On Pseudo-Union Curves in a Hypersurface of a Weyl Space

Nil Kofoğlu

Beykent University
Faculty of Science and Letters
Department of Mathematics
Ayazağa-Maslak, Istanbul, Turkey

Copyright © 2017 Nil Kofoğlu. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract
In this paper, firstly we have obtained the differential equation of pseudo-union curves and then we have defined pseudo-union curves in \( W_n \). Secondly, we have expressed pseudo-asymptotic curves and pseudo-geodesic curves in \( W_n \). Finally, we have given relation among these curves and by means of this relation, necessary theorems have been expressed.

Mathematics Subject Classification: 53B25

Keywords: Weyl space, the pseudo-union curves

1 Introduction
A manifold with a conformal metric \( g_{ij} \) and a symmetric connection \( \nabla_k \) satisfying the compatibility condition

\[
\nabla_k g_{ij} - 2T_k g_{ij} = 0
\]

is called a Weyl space, which will be denoted by \( W_n(g_{ij}, T_k) \). The vector field \( T_k \) is named the complementary vector field. Under a renormalization of the metric tensor \( g_{ij} \) in the form

\[
\tilde{g}_{ij} = \lambda^2 g_{ij}
\]
the complementary vector field $T_k$ is transformed by the law

$$\widetilde{T}_k = T_k + \partial_k \ln \lambda$$  \hfill (3)

where $\lambda$ is a scalar function [3].

The coefficients $\Gamma^i_{kl}$ of the symmetric connection $\nabla_k$ are given by

$$\Gamma^i_{kl} = \left\{^{i}_{kl}\right\} - g^{im}(g_{mk}T_l + g_{ml}T_k - g_{kl}T_m).$$  \hfill (4)

If under the transformation (2), the quantity $A$ is changed according to the rule

$$\tilde{A} = \lambda^p A$$  \hfill (5)

then $A$ is called a satellite of $g_{ij}$ with weight $\{p\}$.

The prolonged derivative and prolonged covariant derivative of $A$ are, respectively defined by [1, 2]

$$\dot{\partial}_k A = \partial_k A - pT_k A$$  \hfill (6)

and

$$\dot{\nabla}_k A = \nabla_k A - pT_k A.$$  \hfill (7)

Let $W_n(g_{ij}, T_k)$ be $n$-dimensional Weyl space and $W_{n+1}(g_{ab}, T_c)$ be $(n+1)$-dimensional Weyl space ($i, j, k = 1, 2, ..., n; \ a, b, c = 1, 2, ..., (n+1)$). Let $x^a$ and $u^i$ be the coordinates of $W_{n+1}(g_{ab}, T_c)$ and $W_n(g_{ij}, T_k)$, respectively. The metrics of $W_n(g_{ij}, T_k)$ and $W_{n+1}(g_{ab}, T_c)$ are connected by the relations

$$g_{ij} = g_{ab}x^a_i x^b_j$$  \hfill (8)

where $x^a_i$ is the covariant derivative of $x^a$ with respect to $u^i$.

The prolonged covariant derivative with respect to $u^k$ and $x^c$ are $\dot{\nabla}_k$ and $\dot{\nabla}_c$, respectively. These are related by the conditions

$$\dot{\nabla}_k A = x^a_k \dot{\nabla}_c A \quad (k = 1, 2, ..., n; \ c = 1, 2, ..., n+1).$$  \hfill (9)

Let the normal vector field $n^a$ of $W_n(g_{ij}, T_k)$ be normalized by the condition $g_{ab}n^a n^b = 1$. The moving frame $\{x^i_a, n^a\}$ and its reciprocal $\{x^a_i, n^a\}$ are connected by the relations [3]

$$n^a n_a = 1, \ n_a x^a_i = 0, \ n^a x^i_a = 0, \ x^a_i x^j_a = \delta^i_j.$$  \hfill (10)
Since the weight of \( x^a_i \) is \( \{0\} \), the prolonged covariant derivative of \( x^a_i \), relative to \( u^k \), is given by [3]

\[ \dot{\nabla}_k x^a_i = \nabla_k x^a_i = w_{ik} n^a, \tag{11} \]

where \( w_{ik} \) are the coefficients of the second fundamental form of \( W_n(g_{ij}, T_k) \).

On the other hand, it is easy to see that the prolonged covariant derivative of \( n^a \) is given by

\[ \dot{\nabla}_k n^a = -w_{kl} g^{il} x^a_i. \tag{12} \]

By means of (10), the prolonged covariant derivative of \( x^j_a \) is found to be [8]

\[ \dot{\nabla}_k x^j_a = \Omega^j_k n_a. \tag{13} \]

Let \( v^i_r \) \((i, r = 1, 2, ..., n)\) be the contravariant components of the vector field \( v \) in \( W_n(g_{ij}, T_k) \). Suppose that the vector fields \( v^i_r \) \((r = 1, 2, ..., n)\) are normalized by the conditions \( g_{ij} v^i_r v^j_r = 1 \).

The reciprocal vector fields \( \hat{v}^i_r \) are defined by the relations [6]

\[ v^i_r \hat{v}^j = \delta^i_j, \quad v^i_r \hat{v}^j = \delta^j_s \quad (i, j, r, s = 1, 2, ..., n). \tag{14} \]

The prolonged covariant derivatives of the vector field \( v \) and its reciprocal \( \hat{v} \) are, respectively, given by [7]

\[ \dot{\nabla}_k v^i = T^s_{ks} v^i, \quad \dot{\nabla}_k \hat{v}^i = -T^s_{ks} \hat{v}^i. \tag{15} \]

Let \( v^a_r \) and \( v^i_r \) be the contravariant components of the vector field \( v \) relative to \( W_{n+1}(g_{ab}, T_c) \) and \( W_n(g_{ij}, T_k) \), respectively. Denoting the components \( \hat{v} \) relative to \( W_{n+1}(g_{ab}, T_c) \) and \( W_n(g_{ij}, T_k) \) by \( \hat{v}_a \) and \( \hat{v}_i \), we have [8]

\[ v^a_r = x^a_i v^i_r, \quad \hat{v}^a = x^a_i \hat{v}_i. \tag{16} \]

If \( K_{rr} \) is the normal curvature of \( W_n(g_{ij}, T_k) \) in the direction of \( v \), we have

\[ K_{rr} = w_{ij} v^i_r v^j_r. \tag{17} \]

Since the weight of \( w_{ij} \) is \( \{1\} \) and that of \( v^i_r \) is \( \{-1\} \), \( K_{rr} \) is a satellite of \( g_{ij} \) of \( \{-1\} \).
The quantities
\[ \frac{\eta}{p} = T_k v^k \quad (r, p = 1, 2, ..., n) \] (18)
are called the geodesic curvature of the lines of the net \((v_1, v_2, ..., v_n)\) relative to \(W_n(g_{ij}, T_k)\) [7].

The vector fields
\[ \frac{c^i}{p} = \frac{\eta}{p} v^i \quad (i, r, p = 1, 2, ..., n) \] (19)
are called the geodesic vector fields of the net \((v_1, v_2, ..., v_n)\) relative to \(W_n(g_{ij}, T_k)\) [7].

If the components of the geodesic vector fields relative to \(W_{n+1}(g_{ab}, T_c)\) are denoted by \(\bar{c}^a\), then we have [8]
\[ v^i \nabla_r \bar{c}^a = \bar{c}^a = (w_{ik} v^i v^k) n^a + c^i x_i^a. \] (20)

Since the net \((v_1, v_2, ..., v_n)\) is orthogonal, we have [7]
\[ T_k = 0, \quad \frac{p}{r} + \frac{r}{p} = 0 \quad (r \neq p). \] (21)

2 Preliminaries

Let \(C : x^i = x^i(s) \quad (i = 1, 2, ..., n)\) be a curve in \(W_n\), where \(s\) is its arc length and \(v\) is the tangent vector field of \(C\) at the point \(P\).

Let \(\lambda\) be a unit vector field in \(W_{n+1}\) and \(\lambda^a\) be the contravariant components of \(\lambda\). \(\lambda\) is a congruence of unit vector fields. It can be expressed as
\[ \lambda^a = x^a_i w^i + zn^a \quad (a = 1, 2, ..., n + 1) \] (22)
where \(w^i\) are the contravariant components of the vector field \(w\) with respect to \(W_n\) and \(z\) is a scalar. Since \(g_{ab}\lambda^a\lambda^b = 1\) we have
\[ g_{ij} w^i w^j + z^2 = 1 \] (23)

or
\[ z^2 = 1 - g_{ij} w^i w^j. \] (24)
Let $N^a$ be the contravariant components of a unit vector field which satisfies the conditions: it is linearly dependent on $\lambda$ and $v_1$ and it is orthogonal to $v_1$ [4]. Hence

$$g_{ab}N^aN^b = 1 \ (b = 1, 2, ..., n + 1) \quad (25)$$

and

$$g_{ab}N^av_1^b = 0. \quad (26)$$

On the other hand, we know that, the relation between $c_1^a$ and $c_1^i$ is as follows:

$$c_1^a = c_1^i x^i_a + (w_{ij}v_1^iv_1^j)n^a \ (j = 1, 2, ..., n) \quad (27)$$

where $c_1^a$ and $c_1^i$ are the geodesic vector fields with respect to $W_{n+1}$ and $W_n$, respectively; $n^a$ are the contravariant components of a unit vector field normal to $W_n$ and $w_{ij}$ are the coefficients of the second fundamental form of $W_n$.

Since $N^a$ is linearly dependent on $\lambda$ and $v_1$, it can be written as

$$N^a = \alpha \lambda^a + \beta v_1^a \quad (28)$$

where $\alpha$ and $\beta$ are scalars.

Multiplying (28) by $g_{ab}N^b$, we get

$$g_{ab}N^aN^b = 1 = \alpha g_{ab} \lambda^a N^b \quad (29)$$

where $g_{ab}v_1^aN^b = 0$, or

$$\alpha = \frac{1}{g_{ab} \lambda^a N^b}. \quad (30)$$

Multiplying (28) by $g_{ab}v_1^b$, we obtain

$$g_{ab}N^av_1^b = 0 = \alpha g_{ab} \lambda^a v_1^b + \beta \quad (31)$$

where $g_{ab}v_1^av_1^b$, or

$$\beta = - \frac{\alpha g_{ab} \lambda^a v_1^b}{\alpha g_{ab} \lambda^a N^b}. \quad (32)$$

From (28), (30) and (32), we have

$$N^a = \frac{1}{g_{cd} \lambda^c N^d} \lambda^a - \frac{g_{cd} \lambda^c \lambda^d}{g_{cd} \lambda^c N^d} v_1^a \ (c, d = 1, 2, ..., n + 1) \quad (33)$$
or

\[ N^a = \frac{x_i^a w^i + zn^a - g_{cd}(x_j^c w^j + zn^c)v_k^d v^a}{g_{cd}(x_j^c w^j + zn^c)N^d} \]  \hspace{1cm} (34)

or

\[ N^a = \frac{x_i^a w^i + zn^a - g_{jk} w^j v_k^a}{zn^c N^d} \]  \hspace{1cm} (35)

where \( v^d = v_i^k x_k^d \), \( g_{cd} x_j^c x_k^d = g_{jk} \), \( g_{cd} x_j^c N^d = 0 \) and \( g_{cd} x_j^c N^d = 0 \).

Using \( v^a = v^i x_i^a \) and (24), we get

\[ N^a = \pm \frac{x_i^a (w^i - g_{jk} w^j v_k^a) + zn^a}{\sqrt{1 - g_{ij} w^i w^j g_{cd} N^d}} \]  \hspace{1cm} (36)

or

\[ N^a = \pm \frac{x_i^a (w^i - g_{jk} w^j v_k^a) + zn^a}{\sqrt{1 - \delta_j^p g_{ip} w^i w^j g_{cd} N^d}} \]  \hspace{1cm} (37)

where \( g_{ij} = \delta_j^p g_{ip} \) or

\[ N^a = \pm \frac{x_i^a (w^i - g_{jk} w^j v_k^a) + zn^a}{\sqrt{1 - \delta_j^p g_{jm} g_{ip} w^i w^j g_{cd} N^d}} \]  \hspace{1cm} (38)

where \( \delta_j^p = g_{pm} g_{jm} \), or

\[ N^a = \pm \frac{x_i^a (w^i - g_{jk} w^j v_k^a) + zn^a}{\sqrt{1 - \sum_{r=1}^n v^p v^m g_{jm} g_{ip} w^i w^j g_{cd} N^d}} \]  \hspace{1cm} (39)

where \( g_{pm} = \sum_{r=1}^n v^p v^m \), or

\[ N^a = \pm \frac{x_i^a (w^i - g_{jk} w^j v_k^a) + zn^a}{\sqrt{1 - v^p v^m g_{jm} g_{ip} w^i w^j}} \]  \hspace{1cm} (40)

The plus sign in (40) is to be taken when \( z > 0 \), the minus sign when \( z < 0 \).

Thus (40) will reduce to \( N^a = n^a \), when \( \lambda \) is linearly dependent on \( v_i^1 \) and \( n^a \); that is \( w^i = k v_i^1 \), \( k \) being any constant different from unity.
Using (40), we have

\[ n^{a} = \frac{N^{a}}{|z|} \sqrt{1 - v^{p}v^{m}g_{jm}g_{ip}w^{i}w^{j}} + x^{a}_{i} \left( g_{jk} \frac{w^{j}}{|z|} - \frac{w^{s}}{|z|} \right) \]  

(41)

or

\[ n^{a} = \frac{N^{a}}{|z|} \sqrt{1 - v^{p}v^{m}g_{jm}g_{ip}w^{i}w^{j}} + x^{a}_{i} \left( g_{jk} \rho^{j}v^{k}v^{i} - \rho^{i} \right) \]  

(42)

where \( \rho^{j} = \frac{w^{j}}{|z|} \).

If we write (42) in (27), we get

\[ c_{1}^{a} = x^{a}_{i} + \frac{K^{a}}{11} \left[ \frac{N^{a}}{|z|} \sqrt{1 - v^{p}v^{m}g_{jm}g_{ip}w^{i}w^{j}} + x^{a}_{i} \left( g_{jk} \rho^{j}v^{k}v^{i} - \rho^{i} \right) \right] \]  

(43)

or

\[ c_{1}^{a} = x^{a}_{i} \left[ c^{i} + \frac{K^{a}}{11} g_{jk} \rho^{j}v^{k}v^{i} - \frac{K^{a}}{11} \rho^{i} \right] + \frac{N^{a}}{|z|} \sqrt{1 - v^{p}v^{m}g_{jm}g_{ip}w^{i}w^{j}} \]  

(44)

where \( K^{a} = w_{ij}v^{i}v^{j} \) is the normal curvature of \( C \).

From (44), we have

\[ c_{1}^{a} = x^{a}_{i}p^{i} + \frac{N^{a}}{11} \]  

(45)

where

\[ p^{i} = c^{i} + \frac{K^{a}}{11} g_{jk} \rho^{j}v^{k}v^{i} - \frac{K^{a}}{11} \rho^{i} \]  

(46)

and

\[ \frac{N^{a}}{11} = \frac{1}{|z|} \sqrt{1 - v^{p}v^{m}g_{jm}g_{ip}w^{i}w^{j}}. \]  

(47)

From (45):

**Definition 2.1** For convenience, \( c_{1}^{a} \) in (45) is called relative first curvature vector field of \( C \) at \( P \) in \( W_{n+1} \) and \( p^{i} \) is called relative first curvature vector field of \( C \) at \( P \) in \( W_{n} \). \( K^{2} = g_{ij}p^{i}p^{j} \) is called relative first curvature of \( C \) at \( P \) in \( W_{n} \).
3 Pseudo Union Curves

Definition 3.1 The totally pseudo-geodesic surface is defined by $v^a$ and $\bar{v}^a$.

Let $\mu$ be a unit vector field in the direction of the curve of congruence of curves, one curve of which passes through each point of $W_n$. $\mu^a$, in general, are not normal to $W_n$ and it can be specified by

$$\mu^a = t^i x_i^a + r N^a$$

(48)

where $t^i$ and $r$ are parameters [5].

Then we have

$$g_{ab} \mu^a \mu^b = 1$$

(49)

and

$$g_{ab} x_i^a N^b = 0.$$  

(50)

With the help of (48), (49) and (50), we get

$$1 = g_{ab} \mu^a \mu^b = g_{ab} (t^i x_i^a + r N^a) (t^j x_j^b + r N^b) = g_{ij} t^i t^j + r^2$$

or

$$g_{ij} t^i t^j = 1 - r^2$$

(51)

where $g_{ab} x_i^a x_j^b = g_{ij}$ and $g_{ab} N^a N^b = 1$.

If the pseudo-geodesic in $W_{n+1}$ in the direction of the curve of the congruence with covariant components $\mu^a$ is to be a pseudo-geodesic of the totally pseudo-geodesic surface, then it is necessary that $\mu^a$ be a linear combination of $v^a$ and $\bar{v}^a$, therefore

$$\mu^a = \alpha v^a + \beta \bar{v}^a$$

(52)

where $\alpha$ and $\beta$ scalars.

From (45), (48) and (52), we get

$$t^i x_i^a + r N^a = \alpha x_i^a v^i + \beta (x_i^a \bar{p}^i + \bar{K}^a N^a)$$

(53)

where $v^a = x_i^a v^i$.

Multiplying (53) by $g_{ab} x_j^b$ and summing for $a$ and $b$, we obtain

$$g_{ij} t^i = \alpha g_{ij} v^i + \beta g_{ij} \bar{p}^i$$

(54)
Pseudo-union curves

where \( g_{ab}x^a_x^b = g_{ij} \) and \( g_{ab}N^a_x^b = 0 \).

Multiplying (54) by \( v^j_i \), we get

\[
g_{ij}t^i_i v^j = \alpha + \beta g_{ij}P^i v^j_i \tag{55}\]

where \( g_{ij}v^i_i v^j = 1 \).

From (46), we obtain

\[
g_{ij}P^i v^j_i = g_{ij}c^i v^j_i + K g_{kk}g^i v^k - K g_{ij}P^i v^j_i \]

\[
= g_{ij}p^i P^i v^j_i \tag{p = 2, 3, ..., n} \]

\[
g_{ij}P^i v^j_i = 0 \tag{56}\]

where \( g_{ij}v^i_i v^j = 1 \) and \( g_{ij}P^i v^j_i = 0 \) \( (p = 2, 3, ..., n) \).

Using (56) in (55), we have

\[
\alpha = g_{ij}t^i_i v^j_i \tag{57}\]

Multiplying (53) by \( g_{ab}N^b \) and summing for \( a \) and \( b \), we get

\[
r = \beta \frac{K}{11} \text{ or } \beta = \frac{r}{\frac{K}{11}} \tag{58}\]

where \( g_{ab}x^a_x^b = 0 \) and \( g_{ab}N^a_x^b = 1 \).

Writing (57) and (58) in (54), we obtain

\[
g_{ij}t^i = \left( g_{kh}t^k v^h \right) g_{ij}v^i_i + \frac{r}{\frac{K}{11}} g_{ij}p^i \tag{h = 1, 2, ..., n} \tag{59}\]

Multiplying (59) by \( g^{jm} \) and summing for \( j \), we obtain

\[
\delta^{m}_{i} t^i = \left( g_{kh}t^k v^h \right) \delta^{m}_{i} v^i_i + \frac{r}{\frac{K}{11}} \delta^{m}_{i} P^i \tag{60}\]

where \( g^{jm} g_{ij} = \delta^{m}_{i} \).

From (60), we have

\[
\frac{t^m}{r} = \left( g_{kh}t^k v^h \right) \frac{v^m}{r} + \frac{1}{\frac{K}{11}} P^m \tag{61}\]
or

$$\ell^m = \left(g_{kh} \ell^k v^h_1\right)v^m_1 + \frac{1}{K_{11}} \bar{p}^m$$  \hspace{1cm} (62)

where \(\ell^m = \ell^m_1\), or

$$\bar{p}^m + K_{11} g_{kh} \ell^k v^h_1 v^m_1 - K_{11} \ell^m = 0 \quad (m = 1, 2, ..., n)$$

$$\bar{p}^m + K_{11} \left(g_{kh} \ell^k v^h_1 v^m_1 - \ell^m\right) = 0.$$ \hspace{1cm} (63)

Equation (63) is the differential equation of the pseudo-union curves. The solutions of the \(n\) equations (63) determine the pseudo-union curves in \(W_n\) to that congruence.

Let us denote the left hand side of (63) by \(\bar{\eta}^m\):

$$\bar{\eta}^m = \bar{p}^m - K_{11} \left(\ell^m - g_{kh} \ell^k v^h_1 v^m_1\right) = \bar{p}^m - K_{11} v^m = 0.$$ \hspace{1cm} (64)

where \(v^m = \ell^m_1 - g_{kh} \ell^k v^h_1 v^m_1\).

\(\bar{\eta}^m\) are called the contravariant components of the pseudo-union curvature vector field.

**Definition 3.2** Pseudo-union curve is defined as a curve whose pseudo-union curvature vector field is a null vector field: \(\bar{\eta}^m = 0\).

If \(\phi = \angle(\mu^a, N^b)\), then \(\cos\phi = g_{ab} \mu^a N^b\) where \(g_{ab} \mu^a \mu^b = 1\) and \(g_{ab} N^a N^b = 1\). Since \(g_{ab} \mu^a N^b = r, \cos\phi = r\) is obtained. Since \(g_{ij} v^i v^j = 1 - r^2 = \sin^2\phi\), we have \(g_{ij} \ell^i \ell^j = g_{ij} v^i v^j = \sin^2\phi / \cos^2\phi = \tan^2\phi\). If \(\alpha = \angle(v^i_1, \ell^i)\), then \(\cos\alpha = \frac{g_{ij} \ell^i v^j}{\sqrt{g_{ij} v^i v^j}}\) or \(\cos\alpha \sin\phi = g_{ij} v^i_1 \ell^j_1\) where \(g_{ij} v^i_1 v^j_1 = 1\). From \(\cos\alpha \sin\phi = g_{ij} v^i_1 \ell^j_1\), we have \(\cos\alpha \cos\phi \tan\phi = g_{ij} v^i_1 \ell^j_1\).

The magnitude \(K_u\) of the vector field \(\bar{\eta}^k\) is

\[K_u^2 = g_{kh} \bar{\eta}^k \bar{\eta}^h\]

\[= g_{kh} \left(\bar{p}^k - K_{11} \ell^k + K_{11} g_{im} \ell^i v^m v^k_1\right) \left(\bar{p}^h - K_{11} \ell^h + K_{11} g_{im} \ell^i v^m v^h_1\right)\]

\[= g_{kh} \bar{p}^k \bar{p}^h - 2K_{11} g_{kh} \bar{p}^k \ell^h + K_{11}^2 g_{kh} \ell^k \ell^h - K_{11}^2 \left(g_{im} \ell^i v^m_1\right)^2\]

\[= K_g^2 - 2K_{11} g_{kh} \bar{p}^k \ell^h + K_{11}^2 \tan^2\phi - K_{11}^2 \cos^2\phi \tan^2\phi\]

\[= K_g^2 - 2K_{11} g_{kh} \bar{p}^k \ell^h + K_{11}^2 \sin^2\phi \tan^2\phi\]

where \(g_{kh} \bar{p}^k \ell^h = 0, g_{kh} \ell^k \ell^h = 1\) and \(K_g^2 = g_{kh} \bar{p}^k \ell^h\).
Multiplying (54) by \( p^j \), we have
\[
g_{ij} t^i p^j = \beta K_g^2
\] (66)
where \( g_{ij} v_i p_j = 0 \) and \( K_g^2 = g_{ij} p^i p^j \), or
\[
\beta = \frac{g_{ij} t^i p^j}{K_g^2}
\] (67)

Writing (57) and (67) in (54), we get
\[
g_{ij} t^i = (g_{kh} t^k v_1^h) g_{ij} v_1 + g_{kh} t^k p^h K_g^2 - g_{ij} p^j.
\] (68)

Multiplying (68) by \( t^j \) and summing for \( i \) and \( j \), we obtain
\[
g_{ij} t^i t^j = \left( g_{kh} t^k v_1^h \right)^2 + \left( g_{kh} t^k p^h \right) K_g^2 - g_{ij} p^i p^j.
\] (69)

\[
\sin^2 \phi = \cos^2 \alpha \sin^2 \phi + \left( g_{kh} t^k p^h \right) K_g^2
\]
\[
\sin \phi \sin \alpha K_g = g_{kh} t^k p^h.
\]

From (69), we have
\[
\tan \phi \sin \alpha K_g = g_{kh} t^k p^h.
\] (70)

Using (70) in (65), we get
\[
K_u^2 = K_g^2 - 2 K_{11} \tan \phi \sin \alpha K_g + K_{11}^2 \sin^2 \alpha \tan^2 \phi
\]
\[
= \left( K_g - K_{11} \sin \alpha \tan \phi \right)^2
\]
\[
K_u = K_g - K_{11} \sin \alpha \tan \phi.
\] (71)

**Definition 3.3**

- If the curve \( C \) is a pseudo-union curve then \( K_u = 0 \).
- If the curve \( C \) is a pseudo-asymptote curve then \( K_{11} = 0 \).
- If the curve \( C \) is a pseudo-geodesic curve then \( K_g = 0 \).
From (71) and Definition 3.3:

**Theorem 3.4** If the curve $C$ has any two of the following properties it also has the third:

- it is a pseudo-union curve,
- it is a pseudo-asymptote curve,
- it is a pseudo-geodesic curve

provided that $v^m$ are not the components of a null vector field.

If $\phi = 0$ or $\alpha = 0$ or $K_{11} = 0$, we obtain $K_u = K_g$.

Hence:

**Theorem 3.5** The necessary and sufficient condition for a pseudo-union curve to be pseudo-geodesic is one of the following

- it is a pseudo-asymptotic curve,
- the congruence consist of the normals,
- the direction of the tangent vector field to $C$ coincides with that of the vector field $\ell^k$.

**References**


Received: July 28, 2017; Published: August 16, 2017