Semi-Boolean Corner Rings

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Abstract

We show that if $R$ is a ring with an arbitrary idempotent $e$ such that both $eRe$ and $(1-e)R(1-e)$ are semi-boolean rings, then $R/J(R)$ is a nil-clean ring. In particular, under certain additional circumstances, $R$ is also nil-clean. These results somewhat improve on achievements due to Diesl in J. Algebra (2013), Koşan-Wang-Zhou in J. Pure Appl. Algebra (2016) and Danchev in Bull. Iran. Math. Soc. (2017).

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1. Introduction and Background

Throughout the present article all rings $R$ under consideration shall be assumed to be associative with identity element 1, which is different from the zero element 0. Standardly, $Id(R)$ stands for the set of all idempotents of $R$ and $Nil(R)$ for the set of all nilpotents of $R$. As usual, $U(R)$ denotes the group of all units in $R$ and $J(R)$ denotes the Jacobson radical of $R$. Note that $1 + J(R) \subseteq U(R)$ is always fulfilled. We also use $E_{ij}$ to denote the $n \times n$ matrix with $(i, j)$-entry 1 and the other entries 0. All unexplained traditional notions and notations may be found in [12]. For instance, a ring $R$ is called boolean, provided $R = Id(R)$.

About the specific terminology, recall that the prime radical $P(R)$ of a ring $R$ is defined as the intersection of all prime ideals in $R$ (note that it coincides with the lower nil-radical $Nil(R)$). A ring $R$ is called 2-primal if $P(R) = Nil(R)$. Notice that every commutative ring as well as every reduced ring (i.e., a ring
with no non-zero nilpotents) has to be 2-primal. We recall also that a ring
$R$ has a *bounded index of nilpotence* if there is $n \in \mathbb{N}$ such that $a^n = 0$ for
every $a \in \text{Nil}(R)$. Moreover, the upper nil-radical $\text{Nil}^*(R)$ of $R$ is defined as
the sum of all two-sided nil-ideals in $R$ and thus it is the largest nil-ideal of
$R$. In conclusion, it follows that the inclusions $\text{Nil}^*(R) = P(R) \subseteq \text{Nil}^*(R) \subseteq
\text{Nil}(R) \cap J(R)$ are true.

On the other vein, we shall say that a ring $R$ is *J-primal* if $P(R) = J(R)$.
Obvious examples of J-primal rings are those rings in which each prime ideal
is maximal – e.g., commutative regular rings.

Moreover, a ring $R$ is known to be *J-reduced* if $\text{Nil}(R) \subseteq J(R)$. It thus
follows that all 2-primal rings are J-reduced. About the converse, all J-primal
J-reduced rings are obviously 2-primal. Likewise, if $R$ is 2-primal and $J(R)$ is
nil, then $R$ is J-primal.

The following two fundamental concepts were defined in [14].

**Definition 1.1.** A ring $R$ is called *exchange* if, for each $x \in R$, there exists
$e \in \text{Id}(R)$ such that $e \in xR$ and $1 - e \in (1 - x)R$.

**Definition 1.2.** A ring $R$ is called *clean* if, for each $x \in R$, there exist $u \in
U(R)$ and $e \in \text{Id}(R)$ such that $x = u + e$. If, in addition, the commutativity
condition $ue = eu$ is satisfied, the clean ring $R$ is said to be *strongly clean*.

It is clear that abelian (in particular, commutative) clean rings are always
strongly clean.

In [9] was introduced the following major concept.

**Definition 1.3.** A ring $R$ is called *nil-clean* if, for each $r \in R$, there are
$q \in \text{Nil}(R)$ and $e \in \text{Id}(R)$ with $r = q + e$. If, in addition, the commutativity
condition $qe = eq$ is satisfied, the nil-clean ring $R$ is said to be *strongly nil-
clean*.

It is obvious that abelian (in particular, commutative) nil-clean rings are always strongly nil-clean. But it was independently established in [7] and [11]
by exploiting different ideas that a ring $R$ is strongly nil-clean if, and only if,$J(R)$ is nil and $R/J(R)$ is boolean.

The next important notion was stated in [15].

**Definition 1.4.** A ring $R$ is called *semi-boolean* if, for each $r \in R$, there are
$j \in J(R)$ and $e \in \text{Id}(R)$ with $r = j + e$. If, in addition, the commutativity
condition $je = ej$ is satisfied, the semi-boolean ring is said to be *strongly
semi-boolean*.

It was proved in [15] that a ring $R$ is semi-boolean if, and only if, $R/J(R)$
is boolean and all idempotents in $R$ lift modulo $J(R)$.

So, the following containment holds:

boolean $\Rightarrow$ strongly nil-clean $\Rightarrow$ strongly clean $\Rightarrow$ semi-boolean $\Rightarrow$ clean $\Rightarrow$
exchange.
Two significant lines of research in noncommutative ring theory are to find to what extent the ring-theoretic properties of $R$ are preserved by its corner ring $eRe$, where $e \in \text{Id}(R)$, or by its full $n \times n$ matrix ring $M_n(R)$, where $n \in \mathbb{N}$, and visa versa. The most important principal known results in these two subjects are these: It was established in [14] that, for any idempotent $e$ of $R$, the ring $R$ is exchange if, and only if, both $eRe$ and $(1 - e)R(1 - e)$ are exchange rings. Also, it was proved in [10] that if $eRe$ and $(1 - e)R(1 - e)$ are both clean rings, then $R$ is a clean ring. However, it was constructed in [16] a clean ring $R$ of characteristic 2 for which $eRe$ is not a clean ring. Nevertheless, it was obtained in [3] that if $R$ is a strongly clean ring, then $eRe$ is again a strongly clean ring. Moreover, it was shown in [9, Corollary 3.26] that if $R$ is a strongly nil-clean ring, then $eRe$ is a strongly nil-clean ring. Likewise, this was extended in [7] to the so-called UU rings which are rings whose units are the sum of 1 and a nilpotent. So, a rather natural question which immediately arises is what we can say about the ring structure of $R$, provided that both $eRe$ and $(1 - e)R(1 - e)$ are semi-boolean rings. We will somewhat settle this in the sequel, thus extending some of the results in [6].

On the other hand, it was proved in [14] and [10] that if $R$ is an exchange ring, respectively a clean ring, then the same is $M_n(R)$. Besides, it was obtained in [1, Corollary 7] that if $R$ is a commutative nil-clean ring, then the ring $M_n(R)$ is nil-clean. This was extended in [11, Theorem 6.1] to 2-primal strongly nil-clean rings and in [11, Corollary 6.8] to strongly nil-clean rings of bounded index of nilpotence. Some more substantial generalizations were established in [6], too.

The objective of this paper is to continue the study on these two directions. Our further work is organized in the next two sections as follows:

2. Main Results

The following technical claim is our non-trivial key instrument (cf. [6], as well).

**Lemma 2.1.** Suppose that $R$ is a ring with $e \in \text{Id}(R)$ for which both $eRe$ and $(1 - e)R(1 - e)$ are boolean rings. Then $R$ is nil-clean.

**Proof.** For any $r \in R$, the equality $r = ere + (1 - e)r(1 - e) + (1 - e)re + er(1 - e)$ holds. Note that $ere \in eRe$ and $(1 - e)r(1 - e) \in (1 - e)R(1 - e)$ are both orthogonal idempotents, while $(1 - e)re$ and $er(1 - e)$ are nilpotents of order 2, because $[(1 - e)re]^2 = (1 - e)re.(1 - e)re = 0 = er(1 - e).er(1 - e) = [er(1 - e)]^2$. On the other side, putting $t = (1 - e)re + er(1 - e)$ and $f = (1 - e)r(1 - e) + er(1 - e)re$, one sees that $t^2 = f$. But by assumption both $(1 - e)r(1 - e) \in (1 - e)R(1 - e)$ and $er(1 - e)re \in eRe$ are idempotents, so that $f$ is again an idempotent being the sum of two orthogonal idempotents. Therefore, $t^2 = f^2$, i.e., $t^2 - f^2 = 0$. Besides, one checks that $tf = (1 - e)r(1 - e)re + er(1 - e)r(1 - e) = ft$ and so $(t - f)(t + f) = 0$. Since $2f = 0$ as $f$ is an element of the sum of two boolean rings, the last equality
is equivalent to $(t - f)^2 = 0$, that is, $t \in f + \text{Nil}(R)$. Next, observing that $r = ere + (1 - e)r(1 - e) + t$, one may also write that
\[ r = [ere + er(1 - e)re] + [(1 - e)r(1 - e) + (1 - e)rer(1 - e)] + q, \]
where $q \in \text{Nil}(R)$. Since $e_1 = ere + er(1 - e)re = e(r + r(1 - e)r)e \in eRe$ and $e_2 = (1 - e)r(1 - e) + (1 - e)rer(1 - e) = (1 - e)(r + rer)(1 - e) \in (1 - e)R(1 - e)$ are idempotents with zero products $e_1.e_2 = e_2.e_1 = 0$, one can infer that $e_1 + e_2 = e'$ is again an idempotent. Consequently, since we may represent $r$ like $r = e' + q$ with $e' \in \text{Id}(R)$ and $q \in \text{Nil}(R)$, we finally obtain by definition that $R$ is nil-clean, as asserted.

Remark 1. It is worthwhile noticing that it cannot be expected that such a ring $R$ will be strongly nil-clean. In fact, it was demonstrated in [9] that every unit in a strongly nil-clean ring must be a unipotent, that is, the sum of 1 and a nilpotent element. However, in the matrix ring $M_2(\mathbb{Z}_2)$ over the boolean ring $\mathbb{Z}_2$, the matrix unit $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ cannot be a unipotent because the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is never a nilpotent. Even more, the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is a unit having the inverse $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$.

Now we can derive our first basic assertion.

Theorem 2.2. Suppose that $R$ is a ring with $e \in \text{Id}(R)$ for which both $eRe$ and $(1 - e)R(1 - e)$ are semi-boolean rings. Then $R/J(R)$ is a nil-clean ring.

Proof. Owing to the mentioned above result from [15], accomplishing it with [12], for any $h \in \text{Id}(R)$ we deduce that the factor-ring $hRh/J(hRh) = hRh/hJ(R)h \cong h'(R/J(R))h'$ is boolean for some idempotent $h' = h + J(R)$ of $R/J(R)$. Hence Lemma 2.1 successfully applies to get that $R/J(R)$ is nil-clean, as stated. □

The next formula is our basic omnibus (cf. [4] and [6], too).

Lemma 2.3. For each ring $R$ and each idempotent $e$, the following equality is fulfilled:
\[ P(eRe) = eP(R)e. \]

Proof. First, observe that if $P$ is any prime ideal of $R$ then either $ePe = eRe$, or $ePe$ is a prime ideal of $eRe$. Therefore, $eP(R)e$ is an intersection of some of the prime ideals of $eRe$, so it is a semiprime ideal of $eRe$. This means that $P(eRe) \subseteq eP(R)e$.

To show that the converse inclusion is valid, it is enough to prove that $eP(R)e \subseteq Q$ for every prime ideal $Q$ of $eRe$. We shall obtain this by demonstrating that $Q = ePe$ for some prime ideal $P$ of $R$. To establish that, note that the set $X = eRe \setminus Q$ is what McCoy called in [13] an "m-system" of $eRe$:
it is nonempty, and for any \( x, y \in X \), there is some \( a \in eRe \) such that \( xay \in X \). Notice however that \( X \) is also an \( m \)-system even in \( R \), and that \( X \) is disjoint from the ideal \( RQR \). Let \( P \supset RQR \) be an ideal of \( R \) which is maximal with respect to being disjoint from \( X \). In [13] was proved that any such ideal has to be prime. Since \( P \) is disjoint from \( X \), we must have \( P \cap eRe = Q \), and consequently \( ePe = Q \), as desired. \( \square \)

We are now in a position to illustrate the truthfulness of the following:

**Corollary 2.4.** If \( R \) is a J-primal ring, then for any idempotent \( e \) of \( R \) the corner \( eRe \) is a J-primal ring, as well.

**Proof.** Utilizing [12] together with Lemma 2.3, one has that \( J(eRe) = eJ(R)e = eP(R)e = P(eRe) \), as needed. \( \square \)

Now, we have all the ingredients to prove the following statement.

**Theorem 2.5.** Suppose that \( R \) is a ring with \( e \in Id(R) \) for which \( eRe \) and \( (1 - e)R(1 - e) \) are both J-primal semi-boolean rings. Then \( R \) is nil-clean.

**Proof.** We foremost see that for \( f \) being either \( e \) or \( 1 - e \), the formula \( J(fRf) = P(fRf) \) is true. Furthermore, one observes with the aid of Lemma 2.3 along with [7] that the ring

\[
\frac{fRf}{J(fRf)} = \frac{fRf}{P(fRf)} = \frac{fRf}{fP(R)f} \cong f'(R/P(R))f'
\]

for \( f' = f + P(R) \in Id(R/P(R)) \), is boolean, so that Lemma 2.1 leads us to the fact that \( R/P(R) \) is nil-clean. Since \( P(R) \) is a nil-ideal of \( R \), we consequently consulting with [9] conclude that \( R \) is nil-clean, as pursued. \( \square \)

The next consequence follows by an immediate combination of Corollary 2.4 and Theorem 2.5.

**Corollary 2.6.** Suppose \( R \) is a J-primal ring whose corners \( eRe \) and \( (1 - e)R(1 - e) \) are semi-boolean rings. Then \( R \) is a nil-clean ring.

**Remark 2.** First of all, note the important fact that J-primal semi-boolean rings are exactly the 2-primal strongly nil-clean rings, and reciprocally. In fact, since J-primal semi-boolean rings have nil Jacobson ideals and these rings modulo their Jacobson radicals are boolean, one of the aforementioned chief results from [7] is applicable to deduce that they are strongly nil-clean. But then the isomorphism \( U(R)/[1 + J(R)] \approx U(R/J(R)) = 1 \) applies to infer that \( U(R) = 1 + J(R) \) and so \( 1 + \text{Nil}(R) \subseteq U(R) \) gives that \( \text{Nil}(R) = J(R) = P(R) \) (see [2, Theorem 2.3(2)] too), whence we obtain the wanted 2-primariness. Conversely, as discussed above, strongly nil-clean rings are themselves semi-boolean. Also, in view of [9], Jacobson radicals have to be nil and thereby the 2-primariness now routinely implies J-primariness, as required.

In this aspect, it is worth noticing that the essence in the proofs of the last theorem and its corresponding analogue from [6] is that in both versions of
J-primal rings and 2-primal rings it must be that \( P(fRf) = \text{Nil}^*(fRf) = J(fRf) \) for the idempotent \( f \in R \) defined as above.

On the other side, appealing to Theorem 2.2, we deduce that the factor-ring \( R/J(R) \) is nil-clean. In order to prove that \( R \) is also nil-clean, with \([9]\) at hand, it suffices to show directly that \( J(R) \) is nil. To that goal, it could be applied the classical Pierce’s direct decomposition for \( J(R) \) into subrings

\[
J(R) = eJ(R)e \oplus (1 - e)J(R)e \oplus eJ(R)(1 - e) \oplus (1 - e)J(R)(1 - e)
\]

used above or, in accordance with \([12]\), we may also use the Pierce’s matrix representation

\[
J(R) \cong \begin{pmatrix} eJ(R)e & eJ(R)(1 - e) \\ (1 - e)J(R)e & (1 - e)J(R)(1 - e) \end{pmatrix} = \begin{pmatrix} J(Re) & eJ(R)(1 - e) \\ (1 - e)J(R)e & J((1 - e)R(1 - e)) \end{pmatrix},
\]

which tells us about the corresponding matrix structure. However, without the extra assumption that \( R \) is commutative, we perhaps will need the validity of the famous Köthe’s conjecture.

Utilizing ordinary induction arguments in the pivotal Lemma 2.1, all statements concerning corners \( eRe \) and \((1 - e)R(1 - e)\) can be expanded to statements on a system of mutually orthogonal idempotents \( \{e_i\}_{i=1}^n \) with \( 1 = e_1 + \cdots + e_n \) such that all corners \( e_iRe_i \) are as above in the case of two idempotents (compare with \([10]\) and \([6]\), too).

With this at hand, we now arrive at the next assertion (compare with \([1]\) and \([6]\), as well).

**Corollary 2.7.** Let \( R \) be a J-primal semi-boolean ring. Then \( M_n(R) \) is nil-clean for each \( n \geq 1 \).

**Proof.** Knowing that \( R \cong E_{11}M_n(R)E_{11} \cong \cdots \cong E_{nn}M_n(R)E_{nn} \) for any \( n \geq 1 \), where \( \{E_{ii}\}_{i=1}^n \) forms a complete system of matrix idempotents (i.e., a set of matrix orthogonal idempotents with sum 1), it suffices to employ the generalized form of Theorem 2.5 to get the desired claim. \( \square \)

**Remark 3.** We cannot expect that, for each \( n \geq 2 \), \( M_n(R) \) is semi-boolean for the semi-boolean ring \( R \), because it can be showed as in \([15]\) that \( M_n(R) \) is semi-boolean if, and only if, \( R = J(R) \). However, this is nonsense when \( 1 \in J(R) \) since it would imply that \( 0 = 1 - 1 \in U(R) \), and hence \( 0 = 1 \), that is impossible. For a more general situation with given details, the interested readers can see \([5]\).

### 3. Left-Open Problems

We close with two problems of interest. We shall say that a ring \( R \) is 2-nil-clean if each its element is the sum of two idempotents and a nilpotent.
**Problem 1.** If $R$ is a ring for which both $eRe$ and $(1-e)R(1-e)$ are commutative semi-boolean rings, does it follow that $R$ is 2-nil-clean? In particular, if $R$ is commutative semi-boolean, is then $M_n(R)$ 2-nil-clean?

A ring is said to be *weakly boolean* if any its element is an idempotent or minus an idempotent (see also [5]). Generalizing this, in [8] were introduced *weakly nil-clean* rings as those rings whose elements are the sum or the difference of a nilpotent and an idempotent. We are now ready to state the following:

**Problem 2.** If $R$ is a ring for which both $eRe$ and $(1-e)R(1-e)$ are weakly boolean rings, is it true that $R$ is weakly nil-clean?

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**References**


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