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Lie Group Structure on the Set $BP(\mathbb{R}^2)$

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Abstract

A ball with one dimension is a circle. Let $BP(\mathbb{R}^2)$ be the set of circles with the same center and r-radius. In this article, we prove that the set of S has a group structure, manifold structure and Lie group structure. If $\{A_i\}_{i\in\mathbb{N}} = A$ is a partition of a non-empty set A and there is a relation among all on A_i, then some geometric properties workable on A using this partition. In this study, using $S^1_{r_k}$ partition of $\mathbb{R}^2$ we establish Lie group structure on this $S^1_{r_k}$ partition.

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1 Introduction

We know that $\mathbb{R}^2$ is a set of (x, y) points, where $x, y \in \mathbb{R}$. Nevertheless we can introduce $\mathbb{R}^2$ as a collection of subsets of $\mathbb{R}^2$ which union of these subsets is $\mathbb{R}^2$. For example, we can treat $\mathbb{R}^2$ as a collection of the parallel straight lines. Similarly, $\mathbb{R}^2$ is a collection of vectors, or P, X tangent vectors, etc. $\mathbb{R}^2$ allows us to build different structures on each different handling $\mathbb{R}^2$. A new structure can be created by treating $\mathbb{R}^2$ as circles with center $(0, 0)$ and radius r, $r \in \mathbb{R}^+$. In this paper, a new structure on $\mathbb{R}^2$ using the ball partition BP of $\mathbb{R}^2$ will be tried to construct. Firstly, we give the definitions and some properties of group, manifold and Lie groups.

Definition 1.1 *A group $(G, *)$ is a monoid, with identity e, which has the additional property that for every element $a \in G$, there exists an element $a' \in G$ such that*

$a * a' = a' * a = e$. Thus a group is a set G together with a binary operation $*$ on G such that

1) $(a * b) * c = a * (b * c)$ for any elements a, b and c in G.

2) There is a unique element e in G such that

$$a * e = e * a \tag{1}$$

for any $a \in G$.

3) For every $a \in G$ there is an element $a' \in G$, called an inverse of a, such that

$$a * a' = a' * a = e \tag{2}$$

[4].

Definition 1.2 *A differentiable or C^∞ (or smooth) structure on a topological manifold M is a family $U = \{U_\alpha, \varphi_\alpha\}$ of coordinate neighborhoods such that:*

1) the $U\alpha$ cover M,

2) for any α, β the neighborhoods $U_\alpha, \varphi_\alpha$and U_β, φ_β are $C^\infty-$compatible,

3) any coordinate neighborhood V, ψ compatible with every $U_\alpha, \varphi_\alpha \in U$ is itself in U.

A C^∞ manifold is a topological manifold together with a C^∞-differentiable structure [1, 2, 3].

Let M be a manifold and N be a set. If there is a 1:1 correspondence between M and N, then we can establish a manifold structure on N. For example, $\mathbb{R}^{m.n}$ is a manifold and $M(n \times m, \mathbb{R})$ is a set of $n \times m$ real matrices. And there is a 1:1 correspondence between $\mathbb{R}m.n$ and $M(n \times m, \mathbb{R})$. Finally, $M(n \times m, \mathbb{R})$ is a differentiable manifold.

Definition 1.3 *A Lie group is a group which is at same time a differentiable manifold such that the group operation $(a, b) \in G \times G \longrightarrow ab^{-1} \in G$ is a manifold mapping of $G \times G$ into G [3].*

Let M be a non-empty set. The set

$$M = \{U_i \mid U_i \subset M, i = 1, ..., n, n \in \mathbb{N}\} \quad (3)$$

is called a covering of the set M. A covering M is called a partition of the set M. If $U_i \cap U_j = \varnothing$ for every i, j, $i, j = 1, ..., n, i \neq j$.

In this study, we will study on $\mathbb{R}^2$, but what we say is true for $\mathbb{R}^n$.

Another partition of $\mathbb{R}^2$ can be given as follows.

$$L(\mathbb{R}^2) = \{A_i \mid A_i = \{(x, y) \mid y = mx + n, m : fixed, n \in \mathbb{R}\}\}. \quad (4)$$

In the partition $P(\mathbb{R}^2)$, subsets are points (x, y). Using the properties of points, we can define some structures on $\mathbb{R}^2$ as affine structure, manifold structure, etc.

In the partition $L(\mathbb{R}^2)$, subsets are parallel straight lines. Using the properties of parallel lines, we can define some structures on $\mathbb{R}^2$.

2 $B^1(\mathbb{R}^2)$ Partition of $\mathbb{R}^2$

In this section we define a special partition of $\mathbb{R}^2$ using circles with center $(0, 0)$.

Definition 2.1 *The partition*

$$BP(\mathbb{R}^2) = \{C_r(0, 0) \mid r \in \mathbb{R}^+\} \quad (5)$$

is called ball partititon of $\mathbb{R}^2$,*where* $C_r(0, 0) = \{x, y\} \in \mathbb{R}^2 \mid x^2 + y^2 = r^2\}$.

It is sufficient to know what is radius r to $C_r(0, 0)$. When $r \in \mathbb{R}^+$ is given we can define $C_r(0, 0)$and $\{re^{i\theta} \mid 0 \leq r \leq \theta\}$. So the following diagram is commutative (Figure 2).

Now we define a product $\otimes$ on $BP(\mathbb{R}^2)$ as follows.

$$BP(\mathbb{R}^2) \times BP(\mathbb{R}^2) \to BP(\mathbb{R}^2)$$
$$(C_{r_1}, C_{r_2}) \to C_{r_1} \otimes C_{r_2} = C_{(r_1.r_2)} \quad (6)$$

where the product $r_1.r_2$ is product on $\mathbb{R}$.

Theorem 2.2 $(BP(\mathbb{R}^2), \otimes)$ *is an abel group.*

1. Associativity: $\forall C_{r_1}, C_{r_2}$ and C_{r_3},

$$(C_{r_1} \otimes C_{r_2}) \otimes C_{r_3} = C_{r_1} \otimes (C_{r_2} \otimes C_{r_3}) \quad (7)$$

2. Existance of unit element: For $r = 1$, C_r is an unit element.
3. Existance of inverse element: For every C_r, $C_{\frac{1}{r}}$ is the inverse element.
4. Commutative or abelian property: For every C_{r_1} and C_{r_2},

$$C_{r_1} \otimes C_{r_2} = C_{r_1.r_2} = C_{r_2.r_1} = C_{r_2} \otimes C_{r_1}. \quad (8)$$

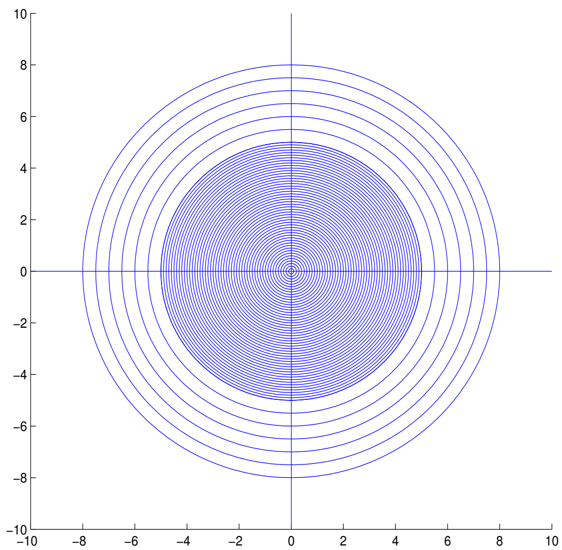

Figure 1: $BP(\mathbb{R}^2)$ Ball Partition of $\mathbb{R}^2$

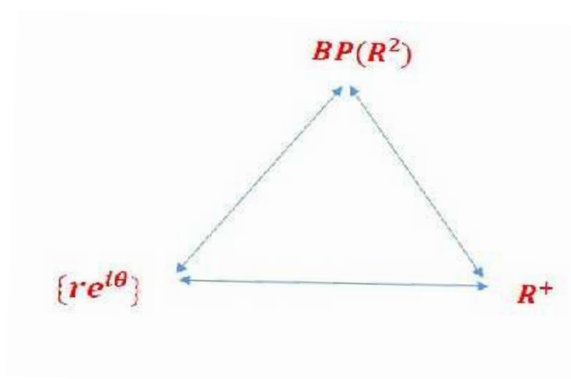

Figure 2: Commutative diagram

Theorem 2.3 *$BP(\mathbb{R}^2)$ is a differentiable manifold with dimension 1.*

Proof 2.4 *There is a 1:1 correspondence between $BP(\mathbb{R}^2)$ and $\mathbb{R}^+$.$\mathbb{R}^+$is an open subset of $\mathbb{R}$ and $\mathbb{R}$ is a differentiable manifold [1].*

We know that every open subset of a manifold is a differentiable manifold with same dimension [1]. So $BP(\mathbb{R}^2)$is a manifold with dimension 1.

Any open subset of $BP(\mathbb{R}^2)$ is

$$\{C_r \mid r_0 < r < r_2, r_0, r, r_1 \in \mathbb{R}\} \tag{9}$$

Now we can give a Lie group structure on $BP(\mathbb{R}^2)$.

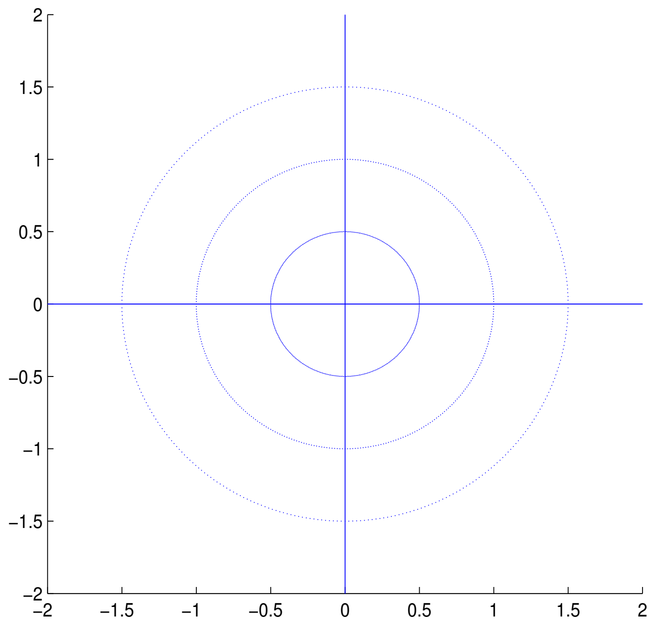

Figure 3: Unit element and inverse element

Theorem 2.5 $BP(\mathbb{R}^2)$ *is a Lie group.*

Proof 2.6 *We showed that* $BP(\mathbb{R}^2)$ *is a differentiable manifold with dimension 1. Also, the group operation*

$$C_{r_1} \otimes C_{r_2} = C_{r_1.r_2} \tag{10}$$

is differentiable such that $r_1 e^{i\theta_1}.r_2 e^{i\theta_2} = r_1 r_2 e^{i\theta}$ *is differentiable.*

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