International Mathematical Forum, Vol. 12, 2017, no. 17, 827 - 833
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2017.7870

Refinements of the Dragomir Inequality for Integrable Functions in a Normed Linear Space

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Abstract

In this paper, we establish a generalization of the so called Dragomir inequality for strongly integrable functions with values in a normed linear space, and then obtain the corresponding upper and lower bounds. As a result, we get some more general inequalities. Some applications will be also given.

Mathematics Subject Classification: 26D15

Keywords: Dragomir inequality, triangle inequality, normed linear space

1 Introduction and preliminaries

The well known triangle inequality is one of the most significant inequalities in mathematics. It has many interesting generalizations, refinements and reverses, which have been studied by many authors, see [2, 3, 5, 9] and references therein. Here, we only point that, in their paper [8], the authors presented the following the following generalized triangle inequalities with n elements in a Banach space X. More precisely, for all nonzero elements $x_1, x_2, \dots, x_n$ in a Banach space X, the following inequalities hold.

$$\begin{aligned} &\left\|\sum_{j=1}^n x_j\right\| + \left(n - \left\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\right\|\right) \min_{1\le j\le n}\{\|x_j\|\} \\ &\quad \le \sum_{j=1}^n \|x_j\| \\ &\quad \le \left\|\sum_{j=1}^n x_j\right\| + \left(n - \left\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\right\|\right) \max_{1\le j\le n}\{\|x_j\|\}. \end{aligned} \tag{1.1}$$

The generalized triangle inequalities are useful to study the geometrical structure of normed spaces. C-Y. Hsu et al. [7] presented these inequalities for strongly integrable functions with values in a Banach space. In fact, they got the following equalities in their paper:

$$\begin{aligned}
&\left\|\int_\Omega a(t)f(t)d\mu\right\| + \left(\|a\|_1 - \left\|\int_\Omega \frac{a(t)f(t)}{\|f(t)\|}d\mu\right\|\right)\operatorname{ess\,inf}(\|f(\cdot)\|)\\
&\quad \le \int_\Omega a(t)\|f(t)\|d\mu \qquad (1.2)\\
&\quad \le \left\|\int_\Omega a(t)f(t)d\mu\right\| + \left(\|a\|_1 - \left\|\int_\Omega \frac{a(t)f(t)}{\|f(t)\|}d\mu\right\|\right)\operatorname{ess\,sup}(\|f(\cdot)\|).
\end{aligned}$$

where f (respectively a) is assumed to be an, almost everywhere nonzero (respectively positive), integrable X-valued (respectively real valued) function on a measure space (Ω, μ) with positive measure μ. Obviously, inequalities (1.1) is a special case of inequalities (1.2).

On the another hand, Pecaric-Rajic [6] obtained the following inequalities which are sharper than inequalities (1.1) above.

$$\begin{aligned}
&\min_{i\in\{1,\cdots,n\}}\left\{\frac{1}{\|x_i\|}\left(\left\|\sum_{j=1}^n x_j\right\| + \sum_{j=1}^n \left|\|x_j\| - \|x_i\|\right|\right)\right\}\\
&\quad \le \left\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\right\| \qquad (1.3)\\
&\quad \le \max_{i\in\{1,\cdots,n\}}\left\{\frac{1}{\|x_i\|}\left(\left\|\sum_{j=1}^n x_j\right\| - \sum_{j=1}^n \left|\|x_j\| - \|x_i\|\right|\right)\right\}.
\end{aligned}$$

But, then, Sever S. Dragomir [4] further proved the following inequalities for an arbitrary number of finitely many elements of a normed linear space X, which also generalized inequalities (1.3) above.

$$\begin{aligned}
&\min_{k\in\{1,\cdots,n\}}\left\{|a_k|\left\|\sum_{j=1}^n x_j\right\| + \sum_{j=1}^n |a_j - a_k|\|x_j\|\right\}\\
&\quad \le \left\|\sum_{j=1}^n a_j x_j\right\| \qquad (1.4)\\
&\quad \le \max_{k\in\{1,\cdots,n\}}\left\{|a_k|\left\|\sum_{j=1}^n x_j\right\| - \sum_{j=1}^n |a_j - a_k|\|x_j\|\right\}.
\end{aligned}$$

Where $a_j \in \mathbb{K}$ and $x_j \in X$ for $j \in \{1, \cdots, n\}$ with $n \ge 2$.

Motivated by inequalities (1.2), in our paper [1], a generalisation of inequalities (1.3) is established for strongly integrable functions with values in a Banach space. In this paper, we shall further consider the continuous versions of the Dragomir inequalities (1.4) in a normed linear space. Some applications will also be given.

2 Dragomir inequalities for integrable functions

Theorem 2.1. *Let X be a normed linear space, (Ω, μ) be a measure space with positive measure μ, and $a(\cdot)$ be an essentially bounded measurable function $a : (\Omega, \mu) \to (-\infty, \infty)$. Let $f \in L^1(\Omega, X)$, and $b(\cdot)$ be an essentially bounded positive integrable function on Ω, then for any fixed $t_1, t_2 \in \Omega$, the following inequalities hold:*

$$
\begin{aligned}
&|a(t_1)| \left\| \int_\Omega b(t)f(t)d\mu \right\| + \int_\Omega |a(t) - a(t_1)|b(t)\|f(t)\|d\mu \\
&\quad \le \left\| \int_\Omega a(t)b(t)f(t)d\mu \right\| \\
&\quad \le |a(t_2)| \left\| \int_\Omega b(t)f(t)d\mu \right\| - \int_\Omega |a(t) - a(t_2)|b(t)\|f(t)\|d\mu.
\end{aligned}
\tag{2.5}
$$

Proof. Obviously, if $a(\cdot)$ is constant almost everywhere in Ω, then both inequalities (2.5) hold with equalities. Therefore, we may assume this is not the case. For the first inequality in (2.5), let us fix $t_1 \in \Omega$, then we have

$$
\begin{aligned}
\left\| \int_\Omega a(t)b(t)f(t)d\mu \right\| &= \left\| \int_\Omega a(t_1)b(t)f(t)d\mu + \int_\Omega (a(t) - a(t_1))b(t)f(t)d\mu \right\| \\
&\le \left\| \int_\Omega a(t_1)b(t)f(t)d\mu \right\| + \left\| \int_\Omega (a(t) - a(t_1))b(t)f(t)d\mu \right\| \\
&\le |a(t_1)| \left\| \int_\Omega b(t)f(t)d\mu \right\| + \int_\Omega |a(t) - a(t_1)|b(t)\|f(t)\|d\mu.
\end{aligned}
$$

From this we can get the first inequality in the first inequality in (2.5). In order to obtain the second inequality in (2.5), we can proceed in a similar way, for a fixed $t_2 \in \Omega$, we can obtain,

$$
\begin{aligned}
\left\| \int_\Omega a(t)b(t)f(t)d\mu \right\| &= \left\| \int_\Omega a(t_2)b(t)f(t)d\mu - \int_\Omega (a(t_2) - a(t))b(t)f(t)d\mu \right\| \\
&\ge \left\| \int_\Omega a(t_2)b(t)f(t)d\mu \right\| - \left\| \int_\Omega (a(t_2) - a(t))b(t)f(t)d\mu \right\| \\
&\ge |a(t_2)| \left\| \int_\Omega b(t)f(t)d\mu \right\| - \int_\Omega |a(t) - a(t_2)|b(t)\|f(t)\|d\mu.
\end{aligned}
$$

Therefore, we obtain two inequalities (2.5). this completes the proof. □

If we choose $a(t_1) = \|f(t_1)\|$ (respectively $a(t_2) = \|f(t_2)\|$) in Theorem 2.1, then it is easy to get the following result.

Corollary 2.1. *Let X be a normed linear space, (Ω,μ) be a measure space with positive measure μ, and $b(\cdot)$ be an essentially bounded positive integrable function on Ω. Let $f\in L^1(\Omega,X)$, then for any fixed $t_1,t_2\in\Omega$, the following inequality holds:*

$$\begin{aligned}
&\|f(t_2)\|\left\|\int_\Omega b(t)f(t)d\mu\right\|-\int_\Omega |\|f(t)\|-\|f(t_2)\||b(t)\|f(t)\|d\mu\\
&\quad\le\left\|\int_\Omega \|f(t)\|b(t)f(t)d\mu\right\| \qquad (2.6)\\
&\quad\le\|f(t_1)\|\left\|\int_\Omega b(t)f(t)d\mu\right\|+\int_\Omega |\|f(t)\|-\|f(t_1)\||b(t)\|f(t)\|d\mu.
\end{aligned}$$

From inequality (2.6), we can also get the following.

Corollary 2.2. *Let X be a normed linear space and (Ω,μ) be a measure space with positive measure μ, and let $b(\cdot)$ be an essentially bounded positive integrable function on Ω, $f\in L^1(\Omega,X)$, then the following inequalities hold:*

$$\begin{aligned}
&\left(\int_\Omega b(t)\|f(t)\|d\mu-\left\|\int_\Omega b(t)f(t)d\mu\right\|\right)\operatorname{ess\,inf}(\|f(\cdot)\|)\\
&\le\int_\Omega b(t)\|f(t)\|^2d\mu-\left\|\int_\Omega \|f(t)\|b(t)f(t)d\mu\right\| \qquad (2.7)\\
&\le\left(\int_\Omega b(t)\|f(t)\|d\mu-\left\|\int_\Omega b(t)f(t)d\mu\right\|\right)\operatorname{ess\,sup}(\|f(\cdot)\|).
\end{aligned}$$

Proof. In order to obtain the results, let us assume that $\operatorname{ess\,inf}(\|f(\cdot)\|)=\|f(t_1')\|$ with $t_1'\in\Omega$. Then, using the second inequality in (2.6) we have

$$\begin{aligned}
&\left\|\int_\Omega \|f(t)\|b(t)f(t)d\mu\right\|\\
&\le\|f(t_1')\|\left\|\int_\Omega b(t)f(t)d\mu\right\|+\int_\Omega |\|f(t)\|-\|f(t_1')\||b(t)\|f(t)\|d\mu,\\
&=\|f(t_1')\|\left\|\int_\Omega b(t)f(t)d\mu\right\|+\int_\Omega b(t)\|f(t)\|^2d\mu-\|f(t_1')\|\int_\Omega b(t)\|f(t)\|d\mu.
\end{aligned}$$

which is clearly equivalent to the first inequality in (2.7). The second part of (2.7) follows likewise and the details are omitted. □

Example 2.2. Let $b(t)\equiv 1$ and let $f\in L^1([-1,1],\mathbb{R}^2)$ be defined by $f(t)=(t,-1)$ for $t\in[-1,0]$ and $f(t)=(t,1+t)$ for $t\in(0,1]$. Then $\|f(t)\|_1=1-t$ for $t\in[-1,0]$ and $\|f(t)\|_1=1+2t$ for $t\in(0,1]$, and so $\inf(\|f(t)\|_1)=1$ and

$\sup(\|f(t)\|_1) = 3$. Elementary calculation shows that

$$\left\|\int_{-1}^{1} f(t)dt\right\|_1 = \frac{1}{2}, \qquad \int_{-1}^{1} \|f(t)\|_1 dt = \frac{7}{2},$$
$$\int_{-1}^{1} (\|f(t)\|_1)^2 dt = \frac{20}{3}, \qquad \left\|\int_{-1}^{1} f(t)\|f(t)\|_1 dt\right\|_1 = 2.$$

Therefore, we have by the inequality (2.7) in Corollary 2.2:

$$(\frac{7}{2} - \frac{1}{2}) \times 1(= 3) < \frac{20}{3} - 2(= \frac{14}{3}) < (\frac{7}{2} - \frac{1}{2}) \times 3(= 9).$$

Example 2.3. Let $b(t) \equiv 1$ and let $f \in L^1([0,1], \mathbb{R}^2)$ be defined by $f(t) = (t, 1-t)$ for $t \in [0,1]$. Then $\|f(t)\|_1 = t + (1-t) = 1$ for $t \in [0,1]$, and so $\inf(\|f(t)\|_1) = \sup(\|f(t)\|_1) = 1$. Elementary calculation shows that all the equalities in (2.7) hold.

3 Application to infinite series

For discrete versions of the results in Section 2, by letting $\Omega = \mathbb{N}$, $\mu(n) := 1$ and $a(n) := a_n$, $b(n) := b_n$ for $n \in \mathbb{N}$. Then using the results established in Theorem 2.1, Corollary 2.1, and Corollary 2.2, we can obtain the following results about the generalized Dragomir inequality and its reverse for infinite series.

Theorem 3.1. *Let $\{a_n\}$ be any sequence of numbers, $\{b_n\}$ be a sequence of nonnegative numbers such that $\sum_{n=1}^{\infty} b_n < \infty$. Then for any sequence $\{x_n\}$ in a normed linear space X such that $\sum_{n=1}^{\infty} b_n\|x_n\| < \infty$, we have*

$$\begin{aligned}
&\sup_i \left\{|a_i| \left\|\sum_{j=1}^{\infty} b_j x_j\right\| - \sum_{j=1}^{\infty} |a_j - a_i| b_j \|x_j\|\right\} \\
&\le \left\|\sum_{j=1}^{\infty} b_j x_j\right\| \\
&\le \inf_i \left\{|a_i| \left\|\sum_{j=1}^{\infty} b_j x_j\right\| + \sum_{j=1}^{\infty} |a_j - a_i| b_j \|x_j\|\right\}.
\end{aligned}$$

Corollary 3.1. *Let $\{b_n\}$ be a sequence of nonnegative numbers such that $\sum_{n=1}^{\infty} b_n < \infty$. Then for any sequence $\{x_n\}$ in a normed linear space X such that $\sum_{n=1}^{\infty} b_n\|x_n\| < \infty$, we have*

$$\begin{aligned}
&\sup_i \left\{\|x_i\| \left\|\sum_{j=1}^{\infty} b_j x_j\right\| - \sum_{j=1}^{\infty} |\|x_j\| - \|x_i\|| b_j \|x_j\|\right\} \\
&\le \left\|\sum_{j=1}^{\infty} b_j x_j\right\| \\
&\le \inf_i \left\{\|x_i\| \left\|\sum_{j=1}^{\infty} b_j x_j\right\| + \sum_{j=1}^{\infty} |\|x_j\| - \|x_i\|| b_j \|x_j\|\right\}.
\end{aligned}$$

Corollary 3.2. *Let $\{b_n\}$ be a sequence of nonnegative numbers such that $\sum_{n=1}^{\infty} b_n < \infty$. Then for any sequence $\{x_n\}$ in a normed linear space X such that $\sum_{n=1}^{\infty} b_n \|x_n\| < \infty$, we have*

$$\begin{aligned}
&\left(\sum_{j=1}^{\infty} b_j \|x_j\| - \left\|\sum_{j=1}^{\infty} b_j x_j\right\|\right) \inf_i \|x_i\| \\
&\quad \leq \sum_{j=1}^{\infty} b_j \|x_j\|^2 - \left\|\sum_{j=1}^{\infty} \|x_j\| b_j x_j\right\| \\
&\quad \leq \left(\sum_{j=1}^{\infty} b_j \|x_j\| - \left\|\sum_{j=1}^{\infty} b_j x_j\right\|\right) \sup_i \|x_i\|.
\end{aligned}$$

4 Conclusions

In this paper, we have considered the continuous versions of the Dragomir inequalities in a normed linear space. As a result, we have obtained upper and lower bounds for the norm estimates. Some applications to series inequalities also presented in our paper. It may be interesting to establish conditions that guarantee equality attainedness for each of our inequalities in a strictly convex Banach space. We would like to propose this issue as one project for further research interest.

Acknowledgements. The research is partly supported by the Science and Technology Research Key Project of Education Department of Henan Province (No. 18A110018).

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Received: August 20, 2017; Published: September 10, 2017