Heronian Mean Labeling of Graphs

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Abstract

A function $f$ is called a Heronian Mean Labeling of a graph $G = (V,E)$ with $p$ vertices and $q$ edges if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $1,2,\ldots,q+1$ in such a way that when each edge $e = uv$ is labeled with

$$f(e = uv) = \left[\frac{f(u) + \sqrt{f(u)f(v) + f(v)}}{3}\right] \quad \text{(OR)} \quad \left[\frac{f(u) + \sqrt{f(u)f(v) + f(v)}}{3}\right]$$

then the edge labels are distinct. In this case, $f$ is a Heronian mean labeling of $G$ and $G$ is called a Heronian Mean Graph. In this paper, we prove that Path, Cycle, Comb, Dragon, Triangular Snake, Quadrilateral Snake, Star $K_{1,n}$ ($n \leq 5$), Complete Graph $K_n$ ($n \leq 4$) are Heronian Mean Graphs.

Mathematics Subject Classification: 05C78

Keywords: Graph, Heronian Mean Graph, Path, Cycle, Comb, Triangular Snake, Quadrilateral Snake, Star $K_{1,n}$, Complete Graph $K_n$
1. Introduction

By a graph we mean a finite undirected graph without loops or parallel edges. For all detailed survey of graph labeling, we refer to J.A. Gallian [1]. For all other standard terminology and notations we follow Harary [2]. The concept of Mean labeling has been introduced by S. Somasundaram and R. Ponraj [3] in 2004. S.Somasundaram and S.S. Sandhya introduced Harmonic mean labeling [4] in 2012. Motivated by the above works we introduce a new type of labeling called Heronian Mean Labeling.

In this paper we investigate the Heronian Mean Labeling of Path, Cycle, Comb, Dragon, Ladder, Triangular Snake, Quadrilateral Snake, Star $K_{1,n}$, Complete Graph $K_n$. We will provide a brief summary of definitions and other information which are necessary for our present investigation.

A Path $P_n$ is a walk in which all the vertices are distinct. A Cycle $C_n$ is a Closed Path. The graph obtained by joining a single pendant edge to each vertex of a Path is called a Comb. A Triangular Snake $T_n$ is obtained from a path $u_1,u_2,\ldots,u_n$ by joining $u_i$ and $u_{i+1}$ to a new vertex $v_i$ for $1 \leq i \leq n - 1$. That is every edge of a path is replaced by a triangle $C_3$. A Quadrilateral Snake $Q_n$ is obtained from a path $u_1,u_2,\ldots,u_n$ by joining $u_i$ and $u_{i+1}$ to two new vertices $v_i$ and $w_i$ respectively and then joining $v_i$ and $w_i$. That is every edge of a path is replaced by a cycle $C_4$. The square $G^2$ of a graph $G$ has $V(G^2) = V(G)$ with $u, v$ adjacent in $G^2$ whenever $d(u, v) \leq 2$ in $G$. A Complete Bipartite graph $K_{m,n}$ is a bipartite graph with bipartition $(V_1,V_2)$ such that every vertex of $V_1$ is joined to all the vertices of $V_2$, Where $|V_1| = m$ and $|V_2| = n$. A Star graph is the complete bipartite graph $K_{1,n}$. A graph $G$ is said to be Complete, if every pair of its distinct vertices are adjacent. A Complete Graph on $n$ vertices is denoted by $K_n$.

Definition 1.1:

A graph $G=(V,E)$ with $p$ vertices and $q$ edges is said to be a Heronian Mean graph if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $1,2,\ldots,q+1$ in such a way that when each edge $e = uv$ is labeled with,

$$f(e = uv) = \begin{cases} \frac{f(u) + \sqrt{f(u)f(v) + f(v)}}{3} & \text{(OR)} \end{cases} \frac{f(u) + \sqrt{f(u)f(v) + f(v)}}{3}$$

then the edge labels are distinct. In this case $f$ is called a Heronian Mean labeling of $G$.

Remark: 1.2

If $G$ is a Heronian mean graph, then ‘1’ must be a label of one of the vertices of $G$. Since an edge should get label ‘1’.
Remark: 1.3
If u gets label ‘1’, then any edge incident with u must get label 1 (or) 2 (or) 3. Hence this vertex must have a degree ≤ 3.

2. Main Results

Theorem: 2.1
Any Path $P_n$ is a Heronian mean graph.

Proof:
Let $P_n$ be a path $u_1u_2u_3\ldots u_n$.
Define a function $f: V(P_n) \rightarrow \{1,2,3,\ldots,q+1\}$ by $f(u_i) = i, 1 \leq i \leq n$.
Then the edge labels are $f(u_iu_{i+1}) = i, 1 \leq i \leq n - 1$.
Hence $P_n$ is a Heronian mean graph.

Example 2.2: A Heronian mean labeling of $P_8$ is shown below.

Figure: 1

Theorem: 2.3
For $n \geq 3$, Any Cycle $C_n$ is a Heronian mean graph.

Proof:
Let $C_n$ be a cycle of length $n$. Let the cycle be $u_1u_2u_3\ldots u_qu_1$.
Take $n = \begin{cases} 2m, & \text{if } n \text{ is even.} \\ 2m + 1, & \text{if } n \text{ is odd.} \end{cases}$
Define a function $f: V(C_n) \rightarrow \{1,2,3,\ldots,q+1\}$ by $f(u_i) = 2i - 1, 1 \leq i \leq m + 1$.

$$f(u_{m+j}) = \begin{cases} n - 2j + 3, & 2 \leq j \leq m, \text{ if } n \text{ is even.} \\ n - 2j + 4, & 2 \leq j \leq m + 1, \text{ if } n \text{ is odd.} \end{cases}$$

The set of labels of edges of $C_n$ are $\{1,2,3,\ldots,n\}$.
Obviously, $f$ is a Heronian mean labeling. Hence $C_n$ is a Heronian mean graph.

Example 2.4: A Heronian mean labeling of $C_6$ is shown below.
Theorem: 2.5
Any Comb $P_n \Theta K_1$ is a Heronian mean graph.

Proof:
Let $P_n \Theta K_1$ be a comb obtained from a path $P_n = u_1u_2 \ldots u_n$ by joining a vertex $u_i$ to $v_i$ ($1 \leq i \leq n$).
Define a function, $f: V(P_n \Theta K_1) \rightarrow \{1, 2, 3, \ldots, q + 1\}$ by $f(u_i) = 2i$, $1 \leq i \leq n$
$$f(v_i) = 2i - 1, \quad 1 \leq i \leq n$$
Edges are labeled by, $f(u_iu_{i+1}) = 2i$, $1 \leq i \leq n - 1$
$$f(u_iv_i) = 2i - 1, \quad 1 \leq i \leq n$$
Clearly, $f$ is a Heronian mean labeling. Hence $P_n \Theta K_1$ is a Heronian mean graph.

Example 2.6: A Heronian mean labeling of $P_5 \Theta K_1$ is shown below.

Theorem: 2.7
Any Triangular Snake $T_n$ is a Heronian mean graph.

Proof:
Let $T_n$ be a Triangular Snake. Let $u_i, v_i$ be the vertices of a Triangular Snake. Join $u_i, v_i$ and $u_{i+1}, v_i$, $1 \leq i \leq n - 1$.
Define a function, $f: V(T_n) \rightarrow \{1, 2, 3, \ldots, q + 1\}$ by $f(u_i) = 3i - 2$, $1 \leq i \leq n$,
$$f(v_i) = 3i - 1, \quad 1 \leq i \leq n - 1.$$
Heronian mean labeling of graphs

Edges are labeled by, \( f(u_iu_{i+1}) = 3i - 1, 1 \leq i \leq n - 1, \)
\[
\begin{align*}
f(u_iv_i) &= 3i - 2, 1 \leq i \leq n - 1, \\
f(u_{i+1}v_i) &= 3i, 1 \leq i \leq n - 1.
\end{align*}
\]
Clearly, \( f \) is a Heronian mean labeling. Hence \( T_n \) is a Heronian mean graph.

**Example 2.8:** A Heronian mean labeling of \( T_5 \) is shown below.

![Figure 4]

**Theorem: 2.9**

Any Quadrilateral Snake \( Q_n \) is a Heronian mean graph.

**Proof:**

Let \( Q_n \) be a quadrilateral snake.
Define a function, \( f: V(Q_n) \rightarrow \{1, 2, 3, \ldots, q + 1\} \) by \( f(u_i) = 4i - 3, 1 \leq i \leq n, \)
\[
\begin{align*}
f(v_i) &= 4i - 2, 1 \leq i \leq n - 1, \\
f(w_i) &= 4i, 1 \leq i \leq n - 1.
\end{align*}
\]
Edges are labeled by, \( f(u_iu_{i+1}) = 4i - 2, \forall 1 \leq i \leq n - 1, \)
\[
\begin{align*}
f(u_iv_i) &= 4i - 3, \forall 1 \leq i \leq n - 1, \\
f(u_{i+1}w_i) &= 4i, \forall 1 \leq i \leq n - 1, \\
f(v_iw_i) &= 4i - 1, \forall 1 \leq i \leq n - 1.
\end{align*}
\]
Clearly \( f \) is a Heronian mean labeling. Hence \( Q_n \) is a Heronian mean graph.

**Example 2.10:** A Heronian mean labeling of \( Q_5 \) is shown below.

![Figure 5]
**Theorem: 2.11**  
Star $K_{1,n}$ is a Heronian mean graph if and only if $n \leq 5$.

**Proof:**  
Star $K_{1,1}$ is same as $P_2$ and Star $K_{1,2}$ is same as $P_3$. Clearly $K_{1,1}$ and $K_{1,2}$ are Heronian Mean graphs. Let the central vertex of the Star be $u$, and the other vertices be $v_1, v_2, v_3, \ldots, v_n$ respectively.

**Case (i):**  
When $2 \leq i \leq 5$, assign the label 3 to the vertex $u$ and $i$ ($1 \leq i \leq 2$) to the vertex $v_i$. Then label $3+i$ to $v_{i+1}$, $1 \leq i \leq 3$. Clearly the above labeling pattern is a Heronian Mean Labeling, Which is shown in the figure: 6.

![Figure 6](image1)

**Case (ii):**  
Assume $n > 5$ and suppose $K_{1,n}$ has Heronian Mean labeling.
Here we consider the following subcases.

**Subcase (ii)(a):**  
Let the label of the central vertex be $u$ be 2.

![Figure 7](image2)

The other vertices $v_1, v_3, v_4, v_5, v_6, v_7, \ldots$ are labeled as $1, 3, 4, 5, 6, 7, \ldots$ respectively. Here the edge labels of $uv_4$ is 3 itself and the edge labels of $uv_5$ and $uv_6$ are from 3 and 4. This is not possible.

**Subcase (ii)(b):**  
Let the label of the central vertex be $u$ be 4.
The other vertices $v_1, v_2, v_5, v_6, v_7, \ldots$ are labeled as $1, 2, 3, 5, 6, 7, \ldots$ respectively. In this case there will be no edge with label 1. Here the edge labels of $uv_3$ is 4 and the edge label of $uv_5$ and $uv_6$ is 4 and 5. This is not possible. From all these, we conclude that $K_{1,n}, n > 5$ is not a Heronian Mean Graph.

Now we investigate the Heronian mean labeling of complete graphs.

**Theorem 2.12**

The graph $P_n^{(2)}$ is a Heronian mean graph.

**Proof:**

Let $P_n$ be the path $u_1, u_2, \ldots, u_n$. Clearly $P_n^{(2)}$ has $n$ vertices and $2n - 3$ edges.

Define a function, $f : V(P_n^{(2)}) \rightarrow \{1, 2, \ldots, q + 1\}$ by $f(u_i) = 2i - 1, 1 \leq i \leq n - 1$.

$$f(u_n) = 2n - 2.$$ 

Edges are labeled by, $f(u_i u_{i+1}) = 2i - 1, 1 \leq i \leq n - 1$

$$f(u_i u_{i+1}) = 2i, 1 \leq i \leq n - 2$$

$$f(u_{n-1} u_n) = 2n - 3.$$ 

Hence $f$ is a Heronian mean labeling of $P_n^{(2)}$.

**Example 2.13:** A Heronian mean labeling of $P_7^{(2)}$ is given below.
Theorem: 2.14
If \( n > 4 \), \( K_n \) is not a Heronian mean graph.

Proof:
Clearly \( K_1 \) is a Heronian mean graph. By Theorem 2.1 and Theorem 2.2, Clearly \( K_2 \) and \( K_3 \) are Heronian Mean graphs. Also \( K_4 \) is also a Heronian mean graph. The labeling pattern of \( K_2, K_3 \) and \( K_4 \) are given below.

If \( n > 4 \), We have repetition of edge labels, which is not possible. Hence to get the edge label 1, we need a vertex \( u \) with label 1. There are four more vertices \( u_1, u_2, u_3, u_4 \) incident with \( u \). This is not possible by Remark 1.2. Hence \( K_n, n > 4 \) is not a Heronian Mean graph.

Remark: 2.15
If \( G \) is a \( k \)-regular graph (with \( k > 4 \)), then \( G \) is not Heronian.

3. Conclusion

The Study of labeled graph is important due to its diversified applications. All graphs are not Heronian mean graphs. It is very interesting to investigate graphs which admit Heronian Mean Labeling. In this paper, we proved that Path, Cycle, Comb, Triangular Snake, Quadrilateral Snake, Star, Square of a Path and Complete Graph are Heronian Mean Graphs. The derived results are demonstrated by means of sufficient illustrations which provide better understanding. It is possible to investigate similar results for several other graphs.

Acknowledgements. The authors are thankful to the referee for their valuable comments and suggestions.

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**Received: August 17, 2016; Published: July 17, 2017**