International Mathematical Forum, Vol. 12, 2017, no. 17, 807 - 817
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2017.7762

Some Properties of the Operator Equation $STS = S^2$ and $TST = T^2$ for Dominant Operator S

Buthainah A. A. Ahmed and Hassan N. Almrayatee

Department of Mathematics
College of Science, University of Baghdad
Baghdad, Iraq

Abstract

In this paper we study the property of the operator equations $STS = S^2$ and $TST = T^2$ where S is dominant operator we show that is S dominant on a finite dimensional Hilbert space and $N(S) = N(ST)$ then ST is normal and if $N(S-\lambda) = N(T-\lambda)$for each $\lambda \in C$ then A is normal operator where $A \in \{S, ST, TS\}$ and we show that if S is polynomial root of dominant then $f(A) \in gW$ for each $f \in H(\sigma(A))$, where $A \in \{ST, TS, T\}$.

1. Introduction

let H be an infinite dimensional separable Hilbert space and let $B(H)$, $B_0(H)$ denote the algebra of bounded linear operator, the ideal of compact operator on H. If $T \in B(H)$ then $N(T)$ and $R(T)$ be the null space and the range of T. Also let $\alpha(T) := dimN(T), \beta(T) := dimN(T^*)$ and let $\sigma(T), \sigma_a(T), \sigma_s(T), \sigma_p(T), \sigma_{p_0}(T)$ *and* $\pi_0(T)$ denote the spectrum, approximate point spectrum , surjective spectrum, point spectrum of T, the set of pole of the resolvent of T and the set of all eigenvalue of T which is isolated in $\sigma(T)$.
Recall that $T \in B(H)$ is dominant if for every $\lambda \in C$ there exists a constant number $M_\lambda > 0$ such that $(T-\lambda)(T-\lambda)^* \leq M_\lambda(T-\lambda)^*(T-\lambda)$, and $T \in B(H)$ is called isoloid if each isolated point of $\sigma(T)$ is an eigenvalue of (T) , an operator $T \in B(H)$ is called normaloid if $r(T) = ||T||$ where $r(T)$ the spectral radius of (T) and it is well known that $r(T) \leq ||T||$. An operator T is said

to be nilpotent if $T^n = 0$ for a natural number n and it is quasinilpotent if $r(T) = 0$ [9, 10]

The operator $E := \frac{1}{2\pi i} \int_{\partial D} (\lambda - T)^{-1}$ is called Riesz idempotent with respect to λ where D is a closed desk centered at λ and $D \cap \sigma(T) = \{\lambda\}$ where $\lambda \in \sigma(T)$ be an isolated point of $\sigma(T)$ [9]

An operator $T \in B(H)$ is said to have the single value extension property $(SVEP)$ at λ_0 if for every analytic solution $f : U \to H$ which is satisfy the equation $(T - \lambda)f(\lambda) = 0 \quad (\lambda \in U)$ is the zero function , where U is open disc centered at λ_0 [14]

An operator $T \in B(H)$ is said have $(SVEP)$ if T has $(SVEP)$ at every λ in C from [2] we recall that for $T \in B(H)$, the asent $a(T)$ and the descent $d(T)$ given by

$$a(T) = inf\{n \geq 0 : N(T^n) = N(T^{n+1})\}$$

and

$$d(T) = inf\{n \geq 0 : R(T^n) = R(T^{n+1})\}$$

An operator $T \in B(H)$ is called Fredholm if it has closed range , finite dimensional null space and its range has finite co-dimensional.the index of a Fredholm operator

$$i(T) = \alpha(T) - \beta(T)$$

T is called Weyle if it is Fredholm of index zero , and Browder if it is Fredholm of finite ascent and descent.The essential spectrum $\sigma_e(T)$, the Wely spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ define as [5, 8]

$$\begin{aligned}
\sigma_e(T) :=& \{\lambda \in C : T - \lambda \quad is \quad not \quad Fredholm\} \\
\sigma_w(T) :=& \{\lambda \in C : T - \lambda \quad is \quad not \quad Weyl\} \\
\sigma_b(T) :=& \{\lambda \in C : T - \lambda \quad is \quad not \quad Browder\} \\
\sigma_e(T) \subseteq& W(T) \subseteq \sigma_b(T) := \sigma_e(T) \cup_{acc} \sigma(T)
\end{aligned}$$

we write $accK$ for the accumulation point of $K \subset C$ if we write $isoK = K \setminus accK$ then we let

$$\pi_{00}(T) := \{\lambda \in iso\sigma(T) : 0 \leq \alpha(T - \lambda) \leq \infty\}$$

$$P_{00}(T) := \sigma(T) \backslash \sigma_b(T)$$

we say that Weyl's theorem hold for T if

$$\sigma(T) \backslash \sigma_w(T) = \pi_{00}(T)$$

and Browder's theorem hold for T if

$$\sigma(T) \backslash \sigma_w(T) = P_{00}(T)$$

An operator $T \in B(H)$ is called B-Freadholm if there exists a natural number n for the induced operator $T_n : R(T) \to R(T^n)$ is Freadholm in the usual sense and B-Weyl's if in addition T_n has zero index .
the B-Fredholm spectrum $\sigma_{BF}(T)$ and B-Weyl spectrum $\sigma_{BW}(T)$ are define by

$$\sigma_{BF}(T) := \{\lambda \in C : T - \lambda \quad is \quad not \quad B - Freadholm\}$$

$$\sigma_{BW}(T) := \{\lambda \in C : T - \lambda \quad is \quad not \quad B - Weyl\}$$

An element x of A is Drazin invertible if there is an element b of A and non-negative integer k such that

$$x^k bx = x^k \quad , bxb = b, \quad , \quad xb = bx$$

[16] the Drazin spectrum of $a \in A$ is define by [6]

$$\sigma_D(a) := \{\lambda \in C : a - \lambda \quad is \quad not \quad Drazin \;\; invertible\}$$

If $T \in B(H)$ that is T is Drazin invertible if and only if it has finite ascent and descent and that is also equivalent to the fact that T decomposed as $T_1 \oplus T_2$ where T_1 is invertible and T_2 is nilpotent and

$$\sigma_{BW}(T) = \cap\{\sigma_D(T + F) : F \in B_0(H)\}$$

[16] the spectrum of B-Browder $\sigma_{BB}(T)$ define as [4]

$$\sigma_{BB}(T) = \cap\{\sigma_D(T + F) : F \in B_0(H) \quad and \quad TF = FT\}$$

Viav [18] study the operator equation $STS = S^2$ and $TST = T^2$ and An, Il Ju and Ko, Eungil [1] study the operator equation $STS = S^2$and $TST = T^2$ for a paranormal operator S

2. Main Results

Let the pair (S, T) of bounded linear operator acting on separable Hilbert space H be a solution of the operator equation $STS = S^2$ and $TST = T^2$, before we give our main results we need the following lemmas

Lemma 2.1. *[7]*

$$(S - \lambda)^{-1}\{0\} = \{0\} \Longleftrightarrow (ST - \lambda)^{-1}(0) = \{0\}$$

$$\Longleftrightarrow (TS - \lambda)^{-1}(0) = \{0\} \Longleftrightarrow (T - \lambda)^{-1}(0) = \{0\}$$

Lemma 2.2. *[11] If A is dominant operator then $N(\lambda I - A)$ reduces A for each $\lambda \in C$*

Theorem 2.3. *Let S be a dominant operator on a finite dimensional Hilbert space H and $N(S) = N(ST)$ then we have*
(1) ST is normal operator
(2) If $N(S - \lambda) = N(T - \lambda)$ for every $\lambda \in C$ then A is normal operator where $A \in \{S, ST, TS\}$

Proof. since $STS = S^2$ and $TST = T^2$ then $\sigma_p(ST) = \sigma_p(S)$ and $N(ST-\lambda) = N(T-\lambda)$ [17]
Let

$$K := \sum_{\lambda\in\sigma_p(ST)} N(ST-\lambda) = \sum_{\lambda\in\sigma_p(S)} N(S-\lambda)$$

since S is dominant operator and $N(S-\lambda)$ reduces S then K reduces S so we can represent S as follows

$$S = S_1 \oplus S_2 : K \oplus K^{\perp}$$

Assume that $K^{\perp} \neq \{0\}$ then $S_2|K^{\perp}$ is also dominant operator and dim ¡∞ , $\sigma_p(S_2) \neq \phi$ then for each $\lambda \in \sigma_p(S_2)$ there exists a nonzero vector $x_\lambda \in K^{\perp}$ such that $\lambda x_\lambda = S_2 x_\lambda = Sx_\lambda$ then $x_\lambda \in K$ but $x_\lambda \in K^{\perp}$ that is $x_\lambda = 0$, which is a contradiction there fore $K^{\perp} = \{0\}$ which is $H = K$ so for every $x \in H$

$$x = \sum_{\lambda\in\sigma_p(S)} x_\lambda = \sum_{\lambda\in\sigma_p(ST)} x_\lambda \quad for \quad some \quad x_\lambda \in N(A-\lambda)$$

$$STx = \sum_{\lambda\in\sigma_p(ST)} \lambda x_\lambda = \sum_{\lambda\in\sigma_p(S)} \lambda x_\lambda = Sx$$

but since $S^*T^*S^* = {S^*}^2$ and $T^*S^*T^* = {T^*}^2$ then

$$T^*S^*x = S^*x = \sum_{\lambda\in sigma_p(S)} \bar{\lambda} x_\lambda = \sum_{\lambda\in\sigma_p(ST)} \bar{\lambda} x_\lambda$$

therefore

$$||STx||^2 = \sum_{\lambda\in\sigma_p(ST)} ||\lambda x_\lambda||^2 = \sum_{\lambda\in\sigma(ST)} |\lambda|||x_\lambda||^2 = \sum_{\lambda\in\sigma_p(ST)} ||\bar{\lambda} x_\lambda|| = ||T^*S^*x||^2$$

so that ST is normal
(2) since $N(S-\lambda) = N(ST-\lambda)$ for every $\lambda \in C$ then

$$N(S-\lambda) = N(ST-\lambda) = N(TS-\lambda) = N(T-\lambda)$$

so that (2) is obvious □

Recall that an operator S is called convexoid if $conv\sigma(S) = \overline{W(S)}$ where $W(S)$is the numerical range of S

Lemma 2.4. *Let S be any operator which is normoloid and $\lambda \in C$ and $\sigma(S) = \{\lambda\}$ then $S = \lambda I$*

Proof. If $\lambda = 0$ then $S = 0$ since S is normoloid.so let $\lambda \neq 0$, which weans that S is invertible but S is normoloid so $||S|| = ||S^{-1}|| = |\lambda||\frac{1}{\lambda}| = 1$ that is S is convexoid so we have $W(S) = \{\lambda\}$and $S = \lambda I$ □

Lemma 2.5. *Let S be a dominant operator which is normoloid and $\sigma(S) = \{\lambda\}$ then we have*
(1) if $\lambda = 0$ then $T^2 = 0$
(2) if $\lambda \neq 0$ then $\lambda = 1$ and S, T the identity operator

Proof. if $\lambda = 0$ then from lemma (2.4) we get $T^2 = 0$
suppose that $\lambda \neq 0$ since S is dominant which is normoliod then $S = \lambda I$, and $STS = S^2$ then $\lambda^2(T - I) = 0$ so that $T = I$ also since $TST = T^2$ then $(\lambda - I)\lambda^2 = 0$and $\lambda = 1$ that is $\sigma(S) = \sigma(T) = \{1\}$, hence S and T are the identity operator .

□

Remark 2.6. Let S be a dominant operator which is normoloid then we have
(1) if $\sigma(S) = 0$ then ST, TS,and T nilpotent
(2) is $\sigma(S - I) = 0$ then $T = I$ that is $ST - \lambda, TS - \lambda$ and $T - \lambda$ are invertible for all $\lambda \in C \backslash \{1\}$

Corollary 2.7. *Let S be a dominant operator which is inevitable on a finite dimensional Hilbert space and $N(S) = N(ST)$ for any real number α then $\alpha ST + (1 - \alpha)S$ is a solution X for all $n \in N$ where $C(A, X)(A^*)$ define as*

$$C(T, X)(T^*) = \sum_{k=0}^{n} n\ k(-1)^k (T^{-1})^{n-k} (T)^*\, X^{n-k}$$

Proof. we have that $[\alpha ST = (1 - \alpha)S]\delta$ where $\delta = S$
since $(\alpha ST)\delta = \alpha STS = \alpha S^2 = \alpha\delta S = \delta(\alpha S)$ and ST, T are normal from theorem (2.3) then by Fulglede-Put nam theorem that $(\alpha SR)^*\delta = \delta(\alpha S)^*$ then

$$[\alpha ST + (1 - \alpha S)]^*\delta = \delta(\alpha S)^* = \delta[(1 - \alpha)S]^* = \delta S^*$$

that is

$$C(S^{-1}, X)(S^*) = \sum_{k=0}^{N} \binom{n}{K} (S^{-1})^{n-k} S^* S^{n-k} = 0$$

□

Corollary 2.8. *If S is dominant operator which is normoloid then $\sigma(A) \subseteq \{0, 1\}$ where $A \in \{S, ST, TS, T\}$*

Proof. Let λ_0 nonzero isolated point of $\sigma(S)$ by Riesz decomposition theorem on $E_{\lambda_0}(S)$ with respect to λ_0 we can act S as a direct sum

$$S = S_1 \oplus S_2 \quad where \quad \sigma(S_1) = \{\lambda_0\} \quad and \quad \sigma(S_2) = \sigma(S)\backslash\{\lambda_0\}$$

since S_1 is dominant operator then $\lambda_0 = 1$ by lemma (2.5) that is $\sigma(A) \subseteq \{0, 1\}$where $A \in \{S, ST, TS, T\}$

□

Lemma 2.9. *If S is a dominant operator and λ_0 is a nonzero isolated point of $\sigma(S,T)$then for Riez idempotent $E_{\lambda_0}(S)$ with respect to λ_0 we have*

$$R(E\lambda_0(S)) = N(ST-\lambda_0) = N(S^*T^*-\overline{\lambda_0})$$

Proof. since S is dominant operator and $\lambda_0 \in \sigma(S)\backslash\{0\}$ then
$R(E_{\lambda_0}(S)) = N(S-\lambda_0) = N(S^*-\overline{\lambda_0})$ for the Riesz idempotent $E_{\lambda_0}(S)$ with respect to λ_0, But the pair $(S,T)$is solution of the operator equation $STS = S^2$and $TST = T^2$ then

$$N(S-\lambda I) = N(ST-\lambda I) = S(N(T-\lambda I))N(T-\lambda I) = N(TS-\lambda I) = T(N(-\lambda I))$$

then $N(S-\lambda_0) = N(ST-\lambda_0)$ and $N(S^*-\bar{\lambda_0}) = N(S^*T^*-\bar{\lambda_0})$ for $\lambda_0 \neq 0$ □

Remark 2.10. We denote the set $\delta$by the collection of every pair (S,T) of operator as
$\delta := \{(S,T)$: S and T are the solution of the operator equation $STS = S^2$ and $TST = T^2$ with $N(S-\lambda) = N(T-\lambda)$ for $\lambda \neq \{0\}$

Proposition 2.11. *Let $(S,T) \in \delta$ and S be a dominant operator if λ_0 is nonzero isolated point of $\sigma(TS)$ then the range is closed*

Proof. Let λ_0 be a nonzero isolated point of $\sigma(TS) \subseteq \{1\}$ by Corollary (2.8) $iso\sigma(TS) = \phi$ then it is obvious that $\sigma(TS)$ has closed range
thus we only consider case which 1 is an isolated point of $\sigma(TS)$ since $STS = S^2$ and $TST = T^2$ then 1 is an isolated point of $\sigma(S)$ [17] then by Riesz idempotent $E_1(S)$ with respect to 1 we can act S as the direct sum

$$S = S_1 \oplus S_2 \quad \sigma(S_1) = \{1\} \quad and \quad \sigma(S_2) = \sigma(S)\backslash\{1\}$$

since $(S,T) \in \delta$and S_1 is dominant then by lemma(2.10)

$$H = R(E) \oplus R(E)^{\perp} = N(TS-I) \oplus N(TS-I)^{\perp}$$

$$TS = C_1 \oplus C_2 \quad where \quad \sigma(C_1) = \{1\} \quad and \quad \sigma(C_2) = \sigma(TS)\backslash\{1\}$$

since S_1 and C_1 are the restriction of S and TS to $R(E_1(S))$ respectively we not that if $T_1 := T|R(E_1(S))$ then $S_1T_1S_1 = {T_1}^2$ and $T_1S_1T_1 = {T_1}^2$ since S_1 is dominant then by lemma(2.5) $C_1 = I$ that is $TS-I = 0 \oplus (C_2-I)$ then

$$R(TS-I) = (TS-I)(H) = 0 \oplus (C_2-I)(N(TS-I)^{\perp})$$

since $C_2 - I$ is invertible , $TS-I$ has closed range □

3. generalized Weyl's theorem for algebraically dominant operators

Definition 3.1. Let $A \in B(H)$ is said to be an algebraically dominant if there exists a non-constant complex polynomial P such that $P(A)$ is dominant

$$M-hyponormal \Rightarrow dominant \Rightarrow algebraically \quad dominant$$

Remark 3.2. Let $A \in B(H)$ be an algebraically dominant then $A - \lambda$ is algebraically dominant for each λinC

Lemma 3.3. *let $S \in B(H)$ be a quasinilpotent algebraically dominant which is normoliod then S is nilpotent*

Proof. Let P a non-constant polynomial such that $P(A)$ is dominant since $\sigma(P(A)) = P\sigma(A))$ then the operator $P(A) - P(0)$ is quasinilpotent by (2.5) since $P(A) - P(0) = 0$ that is

$$P(A) = c(A - \lambda_1)(T - \lambda_2)...........(A - \lambda_n) \quad and \quad P(0) \equiv 0$$

then

$$c(A - \lambda_1)(A - \lambda_2)...........(A - \lambda_n) = 0$$
$$cA[(A - \lambda_1)(A - \lambda_2)...........(A - \lambda_n)] = 0$$
$$cA^2[(A - \lambda_1)(A - \lambda_2)...........(A - \lambda_n)] = 0$$
$$cA^n[(A - \lambda_1)(A - \lambda_2)...........(A - \lambda_n) = 0$$

since $A - \lambda_i$ is invertible for every $\lambda_1 \neq 0$ we must have $A^n = 0$ □

Lemma 3.4. *Let $A \in B(H)$ be an algebraically dominant which is normoliod then A is isoloid*

Proof. Let λ be an isolated point of $\sigma(A)$ then by spectral projection $E := \frac{1}{2\pi i}\int_{\partial D}(\mu - A)^{-1}d\mu$ where D is closed disk of center λ which is contains no other points of $\sigma(A)$, we can act A as the direct sum

$$A = A_1 \oplus T_2 \quad where \quad \sigma A_1 = \{\lambda\} \quad and \quad \sigma(A_2) = \sigma(A)\backslash\{\lambda\}$$

by Riesz decomposition theorem ([15],p31) since A algebraically dominant then $P(A)$ is dominant operator for some non-constant polynomial P since $\sigma(A_1) = \{\lambda\}$ then $\sigma(P(T_1)) = P(\sigma(A_1)) = P(\lambda)$. therefore $P(A_1) - P(\lambda)$ is quasinilpotent since $P(A_1)$ is dominant then by lemma(2.5) $P(A_1) - P(\lambda) = 0$ put $q(z) := p(z) - p(\lambda)$ then $q(A_1) = 0$ and hence A_1 is algebraically dominant since A_1 is quasinilpotent and algebraically dominant then by lemma (3.3) that $A_1 - \lambda$ nilpotent therefore $\lambda \in \sigma_p(A_1)$ and then $\lambda \in \sigma_p(A)$ that is T isoloid □

Theorem 3.5. *Let $A \in B(H)$ be an algebraically dominant operator which is normoloid then $f(A) \in gW$ for each $f \in H(\sigma(A))$*

Proof. Since A is dominant operator then by [12] A has $SVEP$ then by [14, Theorem 3.3.9,p231]$P(A)$ has $SVEP$ hence from [4] that is $f(\sigma_{BW}(A)) = \sigma_{BW}(f(A))$ for each $H(\sigma(A))$ since A is algebraically dominant then by lemma(3.4) T is isoloid therefore [19]

$$\sigma(f(A))\backslash\pi_0(f(T)) = f(\sigma(T)\backslash\pi_0(A)) = f(\sigma_{BW}(A) = \sigma_{BW}(f(A))$$

that is $f(A) \in gw$ □

Lemma 3.6. *we have the following*
(1)$\pi_0(S) = \pi_0(ST) = \pi_0(TS) = \pi_0(T)$
(2)S is isoloid if and only in A is isoloid where $A \in \{S, ST, TS, T\}$

Proof. since by [17] and [7, lemma 2.3] that is $\sigma(S)=\sigma(ST)=\sigma(TS)=\sigma(T)$ and $\sigma_p(S)=\sigma_p(ST)=\sigma_P(TS)=\sigma_p(T)$ that is (2) hold .then for all $\lambda\in C$

$$\alpha(S-\lambda)>0 \iff \alpha(ST-\lambda)>0 \iff \alpha(TS-\lambda)>0 \iff \alpha(T-\lambda)>0$$

that is (1) hold

□

Remark 3.7. Let $(S,T)\in\delta$ and one of the operator S,ST,TS,T be a dominant. If λ_0 is a nonzero isolated point in the spectrum of one of them, then all of the range of $S-\lambda_0$, $TS-\lambda_0$, $ST-\lambda_0$ and $T-\lambda_0$ are closed. Moreover, if λ_0 is a nonzero isolated eigenvalue of the spectrum of one of them with finite multiplicity, then each of the spectral manifold $H_S(\{\lambda_0\}),H_{TS}(\{\lambda_0\}),H_{ST}(\{\lambda_0\})$, and $H_S(\{\lambda\})$ are finite dimensional

Theorem 3.8. *suppose that S or S^* is polynomial root of dominant operator. Then $f(A)\in gW$ for each $f\in H(\sigma(A))$,where $A\in\{ST,TS,T\}$*

Proof. suppose that S is a polynomial root of dominant operator and let $A\in\{ST,TS,T\}$ we must show that A satisfies generalized Weyl's theorem suppose that $\lambda\in\sigma(A)\backslash\sigma_{BW}(A)$ then $A-\lambda$ is B-Weyl but not invertible then by [3, lemma 4.1] that we can act $A-\lambda$ as the direct sum

$$T-\lambda=A_1\oplus A_2 \quad where \quad A_1 \quad is \quad Weyl \quad and \quad A_2 \quad nilpotent$$

since S is polynomial root of dominant operator then by [7, Theorem 2.1] A has $SVEP$ that implies A_1 has $SVEP$ at 0 therefore A_1 is Weyl then A_1 has finite ascent and descent that is $A-\lambda$ has finite ascent and descent . so $\lambda\in\pi_0(A)$

conversely, let $\lambda\in\pi_0(A)$,then then$\lambda\in\pi_0(S)$ by lemma(3.6) but S is polynomial root of dominant operator then $S\in gB$ by [4] then λ is pole of the resolvent of S, then from [7, Theorem 2.11] $A-\lambda$ is Drazin invertible then we can act $A-\lambda$ as the direct sum

$$A-\lambda=A_1\oplus A_2 \quad where \quad A_1 \quad is \quad invertible \quad and \quad A_2 \quad is \quad nilpotent$$

therefore $A-\lambda$ is B-Weyl's , that is $\lambda\in\sigma(A)\backslash\sigma_{BW}(A)$ so $\sigma(A)\backslash\sigma_{BW}(A)=\pi_0(A)$ hence $A\in gW$. We claim that $\sigma_{BW}(f(A))=f(\sigma_{BW}(A))$ for every $F\in H(\sigma(A))$.since $A\in gW$ then $A\in gW$ then by [4, Theorem 2.1] that is $\sigma_{BW}(A)=\sigma_D(A)$. since S is polynomial root of dominant operators ,A has $SVEP$ so $f(A)$ has $SVEP$ by [7, Theorem 3.3.9] for every $f\in H(\sigma(A))$. then $f(A)\in gB$ by [4, Theorem 2.9] then

$$\sigma_{BW}(f(A))=\sigma_D(f(A))=f(\sigma_D(A))=f(\sigma_{BW}(A))$$

since S is polynomial root of dominant operators then by [lemma 3.4] that is S is isoloid hence A is isoloid by [lemma 3.6] so for every $f\in H(\sigma(A))$

$$\sigma(f(A))\backslash\pi_0(f(A))=f(\sigma(A)\backslash\pi_0(A))$$

since $A\in gW$ we have

$$\sigma(g(A))\backslash\pi_0(f(A))=f(\sigma(A)\backslash\pi_0(A))=f(\sigma_{BW}(A))=\sigma_{BW}(f(A))$$

that is $f(A) \in gW$

We suppose that S^* is polynomial root of dominant operator .we must show that $A \in gW$.Let $\lambda \in \sigma(A)\backslash\sigma_{BW}(A)$ hence $\sigma(A*) = \overline{\sigma(A)}$ and $\sigma_{BW}(A^*) = \overline{\sigma_{BW}(A)}$. So $\overline{\lambda} \in \sigma(A^*)\backslash\sigma_{BW}(A^*)$, But $S^*T^*S^* = {S^*}^2$and $T^*S^*T^* = {T^*}^2$ therefore $A^* \in gW$ hence $\overline{\lambda} \in P_0(A^*)$ which is implies that $\overline{\lambda} \in P_0(S^*)$. Since S^* is polynomial root of dominant then $\overline{\lambda}$ is pole of the resolvent of S^* equivalently, λ is pole of the resolvent of A hence $\lambda \in \pi_0(A)$

Conversely, suppose $\lambda \in \pi_0(A)$. Then $\lambda \in \pi_0(S)$. Since $\lambda \in iso\sigma(S^*)$ and S^* is a polynomial root of dominant operators then λ is a pole of the resolvent of S hence $A - \lambda$ is Drazin invertible. Therefore $\lambda \in \sigma(A)\backslash\sigma_{BW}(A)$ thus $\sigma(A)\backslash\sigma_{BW}(A) = \pi_0(A)$ so that $A \in gW$. If S^* is a polynomial root of dominant operators then A is isoloid by lemma(3.6) hence $f(A) \in gW$ □

Corollary 3.9. *Let S is compact dominant operator which is normoloid and suppose that $(S,T) \in \delta$ then we have*

$$TS = I \oplus Q \quad on \quad N(TS - I) \oplus N(TS - I)^{\perp}$$

where Q is quasinilpotent

Proof. suppose that S is compact operator and dominant .then by theorem(3.8) TS satisfies generalized Weyl's theorem. and $iso\sigma(TS) \subseteq \{0,1\}$ by corollary(2.8) then we have

$$\sigma(TS)\backslash\sigma_{BW}(TS) \subseteq \{0,1\}$$

Assume that $\sigma_{BW}(TS)$ is not finite . then $\sigma(TS)$is finite. but S is compact that is $\sigma(TS)$ is countable set $\sigma(TS) = \{0, \lambda_1, \lambda_2, ...\}$,where $\lambda_j \neq 0$ for $j = 1, 2, ..., \lambda_i \neq \lambda_j$ for each$I \neq j$,and$\lambda_i \to 0$ as $\jmath \to \infty$, then from corollary(2.8) $\{\lambda_1, \lambda_2, ...\} \subseteq iso\sigma(TS)\backslash\{0\} \subseteq \{1\}$ but this is a contradiction. Hence $\sigma_{BW}(TS)$ is finite. That is every point is $\sigma_{BW}(TS)$ is isolated therefore $\sigma(TS) \subseteq \{0,1\}$. If $1 \notin \sigma(TS)$,then $\sigma(TS) = \{0\}$ since S is dominant then by lemma(2.4) that $S = 0$ hence $TS = 0$. If $1 \in \sigma(TS)$, then from proposition(2.11)that is
$TS = I \oplus Q$ on $H = N(TS - I) \oplus N(BA - I)^{\perp}$, where Q is quasinilpotent □

Theorem 3.10. *Let S is polynomial root of dominant operator then $f(A)$ satisfies a-Browder's theorem for each $f \in H(\sigma(A))$, where $A \in \{ST, TS, T\}$.*

Proof. First we must show that $\sigma_{ea}(f(A)) = f(\sigma_{ea}(A))$ and $\sigma_w(f(A)) = f(\sigma_w(A))$. Let $f \in H(\sigma(A))$ since the inclusion $\sigma_{ea}(f(A)) \subseteq f(\sigma_{ea}(A))$ hold for each operator, suppose that $\lambda \notin \sigma_{ea}(f(A))$ then $f(A) - \lambda$ is upper semi-Fredholm and $i(f(A) - \lambda) \leq 0$ put

$$f(A) - \lambda = c(A - \mu_1)(T - \mu_2)...(A - \mu_n)g(A)$$

where $c, \mu_1, \mu_2, ..., \mu_n \in C$ and $g(A)$ is invertible, since S is polynomial root of dominant S is has $SVEP$ [12] and [14, Proposition 3.3.9] therefore A has

$SVEP$ [7, Theorem2.1]. Since $A-\mu_i$ is upper semi-Fredholm then $i(A-\mu_i) \leq 0$ for every $i = 1, 2, ..., n$ [13, Proposition2.2], so that $\lambda \neq f(\sigma_{ea}(A))$
Now,suppose that S^* is polynomial root of dominant. since $S^*T^*S^* = S^{*2}$ and $T^*S^*T^* = T^{*2}$ and A^* has $SVEP$ therefore $i(A - \mu_i) \geq 0$ for every $i = 1, 2, ..., n$ by the classical index product theorem, $A - \mu_i$ is Weyl for every $i = 1, 2, ..., n$.hence $\lambda \notin f(\sigma_{ea}(A))$ that is $\sigma_{ea}(f(A)) = f(\sigma_{ea}(A)$ by the same way we prove $\sigma_w(f(A)) = f(\sigma_w(A))$. Since S and S^* is root of dominant operators then A andA^* has $SVEP$ so that a-Browder's holds for$A$ hence

$$f(\sigma_{ab}(A)) = \sigma_{ab}(f(A)) = \sigma_{ea}(f(A)) = f(\sigma_{ea}(A))$$

for each $f \in H(\sigma(A))$ □

References

[1] Il Ju An and Eungil Ko, Paranormal operators and some operator equations, *Filomat*, **29** (2015), no. 6, 1195-1207. https://doi.org/10.2298/fil1506195a

[2] A.E. Taylor and D. Lay, *Introduction to Functional Analysis*, 2nd ed., John Wiley and Sons, New York-Chichester, 1980.

[3] M. Berkani, Index of B-Fredholm operators and generalization of a Weyl theorem, *Proceedings of the American Mathematical Society*, **130** (2002), no. 6, 1717-1723. https://doi.org/10.1090/s0002-9939-01-06291-8

[4] Raúl E. Curto and Young Min Han, Generalized Browder's and Weyl's theorems for Banach space operators, *Journal of Mathematical Analysis and Applications*, **336** (2007), no. 2, 1424-1442. https://doi.org/10.1016/j.jmaa.2007.03.060

[5] H.R. Dowson, R. Harte, Invertibility and Singularity for Bounded Linear Operators (Marcel Dekker Inc., New York and Basel, 1987), *Proceedings of the Edinburgh Mathematical Society, (Series 2)*, **32** (1989), no. 02, 334-335. https://doi.org/10.1017/s0013091500028789

[6] M.P. Drazin, Pseudo-inverses in associative rings and semigroups, *The American Mathematical Monthly*, **65** (1958), no. 7, 506-514. https://doi.org/10.2307/2308576

[7] B.P. Duggal, Operator equations $ABA = A^2$ and $BAB = B^2$, *Funct. Anal. Approx. Comput.*, **3** (2011), no. 1, 9-18.

[8] Robin Harte, Fredholm, Weyl and Browder theory, *Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences*, **85A** (1985), 151-176.

[9] G.H. Heuser, *Functional Analysis*, Jonhn Wiley and Sons Lid, 1982.

[10] Istratusco, *Introduction to Linear Operator Theory*, Marcel Dekker, Inc., New York and Basel, 1981.

[11] C.S. Kubrusly and B.P. Duggal, On posinormal operators, *Advances in Mathematical Sciences and Applications*, **17** (2007), no. 1, 131.

[12] Kjeld Laursen, Operators with finite ascent, *Pacific Journal of Mathematics*, **152** (1992), no. 2, 323-336. https://doi.org/10.2140/pjm.1992.152.323

[13] Mourad Oudghiri, Weyl's and Browder's theorems for operators satisfying the SVEP, *Studia Mathematica*, **163** (2004), 85-101. https://doi.org/10.4064/sm163-1-5

[14] Bebe Prunaru, Kjeld B. Laursen and Michael M. Neumann, *An Introduction to Local Spectral Theory: London Mathematical Society Monographs*, (new series), 2002.

[15] Heydar Radjavi and John A. Nohel, *Invariant Subspaces*, Courier Corporation, 2012.

[16] Steffen Roch and Bernd Silbermann, Continuity of generalized inverses in Banach algebras, *Studia Mathematica*, **136** (1999), no. 3, 197-227.

[17] Christoph Schmoeger, Common spectral properties of linear operators A and B such that $ABA = A^2$ and $BAB = B^2$, *Publications de lInstitut Math. (NS)*, **79** (2006), no. 93, 109-114. https://doi.org/10.2298/pim0693109s

[18] Ivan Vidav, On idempotent operators in a Hilbert space, *Publ. Inst. Math.(Beograd)*, **4** (1964), no. 18, 157-163.

[19] H. Zguitti, A note on generalized Weyl's theorem, *Journal of Mathematical Analysis and Applications*, **316** (2006), no. 1, 373-381. https://doi.org/10.1016/j.jmaa.2005.04.057

Received: July 30, 2017; Published: August 29, 2017