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Perfect Equitable Domination of Some Graphs

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Abstract

Let G be a connected simple graph. A subset D_e of V is called an *equitable dominating set* if for every $v \in V \backslash D_e$ there exists a vertex $u \in D_e$ such that $uv \in E$ and $|\deg(u) - \deg(v)| \leq 1$. The minimum cardinality of such dominating set is called equitable domination number and is denoted by $\gamma_e(G)$. A dominating set $D_p \subseteq V$ is called a *perfect dominating set* of G if each $u \in V \backslash D_p$ is dominated by exactly one element of D_p. The *perfect domination number* of G, denoted by $\gamma_p(G)$, is the minimum cardinality of a perfect dominating set of G. Define the *perfect equitable dominating set* to be the equitable dominating set D_{pe} of G such that for every $v \in V \backslash D_{pe}$ is dominated exactly one element in D_{pe}. The minimum cardinality of the perfect equitable dominating set is called the perfect equitable domination number in G and is denoted by $\gamma_{p_e}(G)$. In this study, we will examine the identites of $\gamma_{p_e}(G)$ of cycles, path, complete graphs and some other special graphs and we show when a perfect domination number is equal to perfect equitable domination number.

Keywords: dominating set, perfect dominating set, equitable dominating set, perfect equitable dominating set

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1. Introduction

A pair $G = (V, E)$ with $E \subseteq E(V)$ is called a graph (on V). The elements of V are the vertices of G, and those of E the edges of G. Suppose $v \in V$, the neighborhood of v is the set $N_G(v) = \{u \in V(G) : uv \in E(G).\}$. Given $D \subseteq V$, the set $N_G(D) = N(D) = \bigcup_{v\in D} N_G(v)$ and the set $N_G[D] = N[D] = D \bigcup N(D)$ are the *open neighborhood* and the *closed neighborhood* of D respectively. We say that D is the *dominating set* of G if for every $v \in (V \backslash D)$, there exists $u \in D$ such that $uv \in E$, that is, u is said to dominate v. Thus, $N[D] = V$. The *domination number* $\gamma(G)$ of G is the smallest cardinality of a dominating set of G.

A dominating set D_p of a graph G is called *perfect dominating set* of G if for every vertex $v \in V \backslash D_p$ is dominated exactly by one vertex $u \in D_p$. The minimum cardinality of the perfect dominating set of G is called the *perfect domination number* of G, and is denoted by $\gamma_p(G)$. A subset D_e of V is called an *equitable dominating set* if for every $v \in V \backslash D_e$ there exists a vertex $u \in D_e$ such that $uv \in E$ and $|\deg(u) - \deg(v)| \leq 1$. The minimum cardinality of such dominating set is called equitable domination number and is denoted by $\gamma_e(G)$.

A set $D_{pe} \subseteq V$ is the *perfect equitable dominating set* in G if it satisfies the following:

1. D_{pe} is a perfect dominating set in G. That is, for every $x \in V \backslash D_{pe}$ is dominated by exactly one element in D_{pe}.
2. D_{pe} is an equitable dominating set in G. Thats is, for every $x \in V \backslash D_{pe}$, there exist $y \in D_{pe}$ such that $xy \in E$ and $|\deg(x) - \deg(y)| \leq 1$.

The minimum cardinality of the perfect equitable dominating set is called *perfect equitable domination number* and is denoted by $\gamma_{pe}(G)$.

Thus, from the above definition, we say that if D_p and D_e be the perfect dominating set and the equitable dominating set in G, respectively. We have the following obvious properties:

1. $D_{pe} \subseteq D_p$
2. $D_{pe} \subseteq D_e$

The graph we consider here is connected simple graph where there should be no loops and no isolated vertex. For some theoretic terms used in this paper, we refer to [1], [2], [4] and [7].

2. Results

The results below also show the graphs whose perfect domination number and equitable domination number are equal to perfect equitable domination number:

Theorem 1.1 *Let K_m be a complete graph of order $m \geq 2$, then $\gamma_{pe}(K_m) = 1$.*

Proof:
Given a perfect equitable dominating set D_{pe} of K_m, assume that $a \in D_{pe}$. Then a dominates all other vertices in K_m and since K_m is a complete graph, then for every $b \in V(K_m)$, a and b are adjacent and $|\deg(a) - \deg(b)| = 0 < 1$. Obviously $D_{pe}(K_m) = \{a\}$ which implies that $\gamma_{pe}(K_m) = 1$. ■

Corollary 1.1 *For all integers $m \geq 2$, $\gamma_{pe}(K_m) = \gamma_e(K_m) = \gamma_p(K_m) = 1$.*

Proof:
This follows from Theorem 2.1. ■

Theorem 1.2 *For any integer $n \geq 2$, $\gamma_{p_e}(P_n) = \left\lceil \frac{n}{3} \right\rceil$.*

Proof: Suppose $V(P_n) = \{a_1, a_2, \cdots, a_n\}$. Let a_1 be the first vertex, a_2 be the second vertex, $\cdots$, a_n be the last vertex where P_n is labeled left to right. Observe that a_1 dominates a_2, a_2 dominates a_1 and a_3 and a_i dominates $a_{i-1}, a_{i+1} \quad i = 2, 3, \cdots, (n-1)$. Let $D_{pe}(P_n)$ be a perfect equitable dominating set of P_n. Consider the following cases:

- Case 1: $P_n = P_{3k-1}, k \in \mathbb{Z}^+$

  If $k = 1$, then $P_{3(1)-1} = P_2$. Clearly $D_{pe}(P_2)$ is $\{a_1\}$ or $\{a_2\}$ and is of minimum order 1. Thus, $\gamma_{pe}(P_2) = 1 = \left\lceil \frac{2}{3} \right\rceil$.

  If $k = 2$, then $P_{3(2)-1} = P_5$. So $D_{pe}(P_5) = \{a_1, a_4\}$ or $\{a_2, a_5\}$, and again are of minimum order. Thus, Thus, $\gamma_{pe}(P_5) = 2 = \left\lceil \frac{5}{3} \right\rceil$.

If $k = 3$, then $P_{3(3)-1} = P_8$. $D_{pe}(P_8) = \{a_1, a_4, a_7\}$ or $\{a_2, a_5, a_8\}$. This is of minimum order. Thus, $\gamma_{pe}(P_8) = 3 = \left\lceil \frac{8}{3} \right\rceil$.

If $k = 4$, then $P_{3(4)-1} = P_{11}$. $D_{pe}(P_8) = \{a_1, a_4, a_7, a_{10}\}$ or $\{a_2, a_5, a_8, a_{11}\}$. Again this is of minimum order. Thus, $\gamma_{pe}(P_{11}) = 4 = \left\lceil \frac{11}{3} \right\rceil$.

In general,

$$\begin{aligned} D_{pe}(P_{3k-1}) &= \{a_{3i-1} | i = 1, 2, \cdots, k \quad k \in \mathbb{Z}^+\} \\ \text{or } &= \{a_{3i+1} | i = 0, 1, 2, \cdots, (k-1) \quad k \in \mathbb{Z}^+\}. \end{aligned}$$

Hence, $\gamma_{pe}(P_{3k-1}) = \left\lceil \frac{3k-1}{3} \right\rceil = \left\lceil \frac{n}{3} \right\rceil$.

- Case 2: $P_n = P_{3k}, \quad k \in \mathbb{Z}^+$

Again, we verify te following:

If $k = 1 \Rightarrow P_{3(1)=P_3}$ and $D_{pe}(P_3) = \{a_2\}$. Obviously, $\gamma_{pe}(P_3) = 1 = \left\lceil \frac{3}{3} \right\rceil$.

If $k = 2 \Rightarrow P_{3(2)} = P_6$ and $D_{pe}(P_6) = \{a_2, a_5\}$. Thus, $\gamma_{pe}(P_6) = 2 = \left\lceil \frac{6}{3} \right\rceil$.

If $k = 3 \Rightarrow P_{3(3)} = P_9$ and $D_{pe}(P_9) = \{a_2, a_5, a_8\}$. Thus, $\gamma_{pe}(P_9) = 3 = \left\lceil \frac{9}{3} \right\rceil$.

If $k = 4 \Rightarrow P_{3(4)} = P_{12}$ and $D_{pe}(P_{12}) = \{a_2, a_5, a_8, a_{11}\}$. Thus, $\gamma_{pe}(P_{12}) = 4 = \left\lceil \frac{12}{3} \right\rceil$.

In general,

$$D_{pe}(P_{3k}) = \{a_{3i-1} | i = 1, 2, \cdots, k \quad k \in \mathbb{Z}^+\}.$$

Thus, $\gamma_{pe}(P_{3k}) = \left\lceil \frac{3k}{3} \right\rceil = \left\lceil \frac{k}{3} \right\rceil = \left\lceil \frac{n}{3} \right\rceil$.

- Case 3: $P_n = P_{3k+1}, \quad k \in \mathbb{Z}^+$

In a similar manner, it can be verified that

$$D_{pe}(P_{3k+1}) = \{a_{3i+1} | i = 0, 1, \cdots, k \quad k \in \mathbb{Z}^+\}.$$

Thus,

$$\gamma_{pe}(P_{3k+1}) = \left\lceil \frac{3k+1}{3} \right\rceil = \left\lceil \frac{n}{3} \right\rceil$$

Hence, in all cases

$$\gamma_{pe}(P_n) = \left\lceil \frac{n}{3} \right\rceil .$$

■

Remark 1.1 *In a path P_n, consecutive vertices of Perfect Equitable Dominating Sets are either adjacent or at a distance 3 apart.*

To see this, supposed that the vertices of the perfect dominating sets are not adjacent and at a distance two apart, then two vertices dominate to exactly one vertex in $V \backslash D_{pe}$ which is a cotradiction to the definition of the perfect equitable dominating sets. ■

Remark 1.2 *Some graphs have unique Perfect Equitable Dominating Set but some have not. The path of order $3k-1$ has different Perfect Equitable Dominating Set.*

The following theorems hold for the cycle C_n.

Theorem 1.3 *For $k \in \mathbb{Z}^+$,*

$$\gamma_{p_e}(C_n) = \begin{cases} 2k & \text{if } n = 6k \\ k+1 & \text{if } n = 3k+1 \text{ or } n = 3k-1 \\ k & \text{if } n = 3k, n \geq 9. \end{cases} \tag{1.1}$$

Proof:

Let $V(C_n) = \{a_1, a_2, \cdots, a_n\}$. Label $V(C_n)$ in a clockwise direction such that a_1 is adjacent to a_n, a_2, a_2 is adjacent to a_1, a_3 and a_{n-1} is adjacent to a_{n-2}, a_n and a_n is adjacent to a_{n-1}, a_1. Consider the following cases:

- Case 1: $n = 6k$

  A perfect equitable dominating set D_{pe} can be obtained as follows: If $a_i \in D_{pe}$ where D_{pe} is perfect equitable dominating set, then a_i dominates its two adjacent vertices a_{i-1} and a_{i+1} modulo n. This means that by selecting one vertex of C_n to be a member of D_{pe}, three vertices are eliminated from the remaining selection for the next choice of a_i. Thus the process of selecting a member of D_{pe} follows the grouping of three consecutive vertices in $6k$ vertices. It is easy to see that grouping of four consecutive vertices containing one dominating vertex is not possible or grouping of four consecutive vertices containing two dominating vertices which are both adjacent to another is possible but implies more elements in D_{pe}. The grouping of three yields a minimum D_{pe}. Thus, $\gamma_{pe}(C_n) = \frac{6k}{3} = 2k, \quad k \in \mathbb{Z}^+$.

- Case 2: $n = 3k + 1$

We also group the vertices of C_n by three. There remains a vertex, say a_j. Either $a_j \in D_{pe}$ or $a_j \notin D_{pe}$. If $a_j \in D_{pe}$, we cannot complete getting the perfect dominating set because there will remain three consecutive vertices which do not belong to D_{pe}. To see this, without loss of generalization, let $a_2, a_5, a_8, \cdots, a_{3k-1}$ be in D_{pe}. Now, $a_{3k+1} \in D_{pe}$ or $a_{3k+1} \notin D_{pe}$.

If $a_{3k+1} \in D_{pe}$, then a_1 is adjacent to a_2, a_{3k+1} which is a contradiction to the definition of perfect dominating set. On the other hand, if $a_{3k+1} \notin D_{pe}$, then there does not exist $a_j \in D_{pe}$ adjacent to a_{3k+1}. Now we let $a_1 \in D_{pe}$ or $a_{3k} \in D_{pe}$. Consequently, $\gamma_{pe}(C_n) = \frac{3k}{3} + 1 = k + 1$. It is easy to see that this is the minimum.

- Case 3: $n = 3k - 1$

We group $3k - 3$ vertices and set aside the remianing two vertices. Let $V(C_n) = \{a_1, a_2, \cdots, a_{3k-4}, a_{3k-3}, a_{3k-2}, a_{3k-1}\}$.

We consider $\{a_1, a_2, \cdots, a_{3k-3}\}$ and set aside a_{3k-2} and a_{3k-1}. Without loss of generalization, let $a_2, a_5, a_8, \cdots, a_{3k-4} \in D_{pe}$ while $a_1, a_3, a_4, a_6, a_7, \cdots, a_{3k-3}, a_{3k-2}, a_{3k-1} \in V(C_n) \backslash D_{pe}$. Consider the remaining vertices a_{3k-2}, a_{3k-1}. If $a_{3k-1} \in D_{pe}$, then a_1 is adjacent to a_2 and a_{3k-1} which is a contradiction to the definition of perfect dominating set. If $a_{3k-2} \in D_{pe}$ then a_{3k-3} is adjacent to a_{3k-4} and a_{3k-2} which is a contradiction to the definition of perfect equitable dominating set. We are forced to let a_{3k-3} and a_{3k-2} be element of D_{pe} and $a_{3k-1} \notin D_{pe}$ or $a_{3k-3}, a_1 \in D_{pe}$ and $a_{3k-2} \notin D_{pe}$. It can be verified that this yields to a minimum number of of perfect dominating set. Thus,

$$\begin{aligned}\gamma_{pe}(C_n) &= \frac{3k-3}{3} + |\{a_{3k-3}, a_{3k-2}\}| \\ &= (k-1) + 2 \\ &= k+1, \quad \text{when } n = 3k-1.\end{aligned}$$

- Case 4: $n = 3k$

  Proof is similar to case 1, when $n = 6k$, but consider instead the $n = 3k$. ■

Corollary 1.2 *Given a graph G, $\gamma_{pe}(G) = 1$ if and only if G is K_m, P_2, P_3 or C_3.*

Definition 1.1 (4) *The tadpole graph $T_{n,k}$ is the graph created by concatenating C_n and P_k with an edge from any vertex of C_n to a pendant of P_k for integers $n \geq 3$ and $k \geq 0$. Below is the perfect equitable domination number of the tadpole graph.*

Theorem 1.4 *For all integers $n \geq 3$ and $k \geq 0$, $\gamma_{p_e}(T_{n,k}) \leq \gamma_{p_e}(C_n) + \gamma_{p_e}(P_k)$.*

Proof:
Note that $T_{n,k}$ is composed of C_n and P_k for $n \geq 3$, $k \geq 0$. Then we will start determining the perfect equitable dominating set at C_n. Then clearly from C_n we have $\gamma_{pe}(C_n)$.

Let $u \in V(C_n)$ and $uv \in E(T_{n,k})$ where $v \in V(P_k)$. Then eiter $u \in D_{pe}(C_n)$ or $u \notin D_{pe}(C_n)$.

- ***Case 1:*** If $u \in D_{pe}(C_n)$

  If $u \in D_{pe}(C_n)$, then v_1 is dominated by u, where v_1 is the first vertex from the left of P_k. Thus the first vertex of P_k which is element of $D_{pe}(P_k)$ is v_3 to dominate v_2 since v_1 is dominated already. But note that the perfect equitable dominating set of P_k alone should start at either first vertex or second vertex. This means that either $\gamma_{pe}(P_k)$ or $\gamma_{pe}(P_k) - 1$ is left to continue the value of the perfect domination number of $T_{n,k}$. Thus, we have $\gamma_{pe}(T_{n,k}) \leq \gamma_{pe}(C_n) + \gamma_{pe}(P_k)$.

- ***Case 2:*** If $u \notin D_{pe}(C_n)$

  If $u \notin D_{pe}(C_n)$, then we select u to be either adjacent to w or at a distance 2 apart from w where $w \in D_{pe}(C_n)$.

  If u is adjacent to w then the perfect equitable dominating set of P_k starts v_2 since v_1 is not allowed anymore to dominate u. Note that the perfect equitable dominating set in P_k starts at either first or second vertex. Thus $D_{pe}(Tn,k) = D_{pe}(C_n) \bigcup D_{pe}(P_k)$ which follows that $\gamma_{pe}(T_{n,k}) = \gamma_{pe}(C_n) + \gamma_{pe}(P_k)$.

  If u is at distance 2 apart from w then we only select $\gamma_{pe}(C_n) - 1$ in C_n part of $T_{n,k}$. Thus v_1 must dominate u and since the perfect equitable dominating set of P_k starts either at first or second vertex, then we select $\gamma_{pe}(P_k)$ of the P_k part of $T_{n,k}$. Thus we have

$$\begin{aligned} \gamma_{pe}(T_{n,k}) &= (\gamma_{pe}(C_n) - 1) + \gamma_{pe}(P_k) \\ &< \gamma_{pe}(C_n) + \gamma_{pe}(P_k). \end{aligned}$$

  Thus either of the subcases, we have

$$\gamma_{pe}(T_{n,k}) \leq \gamma_{pe}(C_n) + \gamma_{pe}(P_k)$$

Therefore, either of the cases, we have $\gamma_{pe}(T_{n,k}) \leq \gamma_{pe}(C_n) + \gamma_{pe}(P_k)$. ■

Definition 1.2 (2) *The ottomar graph, $O_{n,m}$, is the graph C_n, $n \in \mathbb{Z}^+, n \geq 3$, with a vertex connected by a path P_2 to a vertex of C_m, $m \in \mathbb{Z}^+, m \geq 3$. C_n is called the heart while C_m is called a foot (feet in plural). Note there are n copies of C_m .*

Theorem 1.5 *For all integer $n \geq 3, m \geq 3$, $\gamma_{p_e}(O_{n,m}) = n\gamma_{p_e}(C_m)$.*

Proof:
Consider a cycle of $C_m, m \in \mathbb{Z}^+, m \geq 3$. Supposed $a_i \in D_{pe}(C_m), \quad i = 1, 2, \cdots \quad m$ is connected to $a_j \in C_n, \quad j = 1, 2, \cdots \quad n$. Note that there are n copies of C_m and all vertices of C_n are attached to one of the elements of perfect equitable dominating set of C_m. Thus, it follows that all vertices of C_n are elements of $V(O_{n,m}) \backslash D_{pe}(O_{n,m})$. Therefore, $\gamma_{pe}(O_{n,m}) = n\gamma_{pe}(C_m)$. ■

It is worth-noting that some graphs don't have perfect equitable dominating sets. The following are some examples:

Theorem 1.6 *For all integers $n \geq 6$, F_n does not have a perfect equitable dominating set. Moreover, $\gamma_{p_e}(F_n) = 0$.*

Proof:

By the definition found in [7], a fan graph F_n is the joint $K_1 \vee P_{n-1}$ where the vertex come from K_1 is the core. Now consider a vertex a to be the core, then the vertices in P_{n-1} for all $n \geq 6$, say $b_1, b_2, \cdots, b_{n-1}$ are dominated by a. Thus deg $(a) = n - 1$ and for every $b_i \in V(P_{n-1})$ has degree at most 3 (that is, b_i is adjacent to b_{i-1}, b_{i+1} and a. Thus,

$$\begin{aligned} |\deg(a) - \deg(b)| &= |(n-1) - 3| \\ &\geq |(6-1) - 3| \\ &\geq |5 - 3| \\ &= 2 \\ &\geq 1. \end{aligned}$$

Therefore F_n does not have perfect equitable dominating set. Consequently, $\gamma_{p_e}(F_n) = 0$. ■

The following corollary shows the inequality of the perfect domination number and the perfect equitable domination number of a graph.

Corollary 1.3 *For all integers $n \geq 6$, $\gamma_{p_e}(F_n) < \gamma_p(F_n)$.*

Proof: By Theorem 2.6, $\gamma_{p_e}(F_n) = 0$. Suppose a is the vertex core of F_n then a dominates all vertices of P_{n-1} where $V(P_{n-1}) = n - 1$, for all $n \geq 6$. Thus, $\gamma_p(F_n) = 1$ for every $n \geq 6$. Consequently, $\gamma_{p_e}(F_n) < \gamma_p(F_n)$. ■

There are also graphs whose perfect equitable dominating set does not exist. Here are some examples and their proofs are just easy to prove:

Remark 1.3 *For every integer n*

1. *$K_{1,n}$ for $n \geq 3$*

2. *W_n for $n \geq 5$*

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