie group N.

Theorem 3.6. *For $\ell \in \Omega$, let $J_\ell :\ V \to V$ be a real bilinear form defined by*

$$(J_\ell v|v') = 2\mathrm{Re}\gamma\left(\ell\left(\frac{1}{2i}(\{vv'e\} - \{v've\})\right)\right) \text{ for all } v, v' \in V. \tag{3.1}$$

Then $J_\ell v = -2i\{e\ell v\}$ and $\det(J_\ell) = 2^{n_2}\Delta(\ell)^{\frac{n_2}{r}}$.

Proof. By (3.1), Definition 3.5, $\mathrm{Im}\Phi(v, v') = \frac{1}{2i}(\{vv'e\} - \{v've\})$, Definition 2.1 and $L(e, \ell) = L(\ell, e)$ for $\ell \in \Omega \subset X$, we can see that

$$(J_\ell v|v') = 2t \cdot e^*_{(1,\cdots,1)}\left(\frac{1}{2i}(\{vv'e\} - \{v've\})\right) = 2(-i\{e\ell v\}|v')$$

and hence $J_\ell v = -2i\{e\ell v\}$, which together with (2.6) implies that the real determinant of J_ℓ is $\det(J_\ell) = 2^{n_2}\Delta(\ell)^{2b}$. We therefore obtain the desired result. □

As usual, we reuse the symbol $\rho(\ell)$ for the coupling inner product γ and likewise call it the Plancherel measure. Write

$$\rho(\ell) := \det(J_\ell) = 2^{n_2}\Delta(\ell)^{\frac{n_2}{r}}. \tag{3.2}$$

Let $q_\ell(v, v') = 2\gamma(\ell(\Phi(v, v')))$. Then the form q_ℓ is not necessarily sesquilinear on V associated with J for general $\ell \in \mathcal{O}^*$, the following are still valid, for any $v, v' \in V$, $q_\ell(v, v') = \overline{q_\ell(v', v)}$ and $q_\ell(v, v) > 0$. In particular, the coupling inner product γ is a Hermitian inner product, that is, for $\ell \in \Omega$, $\gamma(\ell(x+iy)) = (\ell|x+iy)$, we have $q_\ell(v, v') = 2(\ell|\Phi(v, v'))$. In this case, q_ℓ is a positive definite Hermitian form on the complex vector space V.

For the matrix algebra, Ω is the cone of positive-definite Hermitian matrices. The linear automorphism group $GL(\Omega) := \{g \in GL(X) : \; g(\Omega) = \Omega\}$ is transitive on Ω. Thus $\rho(\ell)$ is an invariant measure.

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