Logarithm of the Exponents in the Prime Factorization of the Factorial

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Abstract

In this note we study the sum $\sum_{2 \leq p \leq n} \log E(p)$ where $E(p)$ is the exponent of the prime $p$ in the prime factorization of $n!$, the sum $\sum_{2 \leq p \leq n} H_{E(p)}$, where $H_k$ denotes the $k$-th harmonic number and another sums. We also consider the generalized harmonic numbers $H_{k,m}$ of order $m$ and obtain a strong connection between these sums and the Riemann zeta function.

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1 Introduction

Let us consider the prime factorization of a positive integer $a$

$$a = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k},$$

where $p_1, p_2, \ldots, p_k$ are the different primes in the prime factorization and $s_1, s_2, \ldots, s_k$ are the exponents. The total number of prime factors in the prime factorization is denoted $\Omega(a)$ (see [1, chapter XXII]), that is,

$$\Omega(a) = s_1 + s_2 + \cdots + s_k.$$
Let us consider the factorial $n!$, the exponent of a prime $p$ in its prime factorization will be denoted $E(p)$. Then we can write the prime factorization of $n!$ in the form

$$n! = \prod_{2 \leq p \leq n} p^{E(p)}$$

since, clearly, the primes that appear in the prime factorization of $n!$ are the primes not exceeding $n$. It is well-known the following asymptotic formula (see [1, chapter XXII])

$$\Omega(n!) = \sum_{2 \leq p \leq n} E(p) = n \log \log n + An + o(n), \quad (1)$$

where $A$ is a constant. Note that $\Omega(n!) = \sum_{2 \leq a \leq n} \Omega(a)$.

In this note we obtain an asymptotic formula for the sequence (compare with (1))

$$L(n!) = \sum_{2 \leq p \leq n} \log E(p).$$

Let us consider the $k$-th harmonic number $H_k$, namely $H_k = \sum_{i=1}^{k} \frac{1}{i}$.

We obtain an asymptotic formula for the sequence (compare with (1))

$$M(n!) = \sum_{2 \leq p \leq n} H_{E(p)}.$$

We also consider the $k$-th generalized harmonic number of order $m \geq 2$, $H_{k,m}$, namely $H_{k,m} = \sum_{i=1}^{k} \frac{1}{i^m}$ and we obtain an asymptotic formula for the sequence (compare with (1))

$$M_m(n!) = \sum_{2 \leq p \leq n} H_{E(p),m}.$$  

where the Riemann zeta function appear.

Finally, we consider the sum

$$\sum_{2 \leq p \leq n} \frac{1}{(E(p) - 1)!}$$

where the $e$ number appear.

2 Main Results

Theorem 2.1 We have the asymptotic formula

$$L(n!) = \sum_{2 \leq p \leq n} \log E(p) = C \frac{n}{\log n} + o \left( \frac{n}{\log n} \right) = C \pi(n) + o(\pi(n)), \quad (2)$$
where the constant $C$ is

$$C = \sum_{j=2}^{\infty} \frac{\log j}{j(j+1)}.$$  \hspace{1cm} (3)

Proof. Note that, if we put

$$A_j = \log j \left( \frac{1}{j} - \frac{1}{j+1} \right) = \frac{\log j}{j(j+1)},$$  \hspace{1cm} (4)

the series of positive terms $A_j$ converges, that is

$$\sum_{j=1}^{\infty} A_j = C.$$  \hspace{1cm} (5)

We have (prime number theorem)

$$\pi(x) = \sum_{2 \leq p \leq x} 1 = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right).$$  \hspace{1cm} (6)

Another more precise well-known formula is

$$\pi(x) = \sum_{2 \leq p \leq x} 1 = \frac{x}{\log x} + f_1(x) \left( \frac{x}{\log^2 x} \right),$$  \hspace{1cm} (7)

where $|f_1(x)| < M$.

We also need the following well-known formula

$$\vartheta(x) = \sum_{2 \leq p \leq x} \log p = x + f_2(x) \frac{x}{\log x},$$  \hspace{1cm} (8)

where $|f_2(x)| < M$.

On the other hand, the exponent $E(p)$ of the prime $p$ in the prime factorization of $n!$ is (Legendre’s theorem)

$$E(p) = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor,$$  \hspace{1cm} (9)

where $\lfloor . \rfloor$ denotes the integer part function. If $p$ satisfies the inequality (where $j$ is a fixed positive integer)

$$\frac{n}{j+1} < p \leq \frac{n}{j},$$  \hspace{1cm} (10)

and the inequality

$$p > \sqrt{n},$$  \hspace{1cm} (11)
then we have
\[
E(p) = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor = \left\lfloor \frac{n}{p} \right\rfloor = j.
\] (12)

Note that if \(n\) is sufficiently large inequalities (10) and (11) are fulfilled since \(\sqrt{n} < \frac{n}{j+1}\).

Let \(\epsilon > 0\), we choose the fixed positive integer \(s\) such that the following inequalities hold
\[
0 \leq \sum_{j=s}^{\infty} A_j \leq \epsilon, \quad (13)
\]
\[
0 \leq \frac{\log s}{s} \leq \epsilon, \quad (14)
\]
\[
\frac{M}{s} \leq \epsilon. \quad (15)
\]

Now, we have (see (4), (5), (6), (10) and (12))
\[
L(n!) = \sum_{2 \leq p \leq n} \log E(p) = \sum_{2 \leq p \leq \frac{n}{2}} \log E(p) + \sum_{\frac{n}{j+1} < p \leq \frac{n}{j}} \log j = \sum_{2 \leq p \leq \frac{n}{2}} \log E(p) + \log n \sum_{j=1}^{s-1} \log j + o \left( \frac{n}{\log n} \right)
\]
\[
= \sum_{2 \leq p \leq \frac{n}{s}} \log E(p) + \frac{n}{\log n} \sum_{j=1}^{s-1} A_j + O \left( \frac{n}{\log n} \right)
\]
\[
= \sum_{2 \leq p \leq \frac{n}{s}} \log E(p) + o(1) \frac{n}{\log n}.
\] (16)

Besides, we have (see (7), (8) and (12))
\[
\sum_{2 \leq p \leq \frac{n}{s}} \log E(p) = \sum_{2 \leq p \leq \frac{n}{s}} \log \left( \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor \right) \leq \sum_{2 \leq p \leq \frac{n}{s}} \log \left( \sum_{k=1}^{\infty} \frac{n}{p^k} \right) = \log n \pi \left( \frac{n}{s} \right)
\]
\[
- \sum_{2 \leq p \leq \frac{n}{2}} \log (p-1) = \log n \pi \left( \frac{n}{s} \right) - \sum_{2 \leq p \leq \frac{n}{2}} \log p + \sum_{2 \leq p \leq \frac{n}{2}} \log \left( \frac{p}{p-1} \right)
\]
\[
\leq \log n \pi \left( \frac{n}{s} \right) - \vartheta \left( \frac{n}{s} \right) + \pi \left( \frac{n}{s} \right), \quad (17)
\]

since \(1 < \frac{p}{p-1} < 2 < e\).

Let us consider the function
\[
f_3(n) = \frac{\log n}{\log \frac{n}{s}}. \quad (18)
\]
Logarithm of the exponents of the primes in the factorial

Note that \( \lim_{n \to \infty} f_3(n) = 1 \).
Therefore (see (8) and (18))

\[
\vartheta \left( \frac{n}{s} \right) = \frac{n}{s} + f_2 \left( \frac{n}{s} \right) \frac{1}{s \log n} f_3(n). \quad (19)
\]

On the other hand, we have (see (7) and (18))

\[
\pi \left( \frac{n}{s} \right) = \frac{n}{s \log \left( \frac{n}{s} \right)} + f_1 \left( \frac{n}{s} \right) \frac{n}{s \log \left( \frac{n}{s} \right)^2} = \frac{n}{s \log n} \left( 1 - \frac{1}{\log s} \log \frac{s}{n} \right)
\]

\[
+ f_1 \left( \frac{n}{s} \right) \frac{1}{s \log^2 n} (f_3(n))^2 = \frac{n}{s \log n} \left( 1 + \frac{\log s}{\log n} f_4(n) \right)
\]

\[
+ f_1 \left( \frac{n}{s} \right) \frac{1}{s \log^2 n} (f_3(n))^2 = \frac{n}{s \log n} + \frac{\log s}{s} \log n \ f_4(n)
\]

\[
+ f_1 \left( \frac{n}{s} \right) \frac{1}{s \log^2 n} (f_3(n))^2 = \frac{n}{s \log n} f_5(n). \quad (20)
\]

where \( \lim_{n \to \infty} f_4(n) = 1, \lim_{n \to \infty} f_5(n) = 1 \) and we have used the formula \( \frac{1}{1-x} = 1 + x(1+o(1)) \ (x \to 0) \).

Substituting (19) and (20) into (17) we obtain (see (14) and (15))

\[
\sum_{2 \leq p \leq \frac{n}{s}} \log E(p) \leq \log n \pi \left( \frac{n}{s} \right) - \vartheta \left( \frac{n}{s} \right) + \pi \left( \frac{n}{s} \right) = f_4(n) \frac{\log s}{s} \frac{n}{\log n}
\]

\[
+ f_1 \left( \frac{n}{s} \right) (f_3(n))^2 \frac{1}{s \log n} - f_2 \left( \frac{n}{s} \right) f_3(n) \frac{1}{s \log n} f_5(n) \frac{1}{s \log n}
\]

\[
\leq \left| f_4(n) \right| \frac{\log s}{s} \frac{n}{\log n} + \left| f_1 \left( \frac{n}{s} \right) \right| (|f_3(n)|)^2 \frac{1}{s \log n} + \left| f_2 \left( \frac{n}{s} \right) \right| |f_3(n)| \frac{1}{s \log n}
\]

\[
+ \left| f_5(n) \right| \frac{1}{s \log n} \leq \left( 2 \frac{\log s}{s} + 4 \frac{M}{s} + 2 \frac{M}{s} + 2 \frac{1}{s} \right) \frac{n}{\log n} \leq 10 \epsilon \frac{n}{\log n} \quad (21)
\]

Note that there exists \( n_0 \) such that if \( n \geq n_0 \) then \( |f_i(n)| \leq 2 \) \((i = 3, 4, 5)\), since \( f_i(n) \to 1 \) \((i = 3, 4, 5)\).

Finally, equations (16), (13) and (21) give

\[
\left| \frac{L(n!)}{n \log n} - C \right| \leq \sum_{2 \leq p \leq \frac{n}{s}} \frac{\log E(p)}{\log n} + \sum_{j=s}^{\infty} A_j + |o(1)| \leq 12 \epsilon \quad (n \geq n_0) \quad (22)
\]

Note that there exists \( n_0 \) such that if \( n \geq n_0 \) the \( o(1) \) (see equation (22)) satisfies \( |o(1)| \leq \epsilon \).

Therefore (2) is proved, since \( \epsilon \) can be arbitrarily small. The theorem is proved.
Theorem 2.2  We have the asymptotic formula
\[ M(n!) = \sum_{2 \leq p \leq n} H_{E(p)} = \frac{\pi^2}{6} \frac{n}{\log n} + o\left(\frac{n}{\log n}\right) = \frac{\pi^2}{6} \pi(n) + o(\pi(n)) \]

Proof. The proof is the same as the proof of Theorem 2.1. Note that
\[ \sum_{j=1}^{\infty} \frac{H_j}{j(j+1)} = \sum_{j=1}^{\infty} \frac{1}{j(j+1)} + \frac{1}{2} \sum_{j=2}^{\infty} \frac{1}{j(j+1)} + \frac{1}{3} \sum_{j=3}^{\infty} \frac{1}{j(j+1)} + \cdots \]
\[ = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \zeta(2) = \frac{\pi^2}{6} \]

since
\[ \sum_{j=s}^{\infty} \frac{1}{j(j+1)} = \sum_{j=s}^{\infty} \left(\frac{1}{j} - \frac{1}{j+1}\right) = \frac{1}{s} \]

Besides we have the inequality
\[ \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} < \int_{1}^{k} \frac{1}{x} \, dx = \log k \quad (k \geq 2) \]

and consequently the inequality
\[ H_k \leq 1 + \log k \quad (k \geq 1) \]

Therefore (see (17))
\[ \sum_{2 \leq p \leq n} H_{E(p)} \leq \sum_{2 \leq p \leq n} (1 + \log E(p)) \leq \log n \pi \left(\frac{n}{\pi}\right) - \vartheta \left(\frac{n}{\pi}\right) + 2\pi \left(\frac{n}{\pi}\right) \]

The theorem is proved.

We have the following generalization of Theorem 2.2 where the generalized harmonic number replaces the harmonic number.

Theorem 2.3  We have the asymptotic formula
\[ M_m(n!) = \sum_{2 \leq p \leq n} H_{E(p),m} = \zeta(m+1) \frac{n}{\log n} + o\left(\frac{n}{\log n}\right) = \zeta(m+1) \pi(n) + o(\pi(n)) \]

Proof. The proof is the same as the proof of Theorem 2.1. Note that
\[ \sum_{j=1}^{\infty} \frac{H_{j,m}}{j(j+1)} = \sum_{j=1}^{\infty} \frac{1}{j(j+1)} + \frac{1}{2^m} \sum_{j=2}^{\infty} \frac{1}{j(j+1)} + \frac{1}{3^m} \sum_{j=3}^{\infty} \frac{1}{j(j+1)} + \cdots \]
\[ = 1 + \frac{1}{2^{m+1}} + \frac{1}{3^{m+1}} + \cdots = \zeta(m+1) \]
Besides
\[
\sum_{2 \leq p \leq \frac{n}{s}} H_{E(p),m} \leq \zeta(m + 1) \sum_{2 \leq p \leq \frac{n}{s}} 1 = \zeta(m + 1)\pi \left(\frac{n}{s}\right)
\]

The theorem is proved.

**Theorem 2.4** We have the asymptotic formula

\[
\sum_{2 \leq p \leq n} \frac{1}{(E(p) - 1)!} = (e - 2) \frac{n}{\log n} + o \left(\frac{n}{\log n}\right) = (e - 2)\pi(n) + o(\pi(n))
\]

Proof. The proof is the same as the proof of Theorem 2.1. The theorem is proved.

In the following theorem we prove that the contribution of the primes \(\frac{n}{s} < p \leq n\) to \(\Omega(n!)\) is negligible (see equation (1)). Besides the \(s\)-th harmonic number \(H_s\) appear.

**Theorem 2.5** Let \(s \geq 2\) an arbitrary but fixed positive integer. We have the asymptotic formula

\[
\sum_{\frac{n}{s} < p \leq n} E(p) = (-1 + H_s) \frac{n}{\log n} + o \left(\frac{n}{\log n}\right)
\]

Proof. We have

\[
\sum_{\frac{n}{s} < p \leq n} E(p) = \sum_{j=1}^{s-1} \sum_{\frac{n}{j+1} < p \leq \frac{n}{j}} j = \left(\sum_{j=1}^{s-1} \frac{j}{j(j + 1)}\right) \frac{n}{\log n} + o \left(\frac{n}{\log n}\right)
\]

\[
= (-1 + H_s) \frac{n}{\log n} + o \left(\frac{n}{\log n}\right)
\]

The theorem is proved.

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References


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