Permanents and Determinants of Tridiagonal Matrices with \((s,t)\)-Pell Numbers

Hasan Huseyin Gulec

Eregli Faculty of Education
Necmettin Erbakan University, Konya, Turkey

Abstract

In this study, we define a \(n \times n\) tridiagonal matrix which have elements of \((s,t)\)-Pell numbers and then investigate the determinantal properties.

Keywords: determinant, permanent, tridiagonal matrix.

1 Introduction

The Fibonacci, Lucas, Pell and Pell-Lucas sequences have been discussed in so many articles and books (see [2-5]). For \(n > 1\), the well-known Fibonacci \(\{F_n\}\), Lucas \(\{L_n\}\), Pell \(\{P_n\}\) and Pell-Lucas \(\{Q_n\}\) sequences are defined as \(F_n = F_{n-1} + F_{n-2}\), \(L_n = L_{n-1} + L_{n-2}\), \(P_n = 2P_{n-1} + P_{n-2}\) and \(Q_n = 2Q_{n-1} + Q_{n-2}\), where \(F_0 = 0, F_1 = 1, L_0 = 2, L_1 = 1, P_0 = 0, P_1 = 1\) and \(Q_0 = 2, Q_1 = 2\).

Further details about the Pell and Pell-Lucas numbers can be seen in [1].

In [6], Kılıç gave the definition of generalized Pell \((p,i)\)-numbers and then presented their generating matrix. He obtained relationships between the generalized Pell \((p,i)\)-numbers and their sums and permanents of certain matrices. Also, he derived the generalized Binet formulas, sums, combinatorial representations. In [7,8], the authors defined a new matrix generalization of the Fibonacci and Lucas numbers, and using essentially a matrix approach they showed properties of these matrix sequences. In [9], the authors gave the following definition.
For any real numbers \((s, t)\) and \(n \geq 2\), let \(s^2 + t > 0\), \(s > 0\) and \(t \neq 0\). The \((s, t)\)-Pell sequence \(\{p_n(s, t)\}_{n \in \mathbb{N}}\) and \((s, t)\)-Pell Lucas sequence \(\{q_n(s, t)\}_{n \in \mathbb{N}}\) are defined respectively by

\[
p_n(s, t) = 2sp_{n-1}(s, t) + tp_{n-2}(s, t),
\]

\[
q_n(s, t) = 2sq_{n-1}(s, t) + tq_{n-2}(s, t)
\]

with initial conditions \(p_0(s, t) = 0\), \(p_1(s, t) = 1\) and \(q_0(s, t) = 2\), \(q_1(s, t) = 2s\).

Then considering these sequences, they defined the matrix sequences which have elements of \((s, t)\)-Pell and \((s, t)\)-Pell Lucas sequences and investigated their properties.

The permanent of an \(n\)-square matrix is defined by

\[
\text{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{\sigma(i)}
\]

where the summation extends over all permutations \(\sigma\) of the symmetric group \(S_n\) [15].

Let \(A = [a_{ij}]\) be an \(m \times n\) matrix with row vectors \(r_1, r_2, ..., r_m\). We say \(A\) contractible on column \(k\), if column \(k\) contains exactly two nonzero elements. Suppose that \(A\) is contractible on column \(k\) with \(a_{ik} \neq 0 \neq a_{jk}\) and \(i \neq j\). Then the \((m - 1) \times (n - 1)\) matrix \(A_{i,j:k}\) obtained from \(A\) replacing row \(i\) with \(a_{jk}r_i + a_{ik}r_j\) and deleting row \(j\) and column \(k\) is called the contraction of \(A\) on column \(k\) relative to rows \(i\) and \(j\). If \(A\) is contractible on row \(k\) with \(a_{ki} \neq 0 \neq a_{kj}\) and \(i \neq j\), then the matrix \(A_{k:i,j} = [A_{i,j:k}]^T\) is called the contraction of \(A\) on row \(k\) relative to columns \(i\) and \(j\). Let we consider the following result (see [16]): Let \(A\) be a nonnegative integral matrix of order \(n > 1\) and let \(B\) be a contraction of \(A\). Then

\[
\text{per} A = \text{per} B.
\]

There are many connections between permanents or determinants of tridiagonal matrices and the Fibonacci and Lucas numbers. For example, Minc [10] defined a \(n \times n\) super diagonal \((0, 1)\)–matrix \(F(n, k)\) for \(n > k \geq 2\) and show that the permanent of \(F(n, k)\) equals to the generalized order-\(k\) Fibonacci numbers. Also he gave some relations involving the permanents of some \((0, 1)\)–Circulant matrices and the usual Fibonacci numbers.

In [11], the authors presented a nice result involving the permanent of an \((-1, 0, 1)\)–matrix and the Fibonacci number \(F_{n+1}\). The authors then explored similar directions involving the positive subscripted Fibonacci and Lucas Numbers as well as their uncommon negatively subscripted counterparts. Finally the authors explore the generalized order-\(k\) Lucas numbers, (see [12] and [13] for more detail the generalized Fibonacci and Lucas numbers), and their permanents.
In [14], Lehmer proved a very general result on permanents of tridiagonal matrices whose main diagonal and super-diagonal elements are ones and whose subdiagonal entries were somewhat arbitrary.

2 Main Results

In this section, we define a tridiagonal matrix and then show that the permanent and determinant of this matrix equal to the \((s,t)\)-Pell number.

**Definition 1** We define a \(n \times n\) tridiagonal \((1,2s,t)\)-matrix \(G_n(s,t) = [g_{ij}]\) with \(g_{ii} = 2s\), \(g_{i+1,i} = 1\), \(g_{i,i+1} = t\) for \(1 \leq i \leq n\) and 0 otherwise. That is,

\[
G_n(s,t) = \begin{bmatrix}
2s & t & 0 \\
1 & 2s & t \\
& 1 & 2s & \ddots \\
& & \ddots & \ddots & \ddots \\
0 & & & 1 & 2s \\
\end{bmatrix}. \tag{4}
\]

Then we give following Theorem.

**Theorem 2** Let the matrix \(G_n(s,t)\) be as in (4). Then for \(n \geq 1\),

\[
\text{per}G_n(s,t) = \text{per}G_{n-2}^{n}(s,t) = p_{n+1}(s,t)
\]

where \(p_n\) is the \(n\)th \((s,t)\)-Pell number.

**Proof.** If \(n = 1\), then \(\text{per}G_1 = \text{per}[2s] = 2s = p_2\).

If \(n = 2\), then

\[
G_2 = \begin{bmatrix}
2s & t \\
1 & 2s \\
\end{bmatrix}
\]

and hence \(\text{per}G_2 = 4s^2 + t = p_3\).

Let \(G_r^n\) be \(r\)th contraction of \(G_n\), \(1 \leq r \leq n - 2\). From the definition of \(G_n\), the matrix \(G_n\) can be contracted on column 1 so that

\[
G_1^n = \begin{bmatrix}
4s^2 + t & 2st & 0 \\
1 & 2s & t \\
& 1 & 2s & \ddots \\
& & \ddots & \ddots & \ddots \\
0 & & & 1 & 2s \\
\end{bmatrix}.
\]
Since the matrix $G_1^n$ can be contracted on column 1 and $p_3 = 4s^2 + t$, $tp_2 = 2st$.

$$G_2^n = \begin{bmatrix}
8s^3 + 4st & 4s^2t + t^2 & 0 \\
1 & 2s & t \\
& 1 & 2s & \ddots \\
& & \ddots & \ddots & t \\
& & & 1 & 2s \\
0 & & & & 1
\end{bmatrix} = \begin{bmatrix}
p_1 & tp_3 & 0 \\
p_1 & 2s & t \\
& 1 & 2s & \ddots \\
& & \ddots & \ddots & t \\
& & & 1 & 2s \\
0 & & & & 1
\end{bmatrix}.$$

Furthermore, the matrix $G_2^n$ can be contracted on column 1 so that

$$G_3^n = \begin{bmatrix}
p_5 & tp_4 & 0 \\
p_5 & 1 & 2s & t \\
& 1 & 2s & \ddots \\
& & \ddots & \ddots & t \\
& & & 1 & 2s \\
0 & & & & 1
\end{bmatrix}.$$

Continuing this process, we obtain

$$G_r^n = \begin{bmatrix}
p_{r+2} & tp_{r+1} & 0 \\
p_{r+2} & 1 & 2s & t \\
& 1 & 2s & \ddots \\
& & \ddots & \ddots & t \\
& & & 1 & 2s \\
0 & & & & 1
\end{bmatrix}$$

for $3 \leq r \leq n - 4$. Hence,

$$G_{n-3}^n = \begin{bmatrix}
p_{n-1} & tp_{n-2} & 0 \\
p_{n-1} & 1 & 2s & t \\
& 1 & 2s & \ddots \\
& & \ddots & \ddots & t \\
& & & 1 & 2s \\
0 & & & & 1
\end{bmatrix}.$$

which, by contraction of $G_{n-3}^n$ on column 1, gives

$$G_{n-2}^n = \begin{bmatrix}
2sp_{n-1} + tp_{n-2} & tp_{n-1} \\
1 & 2s
\end{bmatrix} = \begin{bmatrix}
p_n & tp_{n-1} \\
p_n & 1 & 2s
\end{bmatrix}.$$

By the Eq. (3) and the definition of the $(s,t)$-Pell numbers, we obtain

$$perG_n = perG_{n-2}^n = 2sp_n + tp_{n-1} = p_{n+1}.$$ 

So the proof is complete.  

**Lemma 3** [11] Let $C_1(n)$ be $n \times n$ tridiagonal matrix. That is,

$$C_1(n) = \begin{bmatrix}
c_{1,1} & c_{1,2} & c_{1,3} & \cdots & c_{1,n-1} & c_{1,n} \\
c_{2,1} & c_{2,2} & c_{2,3} & \cdots & c_{2,n-1} & c_{2,n} \\
c_{3,2} & c_{3,3} & \cdots & \cdots & c_{n-1,n} \\
& & \cdots & \cdots & c_{n-1,n} \\
c_{n,n-1} & c_{n,n}
\end{bmatrix}.$$
where the sign of the main diagonal of the matrix \( C_1(n) \) is positive. Then the successive permanents of \( C_1(n) \) are given by the recursive formula
\[
\begin{align*}
\text{per}C_1(1) &= c_{1,1}, \\
\text{per}C_1(2) &= c_{1,1}c_{2,2} + c_{1,2}c_{2,1}, \\
\text{per}C_1(n) &= c_{n,n}\text{per}C_1(n-1) + c_{n-1,n}c_{n,n-1}\text{per}C_1(n-2).
\end{align*}
\]

Now, we can give a different way to prove Theorem 2 by taking advantage of Lemma 3.
\[
\begin{align*}
\text{per}G_1 &= 2s = p_2, \\
\text{per}G_2 &= 2s2s + t = 2sp_2 + tp_1 = p_3, \\
\text{per}G_3 &= 2sp_3 + tp_2 = p_4, \\
&\quad \vdots \\
\text{per}G_n &= 2sp_n + tp_{n-1} = p_{n+1}.
\end{align*}
\]
Hence the result.

[17] A matrix \( A \) is called convertible if there is an \( n \times n \) \((1, -1)\)-matrix \( K \) such that \( \text{per}A = \det(A \circ K) \), where \( A \circ K \) denotes the Hadamard product of \( A \) and \( K \). Such a matrix \( K \) is called a converter of \( A \).

Let \( S \) be a \((1, -1)\)-matrix of order \( n \), defined by
\[
S = \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 \\
-1 & 1 & \cdots & 1 & 1 \\
1 & -1 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & -1 & 1
\end{bmatrix}
\]
Now we denote the matrix \( G_n \circ S \). Thus
\[
G_n \circ S = \begin{bmatrix}
2s & t & 0 \\
-1 & 2s & t \\
-1 & 2s & \ddots \\
\vdots & \ddots & \ddots & t \\
0 & \cdots & \cdots & -1 & 2s
\end{bmatrix}
\]
Then we have
\[
\det(G_n \circ S) = \text{per}G_n = p_{n+1}.
\]

**Acknowledgement**

A small summary of the study was presented at the 4th International Conference on Matrix Analysis and Applications (ICMAA2013) held in Konya in 2013.
References


Tridiagonal matrices with \((s,t)\)-Pell numbers


Received: July 7, 2017; Published: August 5, 2017