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Models of the Solar System in the 16th and 17th Centuries

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Abstract

In this note we discuss some historical characters of the Copernican solar model and some mathematical characters of one of Kepler's planetary models before he reached the ellipse law.

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1. Influence of the Arabic astronomy on Copernicus' "Revolution"

The modern civilization is based on the science and the technology. The first author of this note studies the technology of the human health care (cf. [1]). The second author studies mathematical sciences. Concrete subjects sometimes arise historical, cultural and philosophical problems. The authors think that the bibliographical study, methodological study of early works of the modern sciences will provide many hints to solve our subjects. In this note, we discuss some characters of two scientists, Nicolas Copernicus (1473-1543) and Johannes Kepler (1571-1630). Copernicus' book [2] provided the foundation of the modern heliocentric theory of the solar system. He recognized the three motions of the Earth, (i) daily spin around its axis, (ii) annual rotation around the Sun, and (iii) axial precession. M. Luther's comment in 1539 for Copernican theory was a typical view in that era (cf. [3], pages 216-217). Copernicus also remark the nutation in obliquity of the Earth. Copernicus found that the cause of the precession of equinoxes discovered by Hipparchus was the (astronomical) precession of Earth's axis. As pointed in [4], page 208, this precession is due to the gravitational torques of the Sun and Moon (for the detail [4], 5.8). However, the two types of precession should be distinguished, a heavy symmetrical top model treated in [4] 5.8. The equation of the precession is given by

$$\frac{du}{dt}^2 = (1-u^2)(\alpha - \beta u) - (b - au)^2,$$

where $u = \cos\theta$, θ is the angle of the inclination, t is the time, the letters a, b, α, β denote some constants (cf. [4], (5.62')). So u is expressed as an elliptic function in t.

The origin of Copernicus's idea attracts our attention. As many authors pointed out (cf. [8, 12]), he was not the first person who proposed the heliocentric theory. But it is clear that his fine systematic theory can be viewed as the first step of the modern natural science. Linton [8] pointed that the Islamic astronomers in the medieval period contributed so much to the development of the planetary theory. In [2] Book III, Chapter 4, Copernicus mentioned the idea same with Tusi's one in his [2] without stating its origin (cf. [3]). According to [9] Copernicus cited 4 Islamic (mainly Arabic) astronomers in [2]. We shall consider its influence to his theory of motions of the earth. We recognize the reason he doubted the geocentric theory by reading [2], Book I, Chapter 10, 'Venus and Mercury are located above the sun by some authorities, like Plato's *Timaeus*, but below by others like Ptolemy..., Al-Bitruji places Venus above the sun, and Mercury below it'. By the same chapter, we know that Copernicus supposes that the sun's diameter is not so large relative to the diameters of Venus and Mercury. 'Ibn Rushd reports having seen something blackish when he found a conjunction of the sun and Mercury indicated in the table' (cf. [2]). 'So says Al-Battani of Raqqa, who thinks that the sun's diameter is ten times larger [than Venus]' (cf. [2]). We recall that even Johannes Kepler could not perform fine observations of seasonal changes of the apparent diameter of the sun. By [8], we recognize that Hipparchus and Ptolemy determined astronomical time parameters and positions of the fixed stars and the planets in the heavenly sphere rather correctly. In [2], Book III, Chapter 13, the founder of the modern astronomy mentioned '... by Thabit ibn Qurra. He found its length [the uniform length of the solar year] to be 365 days, plus

...approximately 6 hours , 9 minutes, 12 seconds'. Copernicus found the cause of the precession of the equinoxes. He mentioned the value of the inclination 'the inclination of the axis , ...by Al-Battani to be $23°35'$; by Al-Zarkali the Spaniard, 190 years after him, $23°34'$; and in the same way 230 years later, by Profatius the Jew, about 2' less' ([2], Book III, Chapter 6). Al-Battani (ca.858-929), Al-Zarkali (11th century) Al-Bitruji of Seville (ca. 1200), Ibn-Rushd of Cordova (1126-1298) are known to their respective latinized names Albategnius, Azarquiel, Alpertragius, Averroes. We also pay our attention to the contribution of Jewish scientists in the mediaeval period.

2. Background of the Arabic astronomy, the cosmology of the Islamic holy scripture

Does any Islamic astronomer reach the heliocentric theory of the solar system before Copernicus? There is no clear evidence for the existence of such an astronomer. We recall Gallileo's conflict with the Christian view of the universe. In "Introduction" of [5], page 61, Kepler mentioned his objection against the opinion that the description of 'Joshua' of the Torah is an evidence of the geocentric theory. How about Islamic astronomers? If some Islamic astronomer believed heliocentric theory, the person would have a similar problem. We may obtain some hints for this question from some astronomical description in the Islamic holy scripture.

[11], S$\overline{u}$rah 89,Al-Fajir (the Dawn), 'by the Even and the Odd'. According to the lunar theory of Hipparchus 1 synodic month is

$$29 + \frac{31}{60} + \frac{50}{60^2} + \frac{8}{60^3} + \frac{20}{60^4}$$

days (cf. [8], page 58). Islamic (Hijri) calender is a typical lunar calender. An Islamic year also consists of 12 months. In Islamic calender, every month consists of 29 days or 30 days. Even Ramadan changes its parity depending on the Islamic year. Its mystic change seems to be recognized as one evidence of Almighty God. [11], S$\overline{u}$rah 9, At-Taubah(Repentance), 'the number of months in All$\overline{a}$h's Book of Knowledge is twelve [in a year],it was Ordained by All$\overline{a}$h on the Day when He created the heavens and the earth'. The three-body problem of the Earth, Sun and the Moon is a long living subject of the human being. The motions of the Sun and the Moon relative to fixed stars are fundamental elements of the astronomy. [11], S$\overline{u}$rah 36, Y$\overline{a}$ S$\overline{i}$n, '[According to the Divine Ordainment] it is not right for the Sun to overtake The Moon'. [11], S$\overline{u}$rah 2, Al-Baquarah 'They ask you concerning the phases of the new Moons'. Say: "They are calenders to show fixed periods of time for [the daily life of] the people and for the Hajj pilgrimage as well"'. We recall Dante's

"La Divina Comedia"(the Divine Comedy) described the typical image of the heaven in the Renaissance era of Europe. In the poem, Heaven is depicted as a series of concentric spheres around the Earth, consisting of the Moon, Mercury, Venus, Sun, Mars, Jupiter, Saturn, the fixed stars, the first moved and finally the place of God. A similar vision would be supposed by Islamic peoples in the medieval period. The 5 planets, the Mercury, Venus, Mars, Jupiter and Saturn are treated with the Sun and Moon in [11],S$\overline{u}$rah 71,N$\overline{u}$h (Noah), 'Do you not see how All$\overline{a}$h has created the Seven Heavens on top of another?' Some Arabic star names are used by Europeans, for instance, Aldebaran, Algol, Altair and Deneb are typical examples (cf. [12], page 175). A specific fixed star is mentioned in [11], S$\overline{u}$rah 53, An-Najm (the Star), ' the One who is the Creator of Sirius Star'. It seems that no specific constellation is mentioned in the Quran. But we remark the phrases [11], S$\overline{u}$rah 6, Al-An'$\overline{a}$m(the Cattle) ' All$\overline{a}$h is the one who created the stars for you that you may find your courses by their guidance in the darkness of the land and the sea', [11], S$\overline{u}$rah 15, Al-Hijr, 'we did set constellations in the sky', [11], S$\overline{u}$rah 25, Al-Forq$\overline{a}$n, 'The Source of Blessings is the One who produced constellations in the sky and placed the lamp of Sun for day-light, and Moon for the light of the night'. What is the fate of a shining star? [11], S$\overline{u}$rah 81, Al-Takw$\overline{i}$r (the Folding up, 'When the shining sun is folded up; and when the stars lose their lights', [11], S$\overline{u}$rah 82, Al-Infit$\overline{a}$r (the Cleaving Asunder), 'when the stars are scattered'. The modern theory of the astronomy would supports the view of these phrases (cf. [10]).

3. Kepler's planetary models –the relations among the anomalies–

A German astronomer Johannes Kepler imagined various models of planetary orbits before he reached the ellipse law. The normalized form of his puffy-cheek path ('via buccosa') is parametrized in the Cartesian coordinates as

$$x = e + \{-e\cos\beta + \sqrt{1 + 2e\cos\beta + e^2\cos(2\beta)}\}\cos\beta,$$

$$y = \{-e\cos\beta + \sqrt{1 + 2e\cos\beta + e^2\cos(2\beta)}\}\sin\beta$$

($0 \le \beta \le 2\pi$, $0 \le e < 1/\sqrt{2}$), where e is the eccentricity of the orbit and β is the eccentric anomaly (cf. [7], [13], [8]). The sun is located at the origin $A = (0, 0)$. Denote by B the center $(e, 0)$ of an eccentric circle $(X-e)^2+Y^2 = 1$. We consider the line $\ell = QB$ passing through the $Q = (e + \cos\beta, \sin\beta)$ on the above circle and B. We choose a point R on ℓ as $(e - e\cos^2\beta, \sin\beta)$ so that R is the orthogonal projection of A on ℓ. Then the length of the line segment QR is $1 + e\cos\beta$. This length is called the diametral distance. The position

$P = (x, y) = (x(\beta), y(\beta))$ on the line segment BQ satisfies $AP = QR = 1 + e\cos\beta$. The puffy-cheek path has an implicit expression $F(x, y, 1; e) = 0$ by the polynomial

$$F(x,y,z;e) = (x^2+y^2)^4 - 4exz(x^2+y^2)^3 - 2z^2(x^2+y^2)^2\{(1-2e^2)x^2+(1-e^2)y^2\}$$
$$+4exz^3(x^2+y^2)\{(2+e^2)x^2+2y^2\}+(1-14e^2-10e^4)x^4z^4+2(1-9e^2-6e^4)x^2y^2z^4$$
$$+(1-4e^2-e^4)y^4z^4-4exz^5\{(1-4e^2-e^4)x^2+(1-3e^2-2e^4)y^2\}+2e^2z^6\{(3-7e^2+2e^4)x^2$$
$$+(1-2e^2-e^4)y^2\} - 4e^3(1-e^2)^2xz^7 + e^4(1-e^2)^2z^8 \qquad (3.1)$$

in the above we corrected the error of the coefficient of the term $(1 - 14e^2 + 10e^4)x^4z^4$ in [7]. Kepler's another planetary model was studied in [6]. We remark that the eccentric anomaly satisfies the relation

$$M = \int_0^\beta (1 + e\cos t)dt = \beta + e\beta,$$

which is the relation of the mean anomaly M and the eccentric anomaly β for the true elliptical orbit. Following the classical style we measure these two parameters so that the parameters β, M attain 0 at the aphelion $(x, y) = (1 + e, 0)$ and $\beta = M = \pi$ at the perihelion $(x, y) = (-1 + e, 0)$. In [14] Whiteside compared the puffy-cheek path and the true elliptical orbit by using the equated (true) anomaly $0 \le \phi \le 2\pi$. The former is expressed as

$$r = \tilde{r}(\phi) = 1 + e\cos\phi - e^2\sin^2\phi - \frac{e^3}{2}\sin^2\phi\cos\phi - \frac{e^4}{2}\sin^4\phi + \dots,$$

the latter is expressed as

$$r = r(\phi) = \frac{1-e^2}{1-e\cos\phi} = 1+e\cos\phi - e^2\sin^2\phi - e^3\sin^2\phi\cos\phi - e^4\sin^2\phi\cos^2\phi + \dots.$$

Hence their difference is expressed as

$$\tilde{r}(\phi) - r(\phi) = \frac{e^3}{2}\sin^2\phi\cos\phi + \dots.$$

Since $e \sim 1/11$ for the Mars, the maximal difference is estimated about 1/50000-times the radius of the orbit of the Mars, it amounts 41 second in the arc. Whiteside concluded that this value was less than the limit of Kepler's astronomical observations. If we use the eccentric anomaly β to compare the these two orbit models, the two models satisfy the same equation

$$r = r(\beta) = 1 + e\cos\beta.$$

However it was hard for Kepler to estimate the difference of the mean anomalies for the common eccentric anomaly β, we shall estimate this difference for the puffy-cheek path and the true elliptic orbit. For the true elliptical orbit, the mean anomaly θ is given by $\theta = \beta + e\sin\beta$. For $0 \le \beta \le \pi$, we denote by $I(\beta)/2$ the area of the region of the line segment the Sun-the Mars swept out for the time $0 \le t \le \beta$. It is given by

$$\begin{aligned} I(\beta) &= \int_0^\beta (x(t)y'(t) - y(t)x'(t))\,dt \\ &= \int_0^\beta (1 - \frac{e(e+\cos t)(1+e\cos t)}{\sqrt{1+2e\cos t + e^2\cos(2t)}})dt. \end{aligned}$$

The mean anomaly $\tilde{\theta}$ for the puffy-cheek is given by

$$\tilde{\theta} = \frac{I(\beta)}{I(\pi)}\pi$$

for $0 < \beta \le \pi$. Under the assumption $e = 1/11$, it is expressed as

$$\begin{aligned} \tilde{\theta} = \beta + \frac{1}{6774625020}(&617797620\sin\beta + 14113830\sin(2\beta) + 211750\sin(3\beta) \\ &-14595\sin(4\beta) + 594\sin(5\beta) - 10\sin(6\beta)) + \dots. \end{aligned}$$

Hence the difference is given by

$$\theta_d(\beta) = \tilde{\theta}(\beta) - \beta - \frac{1}{11}\sin\beta$$

$$= \frac{352480}{1242014587}\sin\beta + \frac{470461}{225820834}\sin(2\beta) + \frac{21175}{677462502}\sin(3\beta) - \frac{973}{451641668}\sin(4\beta) + \dots.$$

This amounts about $+8$ minutes in the arc at $\beta = \pi/4$ and -6 minutes in the arc at $\beta = 3\pi/4$. These differences are over the limit of Kepler's observations.

We shall confirm that the two conditions i) the distance of the Sun and the planet is $1 + e\cos\beta$ for the eccentric anomaly $0 \le \beta \le 2\pi$, and ii) the area of the region swept out by the line segment the Sun and the planet for the time $0 \le t \le \beta$ is proportional to $(\beta + e\sin\beta)/2$ imply the orbit of the planet is given by

$$r = r(\phi) = \frac{1-e^2}{1-e\cos\phi}$$

by using the equated anomaly ϕ provided that $\phi = 0$ for $\beta = 0$, $\phi = \pi$ for $\beta = \pi$. In fact we define an orbit

$$r = r_k(\phi) = \frac{1-e^2}{1-e\cos(\phi/k)}$$

for $k > 0$. The area of the region swept out by the line segment of the Sun and the planet for the time $0 \le t \le \beta$ is $k\sqrt{1-e^2}(\beta + e\sin\beta)/2$. The natural relation between ϕ and β implies $k = 1$. The two conditions tell us that the orbit of the planet is an ellipse.

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