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On Properly Discontinuous Actions and Their Foliations

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Abstract

Proper actions particularly admit local models in terms of fibre (vector) bundles, which has important consequences for the structure of the G-manifold and the orbit space M/G. This paper intends to examine an important aspect of discrete actions and their properly discontinuous properties on certain mainfolds. Consequently, we show that the $\mathbb{Z}^d$-action on a smooth connected manifold M defines its foliation as a Riemannian symmetric space.

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Motivation

An important class of discrete groups are the discrete subgroups Γ of Lie groups G. They are of much use in geometric group theory. For the cases where Γ is an infinite discrete group, and X is a topological space or manifold, we have the noncommutative spaces giving rise to noncommutative geometry and algebras. Our interest in this paper is to prove some certain results concerning the actions of these discrete groups.

1 Introduction

Actions by discrete groups on a smooth manifold have peculiar geometric characters. It is our intention to explore these peculiarities of discrete actions on a general manifold and particularly on Riemannian manifold. The major result of this paper is the properly discontinuous action theorem which connects this set of actions to free and proper actions. We will first give a topological analysis of a proper groups action.

2 Proper Actions of Topological Groups

The topological property of being "proper" of an action assures a good behaviour for a topological group action. Such actions particularly admit local models in terms of fibre (vector) bundles; which has important consequences for the structure of the G-manifold and the orbit space M/G. Hence, principal bundles are inherently characterized by proper actions. (cf. [10]). We give the following basic definitions.

Definition 2.1. A topological (Lie) group G is said to define a continuous/smooth left-action on the topological space/manifold M if the map $\psi : G \times M \to M$ is continuous/smooth and (1) $\psi(e, x) = x$, for all $x \in M$; (2) $\psi(g_1, \psi(g_2, x)) = \psi(g_1 g_2, x)$ for all $g_1, g_2 \in G, x \in M$.

We give various equivalent definitions of proper action of a (topological) Lie groups from [5] and [3] which will be very useful in the proof of our main results.

Definition 2.2. The action of a Lie group G on a manifold M is called *proper* if for any two distinct points $p, q \in M$ there exist open neighbourhoods U_p and U_q of p and q in M respectively, so that the set $\{g \in G : gU_p \cap U_q \neq \emptyset\}$ is relatively compact in G. This is equivalent to saying that

$$G \times M \to M \times M, (g, p) \mapsto (p, gp)$$

is a proper map, that is, the inverse image of each compact set in $M \times M$ is also compact in $G \times M$.

Alternatively, we can define a proper action with respect to a fixed point of the manifold as follows.

Definition 2.3. An action of a Lie group G on M is proper at $p \in M$ if and only if there exists a neighbourhood U of p in M such that the set $\{g \in G : g(U) \cap U \neq \emptyset\}$ has a compact closure in G.

Remark 2.4. (1) Every action of a compact Lie group is proper, and the action of any closed subgroup of the isometric group of M is proper.
(2) If G acts properly on M, then M/G–the orbit space–is a Hausdorff space, each orbit $G \cdot p$ is closed in M and therefore embedded submanifold; and each isotropy group G_p is compact.

The following definitions follow from the action of a group G on a manifold M.

Definition 2.5. (1) If $G(x) = \{x\}$, then x is the *fixed point* of the action.
(2) The subgroup $\underset{x \in M}{\cap} G_x$ is called the *ineffective kernel* of the action. If it is the trivial subgroup $\{e\}$, then the action is said to be *effective*. If $G_x = \{e\}$ for all $x \in M$, the action is said to be *free*.
(3) The isotropy group G_x of a fixed point is the entire group G, and an action with fixed points cannot be free.

Remark 2.6. Every G-action can be reduced to an effective action of the quotient of G by the ineffective kernel $\underset{x \in M}{\cap} G_x$, which is a normal subgroup of G.

Lemma 2.7. *Let H be a closed subgroup of the Lie group G. Then right multiplication $(g, h) \mapsto gh; G \times H \to G$ is a free and proper right H action on G.*

Proof. Since H is a closed subgroup of the Lie group G, then $h \in H$ defines a right action of H on G by the map

$$R_h : G \times H \to G, x \mapsto xh,$$

which is the right multiplication or translation of H on G. The orbits of the action constitute the set of left cosets of G modulo H or the translates of H in G given as $G/H = \{xH : x \in G\}$. Next, we define an equivalent G-manifold.

Let $H \subset G$ be a closed subgroup then the quotient space $G/H = X$ is a smooth manifold. We define a left G-action on X by left translation

$$L_a : G \times G/H \to G/H, (a, gH) \mapsto (ag)H,$$

where $a, g \in G$ and $gH \in G/H$. The map is transitive and the stabilizer of gH is gHg^{-1}, since $gHg^{-1} \cdot gH = gH$. From this, we have that the projection map $\pi : G \to G/H$ is an equivariant map between the two actions, since

$$\pi \circ R_h(g) = L_a \circ \pi(g).$$

We therefore conclude that the right multiplication $G \times H$ is a free and proper right H-action since the left G-action on G/H is free, for $g_1 H = g_2 H \implies g_1^{-1} g_2 H = eH$; and proper since the action of the isotropy group is always proper. □

Next we show that left and right translations (multiplications) are equivalent.

Lemma 2.8. *A left G-action is equivalent to a right G-action on a smooth manifold M*

Proof. Let $\alpha_L : G \times M \to M$ be a left G-action, and define $\alpha_R : M \times G \to M$ by setting $\alpha_R(x, g) := \alpha_L(g^{-1}, x)$. Then α_R is a right G-action, and the identity map on M is a G-equivariant diffeomorphism with respect to these actions. Hence,

$$I_M(\alpha_R(x, g)) = \alpha_L(g^{-1}, I_M(x)).$$

Right actions can also be analogously transformed into left actions. □

Remark 2.9. The importance of this lemma lies in the fact that it shows the conjugation invariance of the isotropy subgroups G_x, which defines the isotropic and slice representations of G via a subaction. A subaction of an action of a group G is obtained by restricting the G-action on M to an action of a subgroup H of G.

Definition 2.10. A fibre bundle (P, π, B, F, G) is called a *principal G bundle*, or principal bundle, if $F = G$ and the action of G is by left translations. A fibre bundle is principal if and only if its fibres are orbits of free proper action of G.

Definition 2.11. An action $G \times M \to M$ is called *properly discontinuous* if for all $x \in M$, there exists a neighbourhood U of x such that $gU \cap U = \emptyset$ for all $g \in G \setminus \{e\}$.

3 Discrete Subgroups Γ of a Topological Group G

The study of discrete subgroups Γ of a topological group G is very important for the study of noncommutative algebras, for their actions result in noncommutative spaces.[6]. A concrete example is the $\mathbb{Z}$ action on a manifold. Let $\mathbb{Z} \times \mathbb{T} = \mathbb{T}$ be the action of $\mathbb{Z}$ on the circle or torus by rotation through a fixed circle $2\pi\theta$. To really understand the geometry structure of this noncommutative torus and similar noncommutative spaces, we need to understand the topological features of discrete groups and discrete subgroups of topological (Lie) groups. From [15] we define discrete topology of a topological space as follows.

Definition 3.1. The discrete topology is the finest topology that can be given to a set. It defines all subsets as open sets. In particular, singletons are open sets in the discrete topology.

Definition 3.2. A discrete group is a group with the discrete topology.

Definition 3.3. Let G be a topological group (assume G is also locally compact). Then a subgroup $\Gamma \subset G$ is a discrete subgroup if the induced topology

$$(\Gamma, \tau) = \{\Gamma \cap U : \ U \in (G, \tau)\}$$

is discrete.

For example, $\mathbb{Z}$ with the natural topology is a discrete group. The inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ induces the discrete topology on $\mathbb{Z}$.

4 Discrete Subgroup and Lattice

We next trace the relationship between the action of a discrete subgroup Γ and the notion of lattice. Much of the materials here are from [7], [2], [11]. One can also make reference to [9].

Definition 4.1. A discrete subgroup Γ of a topological group G is called a uniform lattice if $\Gamma \setminus G$ is compact. Hence we have the following.
(a) Every discrete subgroup Γ of $\mathbb{R}^n$ is of the form $\Gamma = \mathbb{Z}r_1 + \cdots + \mathbb{Z}r_k$, where r_i's are linearly independent.
(b) Γ is a uniform lattice if and only if $vol(\Gamma \setminus \mathbb{R}^n)$ is finite.
The notion of volume of the quotient space $\Gamma \setminus G$ is made precise using the idea of fundamental set and the induced Haar measure of the Γ-action on G. This is defined as follows.

Definition 4.2. A subset $\mathfrak{F}$ of G is called a fundamental set for the Γ-action on G if $\mathfrak{F}$ meets every Γ-orbit exactly once, this implies

$$\bigcup_{\gamma \in \Gamma} \gamma \mathfrak{F} = G, \text{ and } \gamma_1 \mathfrak{F} \cap \gamma_2 \mathfrak{F} = \emptyset \text{ for } \gamma_1 \neq \gamma_2 \text{ in } \Gamma.$$

Remark 4.3. The axiom of choice guarantees the existence of $\mathfrak{F}$; that is, by picking an element from every orbit in such a way as to have a Borel set. By definition, any measurable partition of G which can span a sigma-algebra of G is a fundamental set. To show the existence of a fundamental set $\mathfrak{F}$ which is a Borel set, we need to define a Haar measure on $\Gamma \setminus G$. That a Haar measure on G induces a measure on the quotient space $\Gamma \setminus G$ is proved in [7].

Proposition 4.4. *Given G a second countable locally compact topological group and ν a left invariant Haar measure on G, then for any discrete subgroup Γ of G, there exists an induced measure on $\Gamma \setminus G$.*

Proof. This proof starts with a construction of a fundamental set $\mathfrak{F}$ for the Γ-action that is a Borel set. Γ being a discrete subgroup of G, there is $V \subseteq G$ containing $e \in G$ such that $V \cap \Gamma = \{e\}$. G being a topological group, there exists an open neighbourhood U of e such that $UU^{-1} \subseteq V$. Then the Γ-translates of U are disjoint, that is, $\gamma_1 U \cap \gamma_2 U = \emptyset, \ \forall \ \gamma_1, \gamma_2 \in \Gamma, \gamma_1 \neq \gamma_2$. The second countability property of G implies the existence of of a sequence $g_1, g_2, \cdots$, of elements of G such that $G = \bigcup\limits_{i=1}^{\infty} U g_i$. Then let

$$\mathfrak{F} = \bigcup_{n=1}^{\infty} \left(U g_n - \bigcup_{i<n} \Gamma U g_i \right).$$

$\mathfrak{F}$ is verifiably a Borel set, for it forms a measurable partition of G. Hence, if $\pi : G \to \Gamma \setminus G$ is the orbit projection, then $\pi|_{\mathfrak{F}} : \mathfrak{F} \to \Gamma \setminus G$ is one to one. An induced measure μ is therefore defined on $\Gamma \setminus G$ by

$$\mu(B) = \nu(\mathfrak{F} \cap \pi^{-1}(B)),$$

where B is a Borel subset of $\Gamma \setminus G$.

To show that the induced measure μ is independent of the choice of $\mathfrak{F}$, let $\mathfrak{F}, \mathfrak{F}'$ be two Borel fundamental sets, the task is to show that $\nu(\mathfrak{F} \cap A) = \nu(\mathfrak{F}' \cap A)$ for any subset $A \subset G$ which is Γ-invariant. By the invariance of A and definition of $\mathfrak{F}, \mathfrak{F}'$, we obtain

$$\nu(\mathfrak{F} \cap A) = \sum_{\gamma \in \Gamma} \mu(\gamma \mathfrak{F}' \cap \mathfrak{F} \cap A) = \sum_{\gamma \in \Gamma} \mu(\mathfrak{F}' \cap \gamma^{-1} \mathfrak{F} \cap \gamma^{-1} A)$$

$$= \sum_{\gamma \in \Gamma} \mu(\gamma \mathfrak{F} \cap \mathfrak{F}' \cap A) = \nu(\mathfrak{F}' \cap A).$$

□

With this characterization of the discrete subgroup Γ, the fundamental set $\mathfrak{F}$ and the induced Haar measure μ on $\Gamma \setminus G$, a lattice subgroup of a topological group G is defined as follows.

Definition 4.5. A discrete subgroup Γ of a locally compact topological group G is called a lattice (lattice group) if $vol(\Gamma \setminus G)$ is finite. Alternatively, if ν is a left invariant Haar measure on G, then Γ is called a lattice subgroup of G if $\mu(\Gamma \setminus G)$ is finite, where μ is the ν-induced measure on $\Gamma \setminus G$.

4.1 Deck Transformations

In a covering projection $p : \tilde{M} \to M$, the deck transformations are usually taken to be the symmetries of the covering space. Assuming the spaces are connected and locally path connected, we have a formal definition from [4] as follows.

Definition 4.6. Let $p : \tilde{M} \to M$ be a covering projection of connected and locally path-connected spaces. A deck transformation is a homeomorphism $\phi : \tilde{M} \to \tilde{M}$ such that $p \circ \phi = p$, in other words, ϕ is a lift of p.

The set of deck transformations form a group which acts freely on $\tilde{M}$. The basic properties of these transformations are summed by the following theorem.

Theorem 4.7. *[19] Let $p : \tilde{M} \to M$ be a covering projection and ϕ a deck transformation. Then*
(i) ϕ is uniquely determined by its value at a point of $\tilde{M}$.
(ii) $\phi(\bar{x}_o) \in p^{-1}(x_o)$ whenever $\bar{x}_o \in p^{-1}(x_o)$.
(iii) If $\phi(\bar{x}_1) = \bar{x}_2$, where $\bar{x}_1, \bar{x}_2 \in p^{-1}(x_o)$, then $p_*\pi_1(\tilde{M}, \bar{x}_1) = p_*\pi_1(\tilde{M}, \bar{x}_2)$
(iv) Conversely if (i) holds then there exists a unique deck transformation ϕ such that $\phi(\bar{x}_1) = \bar{x}_2$.

Proof. (i) follows from the uniqueness of lifts. (ii) follows immediately from definition.
(iii) Applying the lifting necessity criterion we have

$$p_*\pi_1(\tilde{M}, \bar{x}_1) = \pi_1(p(\tilde{M}, \bar{x}_1) \ \text{ homomorphism property}$$

$$= \pi_1(p(\tilde{M}, \phi^{-1}(\bar{x}_2)) = \pi_1(M, p \circ \phi^{-1}(\bar{x}_2)) = \pi_1(M, p(\bar{x}_2)) \text{ lifting property}$$

$$= \pi_1(M, x_o) = p_*\pi_1(\tilde{M}, \bar{x}_2).$$

(iv) We apply the lifting sufficiency criterion to get continuous functions $\phi : \tilde{M} \to \tilde{M}$ and $\psi : \tilde{M} \to \tilde{M}$ such that $p \circ \phi = p, \phi(\bar{x}_1 = \bar{x}_2;\ p \circ \psi = p, \psi(\bar{x}_2) = \bar{x}_1$. Then $\phi \circ \psi$ and $\psi \circ \phi$ are both lifts of the map $p : \tilde{M} \to M$ such that

$$\phi \circ \psi(\bar{x}_2) = \bar{x}_2, \psi \circ \phi(\bar{x}_1) = \bar{x}_1$$

The identity map on $\tilde{M}$ is also a lift of p with these initial conditions. By uniqueness, we see that both $\phi \circ \psi$ and $\psi \circ \phi$ must be the identity map on $\tilde{M}$, proving that ϕ and ψ are homeomorphism. The uniqueness clause follows from that of lifts. □

The set of deck transformations of a covering projection $p : \tilde{M} \to M$ forms a group under composition of maps denoted by $Deck(\tilde{M}, M)$.

Remark 4.8. If $\phi : \tilde{M} \to \tilde{M}$ is a continuous map such that $p \circ \phi = p$, it can be shown that ϕ is a homeomorphism in the following cases: (i) $\pi_1(\tilde{M})$ is a finite group; (ii) $\tilde{M}$ is a regular cover of M.
If H is a subgroup of G and $gHg^{-1} \subset H$, then $gHg^{-1} = H$ in case H is finite or has finite index or is normal. The following propositions from [1] and [17] indicate the a relation between discrete subgroups and covering spaces and demonstrate some of these.

Proposition 4.9. *Let $\bar{G}, G$ be connected Lie groups, then*
(a) If $\pi : \bar{G} \to G$ is a covering of Lie groups, then $\ker \pi$ *is a discrete subgroup of $\mathfrak{Z}(\bar{G})$, the centre of $\bar{G}$.*
(b) If $\Gamma \subset G$ is a discrete subgroup of $\mathfrak{Z}(G)$, then G/Γ is a Lie group and the projection $\pi : G \to G/\Gamma$ is a (normal) covering with deck group $\{L_\gamma : \gamma \in \Gamma\}$, where L_γ is the left action by element of Γ.

Proof. (a) Set $\Gamma = ker\ \pi$. Since π is a local diffeomorphism, there exists a neighbourhood U of $e \in \hat{G}$ such that $U \cap \Gamma = \{e\}$. For any arbitrary element $a \in \bar{G}$, we have $aU \cap \Gamma = \{a\}$. Since $au = a' \implies u = a^{-1}a'$, it follows that Γ is a discrete normal subgroup of $\hat{G}$ which lies in $\mathfrak{Z}(\hat{G})$. This is verified by fixing $g \in \hat{G}$ and $a \in \Gamma$, and letting g_t be a path with $g_0 = e$ and $g_1 = g$, then $g_t a g_t^{-1} \in \Gamma$ which starts at a and by discreteness is equal to a for all t.

(b) By definition, an action of Γ on a manifold M is called properly discontinuous if it satisfies the following two conditions:

1 For any $p \in G$ there exists a neighbourhood U of p such that the open sets $L_g U$ are all disjoint.

2 For any $p, q \in G$ with $p \notin \Gamma q$ there exist neighbourhoods U of p and V of q such that $\gamma U \cap \gamma' V = \emptyset$ for all $\gamma, \gamma' \in \Gamma$.

While condition (1) guarantees that $G \to G/\Gamma$ is a covering since the image of U is an evenly covered neighbourhood, (2) ensures that the quotient group is Hausdorff. Given these, we reformulate (b) as follows:
If Γ acts properly discontinuously on M, then M/Γ is a manifold and $M \to M/\Gamma$ a covering with deck (discrete) group Γ
Let $\Gamma \subset \mathfrak{Z}(G)$ be a discrete subgroup. To show (1), let U be a neighbourhood of $e \in G$ such that $\Gamma \cap U = \{e\}$, which is possible since Γ is discrete. We then choose V such that $e \in V \subset U$ and $V.V^{-1} \subset U$. Then we claim that $L_g V$ are all disjoint. This is so because if $g_1 u = g_2 v$, for some $u, v \in V$, then $g_2^{-1} g_1 = vu^{-1} \in \Gamma \cap U \implies g_1 = g_2$.

For (2), fix $g_1, g_2 \in G$ with $g_1 \notin \Gamma g_2$. Let $V \subset U$ be neighbourhoods of e as above, which in addition satisfy $g_2^{-1} \Gamma g_1 \cap U = \emptyset$, which is possible since $g_2 \Gamma g_1$ is discrete and does not contain e by assumption. Then $g_1 V$ and $g_2 V$ are the desired neighbourhoods of g_1 and g_2. To show that they are disjoint, let

$y_1 g_1 u = y_2 g_2 v$ for some $y_1, y_2 \in \Gamma$ and $u, v \in V$, then $g_2^{-1} y_2^{-1} y_1 g_1 = vu^{-1} \in g_2^{-1} \Gamma g_1 \cap U$ which is impossible. Thus, the projection $G \to G/\Gamma$ is a covering.

Since $\Gamma \subset \mathfrak{Z}(G), G/\Gamma$ is a group and since π is a covering, it is a manifold as well. Since π is a local diffeomorphism, multiplication and inverse is smooth. Furthermore, the deck or discrete group is $\{L_\sigma = R_\sigma | \sigma \in \Gamma\} \cong \Gamma$ since $\pi(a) = \pi(b)$ implies $\pi(ab^{-1}) = e$, that is, $a = \sigma b$ for some $\sigma \in \Gamma$. Γ acts transitively on the fibres of π, which is the definition of a normal cover. □

Remark 4.10. From the above, the deck transformation group is discrete and isomorphic to the fundamental group of the base topological space (manifold) M, that is, $Deck(\tilde{M}, M) \simeq \pi_1(M)$.

Many natural spaces result from the quotient space $\Gamma \setminus M$ of the action of a discrete group on a topological space M which are usually proper actions. Example of such spaces is locally symmetric spaces. Next, we consider proper and discontinuous actions of Γ.

Definition 4.11. Let X be a (locally compact) Hausdorff space with a homeomorphic action of Γ: (i) We say that Γ acts discontinuously if for any $x \in X$, the orbit $\Gamma \cdot x$ of x is a discrete subset of X; in other words, $\Gamma \cdot x$ is a discrete subset of $\Gamma \times X = X$.
(ii) Again, the action of Γ is said to be properly discontinuous if for any compact subset $C \subseteq X$, the set $\{\gamma \in \Gamma : \gamma C \cap C \neq \emptyset\}$ is finite.

Proposition 4.12. *If the Γ-action on X is properly discontinuous, then the quotient space $\Gamma \setminus X$ with the quotient topology is Hausdorff.*

Proof. The proof follows from the fact that the quotient space of all proper actions are Hausdorff, as already seen above. □

Remark 4.13. The importance of discrete subgroup comes from their natural occurrence. For a manifold M with nontrivial (discrete) fundamental group, $\pi_1(M)$; the discrete group acts on the cover of M, denoted $\bar{M}$, as deck transformations. We give the following theorem as the first result of this paper.

Theorem 4.14. *Let G be a discrete group that acts on a smooth manifold M. Then this action is properly discontinuous if and only if it is free and proper.*

Proof. By hypothesis, G is discrete and defines an action on M. First, we assume the action to be properly discontinuous. This means that for each point $x \in M$ there is a neighbourhood U of x such that $gU \cap U = \emptyset$, $\forall\, g \in G$. To show that the action is free we proceed as follows. Since G is discrete, G_x is discrete for all $x \in M$. Thus, $G_x \cap G_y = \{e\}$, for any $x, y \in M$. Assuming this is not the case, then there exists a neighbourhood V of e in G_x, such that $G_y \cap V = \{e\}$. We choose a $W \subset V$ such that $W \cdot W^{-1} \subset V$. Then $L_g W$ are all disjoint for all $g \in G$, for if $g_1 u = g_2 v$, for some $u, v \in W$, then

$g_2g_1^{-1} = vu^{-1} \in G_y \cap U \implies g_1 = g_2$. Thus, G acts freely. The action is also proper since the isotropy group G_x is compact for all $x \in M$.

Conversely, let the action of G be free and proper, then we show that it is properly discontinuous. The freeness implies $\underset{x \in M}{\cap} G_x = \{e\} = V$. From this, we have that $G/\{e\}$ acts effectively on M. Thus, $G/\{e\} \simeq M$. It follows therefore that $L_gV \cap V = \emptyset$, for all $g \in G$. □

Remark 4.15. The Theorem above has an exception in some general situations. For example, if we consider a $\mathbb{Z}-$action on $\mathbb{R} \setminus \{(0,0)\}$ generated by a diffeomorphism $f : (x,y) \mapsto (2x, y/2)$, this action is properly discontinuous and free. This gives a contradiction but the exception here is that in the coordinate $(2x, y/2)$, $1/2 \notin \mathbb{Z}$. Thus, the result remains valid for all diffeomorphisms $f : (x,y) \mapsto (mx, ny)$, $m, n \in \mathbb{Z}$.

Example 4.16. A typical example of manifold with properly discontinuous action of a discrete group is a principal G-bundle. A universal covering of a smooth manifold $\rho : \tilde{M} \to M$ is a principal G-bundle over M, where $G = \pi_1(M)$-the fundamental group of M, which determines a properly discontinuous action on the cover $\tilde{M}$ by deck transformation.

5 Foliations by Discrete Actions

We now consider the foliation of a smooth manifold by the action of a discrete group Γ. As a way of definition, a foliation of a manifold M is the decomposition of the manifold into leaves that fit nicely together, which are of the same dimension. According to [8], there are different and equivalent ways of giving a foliation of a manifold; one of these ways, which we will employ here is the definition of an integrable subbundle of the tangent bundle.

Let M be a smooth manifold; we take $\mathbb{R}^n$ for a typical example. Then the discrete subgroup of $\mathbb{R}^n$ is of the form $\Gamma = \mathbb{Z}r_1 + \cdots + \mathbb{Z}r_k$, where $r_1, \cdots, r_k$ are linearly independent in $\mathbb{R}$. Assuming $k = n$, then the action $\Gamma \times \mathbb{R}^n = \mathbb{R}^n$ gives rise to the usual n-torus $\mathbb{T}^n$ as its orbit space $\mathbb{R}^n/\Gamma$. The natural projection $p : \mathbb{R}^n \to \mathbb{T}^n = \mathbb{R}^n/\Gamma$ is a covering map.

Since both $\mathbb{R}^n$ and $\mathbb{T}^n$ are connected Lie groups, but not path-connected, we employ the functor π_1 which goes from the category of pointed topological spaces Top_o with objects given as the pairs $(\mathbb{R}^n, x_o)$, $(\mathbb{T}^n, t_o)$, of the spaces $\mathbb{R}^n$ and $\mathbb{T}^n$; with morphisms as $C(\mathbb{R}^n, \mathbb{T}^n)$–the space of all continuous functions from $\mathbb{R}^n$ to $\mathbb{T}^n$– such that for $f \in C(\mathbb{R}^n, \mathbb{T}^n), f(x_o) = t_o$. Each $f : (\mathbb{R}^n, x_o) \to (\mathbb{T}^n, t_o)$ defines a group morphism

$$f_* : \pi_1(\mathbb{R}^n, x_o) \to \pi_1(\mathbb{T}^n, t_o) : [\alpha] \mapsto [f \circ \alpha],$$

where α is a loop in $\mathbb{R}^n$ with x_o as base, and $f \circ \alpha$ is a loop in $\mathbb{T}^n$ with t_o as base.

The morphism f_* is well defined, since given two homotopic loops α_1, α_2 in $I\!R^n$ based at x_o, with their homotopy given by F, then $F \circ f$ defines the homotopy between $f \circ \alpha_1$ and $f \circ \alpha_2$ in $\mathbb{T}^n$ with base as t_o. We therefore have a group homomorphism given by

$$f \circ (\alpha_1 \star \alpha_2) = (f \circ \alpha_1) \star (f \circ \alpha) \implies f_*([\alpha_1][\alpha_2]) = f_*([\alpha_1]) f_*([\alpha_2]).$$

which is induced by f on $\pi_1(I\!R^n, x_o)$ and $(\pi_1(\mathbb{T}^n, t_o)$. Thus, $\Pi_1 : \mathbf{Top}_o \to \mathbf{Gr}$ is therefore a covariant functor. We now give the second major result of this paper.

Theorem 5.1. *The functor* Π_1 *gives a foliation of a smooth connected manifold* M.

Proof. This is seen from the fact that given a connected Lie group G and a cover $\bar{G}$, the covering map $p : \bar{G} \to G$ is a local diffeomorphism. Hence, for any morphism of Lie groups $f : \bar{G} \to G$, the $\ker(f)$ is a normal Lie subgroup in $\bar{G}$. This means that its Lie algebra is trivial; that is, the Lie bracket is zero on $T_e(\ker(f))$. By construction, the $\ker f \subseteq \Gamma$ for every Lie morphism $f \in C(\bar{G}, G)$, where $G \simeq \bar{G}/\Gamma$.

Thus, we have the tangent map $df : T_{\bar{e}}(\bar{G}/\ker f) \to T_e G$ to be injective. Thus, the orbits of Γ-action form the closed submanifolds of G. Hence, it follows from the injectivity of $df : T_1(I\!R^n/\Gamma) \to T_1(\mathbb{T}^n)$ that the action of the discrete subgroup Γ on $I\!R^n$ (true for any connected smooth manifold) causes the foliation of the manifold by its connected components which are the images of the functor Π_1. □

This is the converse of the following theorem by [17].

Theorem 5.2. *If* G *is a connected Lie group then its universal cover* $\bar{G}$ *has a canonical structure of a Lie group such that the covering map* $\phi : \bar{G} \to G$ *is a morphism of Lie groups, and* $\ker \phi = \pi_1(G)$ *is a group. Morever, in this case* $\ker \phi$ *is a discrete central subgroup in* $\bar{G}$.

Proof. In topology if M, N are connected manifolds or nice enough topological spaces, then any continuous map $f : M \to N$ can be lifted to a map $\bar{f}$ between their covers $\bar{M}, \bar{N}$ such that $\bar{f} : \bar{M} \to \bar{N}$. Morever, if we choose $m \in M, n \in N$ such that $f(m) = n$ and choose liftings $\bar{m} \in \bar{M}, \bar{n} \in \bar{N}$ such that $p(\bar{m}) = m, p(\bar{n}) = n$, then there is a unique lifting $\bar{f}$ of f such that $\bar{f}(\bar{m}) = \bar{n}$.

We now choose some element $\bar{1} \in \bar{G}$ such that $p(\bar{1}) = 1 \in G$. Then by the above theorem, there is a unique map $\bar{i} : \bar{G} \to \bar{G}$ which lifts the inversion map $i : G \to G$ satisfying $\bar{i}(\bar{1}) = \bar{1}$; and a unique multiplication $\bar{m} : \bar{G} \times \bar{G} \to \bar{G}$ which lifts multiplication $m : G \times G \to G$ and satisfies $\bar{m}(\bar{1}, \bar{g}) = \bar{g}$.

Finally, to show that $\ker p$ is central. Let $\bar{h} \in \ker p$, for any $\bar{g} \in \bar{G}$, we have

$$p(\bar{g}\bar{h}\bar{g}^{-1}) = p(\bar{g})p(\bar{h})p(\bar{g}^{-1}) = p(\bar{g})p(\bar{g})^{-1} = e.$$

So $\bar{g}\bar{h}\bar{g}^{-1} \in \ker p$. Hence, $\ker p$ is central. □

6 Conclusion

As we have seen, all the $\mathbb{Z}^n$ actions are always proper on smooth and connected manifolds. We note in conclusion, that for the foliation of a smooth connected manifold by an action of the discrete group Γ, which we have described here, to be possible, the loops α, $f \circ \alpha$ in M and in M/Γ (in particular, $\mathbb{R}^n$ and $\mathbb{R}^n/\mathbb{Z}^n$) respectively, must be closed loops. This condition is known to be governed by the nature of the numbers $r_1, \cdots, r_n \in \mathbb{R}$ which generate the discrete subgroup $\Gamma \subset M$. When they form a diophantine vector in M, then the foliation is as given above. The case of a Louiville vector results in noncommutative space, which we will consider with groupoid facilities. In the above case, a Haar measure on M is invariant under translation by Γ. Thus, the leaves of the foliation caused by the Γ-action are integrable; they form immersed submanifolds of M.

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