Some Results about the Bruhat Ordering\(^1\)

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Abstract

In this paper, we consider the Bruhat ordering in a Coxeter group, and we get some results about it.

Keywords: Coxeter system; Bruhat ordering; the length function

1. INTRODUCTION

Let \(W = (W, S)\) be a Coxeter system. We can define \(\leq\) on \(W\) as following. \(y \leq w\) for \(y, w \in W\) if and only if \(y\) is a subexpression of any reduced expression of \(w\). Clearly \(\leq\) is a partial order on \(W\) which is called the Bruhat (or Bruhat-Chevalley) ordering on \(W\) (See [4]). In particular, let \(W\) be the dihedral group \(D_m\), for any \(y, w \in W\), we can get that \(y < w\) if and only if \(l(y) < l(w)\).

In [1], Shi have that if \(s \notin \mathcal{R}(X) \cup \mathcal{L}(Y)\). Then \(XY < XsY\) for \(X, Y \in W\) and \(s \in S\). Enlightened by Shi in [1], we generalize this result as follows.

Let \(X, Y \in W\), \(s \in S\). Then \(XY < XsY\) if and only if either \(s \notin \mathcal{R}(X) \cup \mathcal{L}(Y)\) or \(s \in \mathcal{R}(X) \cap \mathcal{L}(Y)\). We get this result in Section 3.

And in section 4, we also consider the question about the Bruhat ordering: Let \(X, Y, Z \in W\), \(s, t \in S\), when does \(XYZ < XsYtZ\) hold? \(^2\)

\(^1\)This work was supported in part by the NSF of P. R. China (no. 11261021) and the NSF of JX Province (no. 20142BAB211011).
This work was also supported in part by the Science Foundation of Education Department of JX Province (no. GJJ10396).
2. Preparation

Let $W = (W, S)$ be a Coxeter system with $S$ the set of its Coxeter generators, subject only to relations of the form

$$(ss')^m(s, s') = 1,$$

where $m(s, s) = 1$, $m(s, s') \geq 2$ for $s \neq s'$ in $S$.

For $w \in W$, let $l(w)$ be smallest integer $q \geq 0$ such that $w = s_1s_2 \cdots w_q$ with $s_1, s_2, \ldots, s_q$ in $S$. At the same time we say that $s_1s_2 \cdots s_q$ is a reduced expression of $w$ and $l(w)$ is the length of $w$.

Let “$\leq$” be the Bruhat ordering on $W$ and $w = s_1s_2 \cdots s_r$ be reduced, $s_i \in S$. We say that the form $s_{i_1}s_{i_2} \cdots s_{i_q}$ $(1 \leq i_1 < i_2 < \cdots < i_q \leq r)$ is a subexpression of $w$, and we write that $s_{i_1}s_{i_2} \cdots s_{i_q} \leq s_1s_2 \cdots s_r$.

Now let $w \in W$, $s \in S$ and

$$\mathcal{L}(w) = \{ s \in S \mid sw < w \}, \quad \mathcal{R}(w) = \{ s \in S \mid ws < w \}.$$

Let $x_i \in S$, $y_j \in S$. Then $x_1x_2 \cdots x_a \equiv y_1y_2 \cdots y_b$, if $a = b$, $x_i = y_i$, for each $i$. Let $X = x_1x_2 \cdots x_a$, $Y = y_1y_2 \cdots y_b$ and $X = Y$. Then there exist (A), (B), (C) Coxeter transformations, such that $X$ can be passed to $Y$.

(A) If there exist some $s, t \in S$, with $s \neq t$ and $1 \leq i < j \leq a$ such that

$$x_i x_{i+1} \cdots x_j \equiv stst \cdots, \quad j - i + 1 = m_{s,t}.$$

Where $m_{s,t}$ is the order of $st$ and $i, j$ are integer. Then we can define a transformation

$$x_1x_2 \cdots x_a \mapsto x_1x_2 \cdots x_{i-1} (tst \cdots) x_{j+1} \cdots x_a.$$

(B) If there exist some $i \in Z$, $1 \leq i < a$ such that $s_i = s_{i+1}$. Then we define transformation

$$x_1x_2 \cdots x_a \mapsto x_1x_2 \cdots x_{i-1}x_{i+2} \cdots x_a.$$

(C) For any $i \in Z$, $s \in S$ and $0 \leq i \leq a$. Then we define transformation

$$x_1x_2 \cdots x_a \mapsto x_1x_2 \cdots x_{i}(ss)x_{i+1} \cdots x_a.$$

Thus if $X$ and $Y$ are reduced. Then $X$ can be only passed to $Y$ by (a).

For $X, Y, Z \in W, s, t \in S$, Let

$$P(X, s, Y) = l(X) + l(Y) + 1 - l(XsY),$$

$$P(X, s, Y, t, Z) = l(X) + l(Y) + l(Z) + 2 - l(XsYtZ).$$

3. Some Generalized Conclusions about $XY < XsY$

**Lemma 3.1.** (See [1]) Let $X, Y \in W$, $s \in S$, $X \equiv x_1x_2 \cdots x_a, Y \equiv y_1y_2 \cdots y_b$ and they are reduced. Let $g \equiv x_1x_2 \cdots x_asy_1y_2 \cdots y_b$, $s \notin \mathcal{R}(X) \cup \mathcal{L}(Y)$ and $P(X, s, Y) > 0$. Then there exists a sequence of expressions $g \equiv g_0, g_1, \cdots, g_{u_1}, \cdots, g_{u_r}$ of $XsY$ for some $h, u_1, \cdots, u_r$ such that for each $i, 1 \leq i \leq u_r$, $g_i$ is obtained from $g_{i-1}$ by Coxeter transform of kind
Proof. \( g_i \equiv x(i, 1) \cdots x(i, k_i)s(i)y(i, 1) \cdots y(i, m_i) \) for \( 0 \leq i \leq u_r \).

(ii) There exists some integer \( 1 \leq h < u_i \) such that the expressions \( x(i, 1) \cdots x(i, k_i) \) and \( y(i, 1) \cdots y(i, m_i) \) are reduced for all \( i, 0 \leq i < h \).

(iii) Either \( x(h, 1) \cdots x(h, k_h) \) or \( y(h, 1) \cdots y(h, m_h) \) is not reduced expression, for \( h \) in (ii).

(iv) Let \( X(i) \equiv x(i, 1) \cdots x(i, k_i), Y(i) \equiv y(i, 1) \cdots y(i, m_i) \), for \( 0 \leq i \leq u_r \). Then \( XY = X(i)Y(i) \), \( XsY = X(i)s(i)Y(i) \), \( s(i) \notin \mathbb{R}(X(i)) \cup \mathcal{L}(Y(i)) \), for \( 0 \leq i \leq u_r \) and \( P(X(i), s(i), Y(i)) \)

\[ = P(X, s, Y) \] when \( 0 \leq i < h \), \( P(X(i), s(i), Y(i)) < P(X, s, Y) \) with \( h \leq i \leq u_r \).

(v) \( g_{u1} \) is not reduced. \( g_{u1}, \ldots, g_{ur} \) are reduced and they contain all reduced expressions of \( XsY \). Then \( P(X(i), s(i), Y(i)) = 0 \), for \( u_1 \leq i \leq u_r \).

**Theorem 3.2.** (See [1]) Let \( X, Y \in W \) and \( s \in S \). If \( s \notin \mathbb{R}(X) \cup \mathcal{L}(Y) \). Then \( XY < XsY \).

**Theorem 3.3.** Let \( X, Y \in W, s \in S, s \in \mathbb{R}(X) \cap \mathcal{L}(Y) \) and \( P(X, s, Y) > 2 \). Then there exist \( g(h) \equiv x(h, 1) \cdots x(h, k_h)s(h)y(h, 1) \cdots y(h, m_h) \) for some integer \( h \), where \( X(h) \equiv x(h, 1) \cdots x(h, k_h), Y(h) \equiv y(h, 1) \cdots y(h, m_h) \) which satisfy that

(i) \( XY = X(h)Y(h) \),

(ii) \( XsY = X(h)s(h)Y(h) \),

(iii) \( s(h) \in \mathbb{R}(X(h)) \cap \mathcal{L}(Y(h)) \),

(iii) \( P(X(h), s(h), Y(h)) < P(X, s, Y) \).

**Proof.** Let \( X' = Xs, Y' = sY \). Then \( XsY = X'sY' \), and \( s \notin \mathbb{R}(X') \cup \mathcal{L}(Y') \). By Lemma 1, we have that \( XsY = X'(h)s(h)Y'(h) = (X'(h)s(h))s(h)(s(h)Y'(h)) = X(h)s(h)Y(h) \), where \( X(h) = X'(h)s(h), Y(h) = s(h)Y'(h) \).

Then we have that \( XY = X'Y' = X'(h)Y'(h) = (X'(h)s(h))s(h)(s(h)Y'(h)) = X(h)s(h)Y(h) \).

Since \( s(h) \notin \mathbb{R}(X'(h)) \cup \mathcal{L}(Y'(h)) \), then \( s(h) \in \mathbb{R}(X(h)) \cap \mathcal{L}(Y(h)) \). \( P(X(h), s(h), Y(h)) = P(X'(h)s(h), s(h), s(h)Y'(h)) = P(X'(h), s(h), Y'(h)) + 2 < P(X', s, Y') + 2 = P(X, s, Y) + 2 = P(X, s, Y) \).

\( \square \)

**Lemma 3.4.** Let \( X, Y \in W, s \in S \). Then

(i) \( XY < XsY \) if and only if \( l(XsY) > l(XY) \).

(ii) \( XsY < XY \) if and only if \( l(XY) > l(XsY) \).

**Proof.** Since \( XY = XsY(Y^{-1}sY) \), \( XsY = XY(Y^{-1}sY) \). Then the result is clear. \( \square \)

**Proposition 3.5.** Let \( X, Y \in W, s \in S \). If \( l(XsY) > l(XY) \) and \( s \notin \mathbb{R}(X) \). Then \( s \notin \mathcal{L}(Y) \).

**Proof.** If \( s \in \mathcal{L}(Y) \). Let \( Y' = sY \), then \( s \notin \mathcal{L}(Y') \). By Lemma 3.4 and Theorem 3.2, we know \( XsY = XY' > XsY' = XY \). Since \( s \notin \mathbb{R}(X) \cup \mathcal{L}(Y') \). This is a contradiction. \( \square \)

**Proposition 3.6.** Let \( X, Y \in W, s \in S \). If \( l(XsY) > l(XY) \) and \( s \notin \mathcal{L}(Y) \). Then \( s \notin \mathbb{R}(X) \).
Proof. The proof is similar to proof of Proposition 3.5.

Corollary 3.7. Let $X, Y \in W$, $s \in S$. If $l(XsY) > l(XY)$. Then either $s \notin \mathcal{R}(X) \cup \mathcal{L}(Y)$ or $s \in \mathcal{R}(X) \cap \mathcal{L}(Y)$.

Proof. We can get the result easily by Proposition 3.5 and Proposition 3.6.

Theorem 3.8. Let $X, Y \in W$, $s \in S$. If $s \in \mathcal{R}(X) \cap \mathcal{L}(Y)$. Then $XY < XsY$.

Proof. Since $s \in \mathcal{R}(X) \cap \mathcal{L}(Y)$, then $XsY = (Xs)sY = X'sY'$. Thus $s \notin \mathcal{R}(X') \cup \mathcal{L}(Y')$. Hence $XsY = X'sY' > X'Y' = (X'(s)(s') = XY$.

Corollary 3.9. Let $X, Y \in W$, $s \in S$. Then $XY < XsY$ if and only if either $s \notin \mathcal{R}(X) \cup \mathcal{L}(Y)$ or $s \in \mathcal{R}(X) \cap \mathcal{L}(Y)$.

Proof. It is easy from Theorem 3.8, Theorem 3.2 and Corollary 3.7.

Lemma 3.10. Given $Y \in W$, let $w = s_1s_2 \cdots s_r$ be reduced, $s_i \in S$. Then $l(s_1s_2 \cdots s_rY) = l(Y) - r$ if and only if $s_i \in \mathcal{L}(s_{i+1} \cdots s_rY)$ for each $1 \leq i \leq r$.

Proof. If $s_i \in \mathcal{L}(s_{i+1} \cdots s_rY)$, for each $1 \leq i \leq r$, then we obtain easily the result. Assume that $l(s_1s_2 \cdots s_rY) = l(Y) - r$. We can apply induction on $r$. If $r = 1$, then it is trivial. Now suppose that $r > 1$. By the inductive hypothesis, $s_i \in \mathcal{L}(s_{i+1} \cdots s_rY)$ for each $2 \leq i \leq r$, and $l(s_2 \cdots s_rY) = l(Y) - r + 1$, since $l(s_2 \cdots s_r) < r$. If $s_1 \notin \mathcal{L}(s_2 \cdots s_rY)$, then $l(s_1s_2 \cdots s_rY) = l(s_2 \cdots s_rY) + 1 = l(Y) - r + 1 + 1 = l(Y) - r + 2$. This is a contradiction.

Theorem 3.11. Let $X, Y \in W$, $J = \mathcal{R}(x)$ and $w \in W_J$, $l(wY) = l(Y) - l(w)$. Where $(W_J, J)$ is a Coxeter system. Then $XY < XwY$.

Proof. Let $w = s_1s_2 \cdots s_r$ be reduced, $s_i \in S$. We can apply induction on $r$. If $r = 1$, it is trivial. Now suppose that $r > 1$. By the inductive hypothesis, then $XY < Xs_2 \cdots s_rY$ by Lemma 3.10, since $l(s_2 \cdots s_r) < r$. We know that $s_1 \in \mathcal{L}(s_2 \cdots s_rY)$ and $s_1 \in \mathcal{R}(X)$, hence $Xs_2 \cdots s_rY < XwY$. Then $XY < XwY$.

4. RESULTS ABOUT $X_{YZ} < Xs_{YtZ}$

Theorem 4.1. Let $X, Y, Z \in W$, $s, t \in S$. If $t \notin \mathcal{R}(XsY) \cup \mathcal{L}(Z)$, $s \notin \mathcal{R}(X) \cup \mathcal{L}(YZ)$. Then $X_{YZ} < Xs_{YtZ}$.

Proof. If $t \notin \mathcal{R}(XsY) \cup \mathcal{L}(Z)$, then $Xs_{YZ} < Xs_{YtZ}$. If $s \notin \mathcal{R}(X) \cup \mathcal{L}(YZ)$, then $X_{YZ} < Xs_{YZ}$ from Corollary 3.9. Thus $X_{YZ} < Xs_{YtZ}$.

Similarly, if $t \in \mathcal{R}(XsY) \cap \mathcal{L}(Z)$, $s \notin \mathcal{R}(X) \cup \mathcal{L}(YZ)$, then $X_{YZ} < Xs_{YtZ}$.

If $t \notin \mathcal{R}(XsY) \cup \mathcal{L}(Z)$, $s \in \mathcal{R}(X) \cap \mathcal{L}(YZ)$, then $X_{YZ} < Xs_{YtZ}$.

If $t \in \mathcal{R}(XsY) \cap \mathcal{L}(Z)$, $s \in \mathcal{R}(X) \cap \mathcal{L}(YZ)$, then $X_{YZ} < Xs_{YtZ}$.

Theorem 4.2. Let $X, Y, Z \in W$, $s, t \in S$. If $s \notin \mathcal{R}(X) \cup \mathcal{L}(YtZ)$, $t \notin \mathcal{R}(XY) \cup \mathcal{L}(Z)$. Then $X_{YZ} < Xs_{YtZ}$.
Let $s \notin \mathcal{R}(X) \cup \mathcal{L}(YtZ)$. Then $XYtZ < XsYtZ$. If $t \notin \mathcal{R}(XY) \cup \mathcal{L}(Z)$, then $XYtZ < XYtZ$ by Corollary 3.9. Thus $XYZ < XsYtZ$. Similarly, if $s \in \mathcal{R}(X) \cap \mathcal{L}(YtZ)$, $t \notin \mathcal{R}(XY) \cup \mathcal{L}(Z)$, then $XYZ < XsYtZ$. If $s \notin \mathcal{R}(X) \cup \mathcal{L}(YtZ)$, $t \in \mathcal{R}(XY) \cap \mathcal{L}(Z)$, then $XYZ < XsYtZ$. If $s \in \mathcal{R}(X) \cap \mathcal{L}(YtZ)$, $t \in \mathcal{R}(XY) \cap \mathcal{L}(Z)$, then $XYZ < XsYtZ$. □

Let $W = D_{10} = (s, t)$ be the with $m_{st} = 10$, $X = tst$, $Y = sts$, $Z = ststs$. Then $XYZ = t$, $XsYtZ = ststs$. Thus $XYZ < XsYtZ$. Clearly they do not satisfy these conditions above.

**Theorem 4.3.** Let $X, Y, Z \in W$, $s, t \in S$. If $s \notin \mathcal{R}(X) \cup \mathcal{L}(Y)$ and $P(X, s, Y) = P(X, s, Y, t, Z)$. Then $XYZ < XsYtZ$.

**Proof.** We can apply induction on $P(X, s, Y, t, Z)$. Since $l(XsYtZ) \equiv l(X) + l(Y) + l(Z) + 2 \mod 2$, $l(XsY) \equiv l(X) + l(Y) + 1 \mod 2$, therefore $P(X, s, Y) = P(X, s, Y, t, Z)$ are even. Now if $P(X, s, Y) = 0$, then it is trivial. Now suppose that $P(X, s, Y, t, Z) > 0$, hence $P(X, s, Y) > 0$. We know that there exist $g(h) \equiv x(h, 1) \cdots x(h, k_h)s(h)g(h, 1) \cdots y(h, m_h) \equiv X(h)s(h)Y(h)$ by Lemma 1, where $X(h) \equiv x(h, 1) \cdots x(h, k_h)$, $Y(h) \equiv y(h, 1) \cdots y(h, m_h)$. They satisfies that $s(h) \notin \mathcal{R}(X(h)) \cup \mathcal{L}(Y(h))$, $P(X(h), s(h), Y(h)) < P(X, s, Y)$ and $s(h) \notin \mathcal{R}(X(h)) \cup \mathcal{L}(Y(h))$. Let $P(X(h), s(h), Y(h)) = P(X, s, Y) - 2m$, then $l(X(h)) + l(Y(h)) = l(X) + l(Y) = 2m$ and $P(X, s, Y, t, Z) = l(X) + l(Y) + l(Z) + 2 - l(XsYtZ) = l(X(h)) + l(Y(h)) + l(Z) + 2 - l(XsYtZ) = 2m = P(X, s, Y, t, Z) + 2m = P(X, s, Y) - 2m = P(X, s, Y) - 2m$. Hence $P(X(h), s(h), Y(h), t, Z) = P(X, s, Y, t, Z) - 2m = P(X, s, Y) - 2m = P(X, s, Y) - 2m$. Since $s(h) \notin \mathcal{R}(X(h)) \cup \mathcal{L}(Y(h))$, $s(h) \notin \mathcal{R}(X(h)) \cup \mathcal{L}(Y(h))$. Then $XYZ < XsYtZ$.

**Corollary 4.4.** Let $X, Y, Z \in W$, $s, t \in S$. If $t \notin \mathcal{R}(Y) \cup \mathcal{L}(Z)$ and $P(Y, t, Z) = P(X, s, Y, t, Z)$. Then $XYZ < XsYtZ$.

**Proof.** The proof is similar to proof of Theorem 4.3.

**Theorem 4.5.** Let $X, Y, Z \in W$, $s, t \in S$, $s \in \mathcal{R}(X) \cap \mathcal{L}(Y)$. If $P(X, s, Y) = P(X, s, Y, t, Z)$. Then $XYZ < XsYtZ$.

**Proof.** Since $s \in \mathcal{R}(X) \cap \mathcal{L}(Y)$, hence $P(X, s, Y) = P(X, s, sY, t, Z)$ and $s \notin \mathcal{R}(X) \cup \mathcal{L}(sY)$. We have $(Xs)(sY)Z < (Xs)YtZ$ by Theorem 3.4. Thus $XYZ < XsYtZ$.

**Corollary 4.6.** Let $X, Y, Z \in W$, $s, t \in S$, $t \in \mathcal{R}(Y) \cap \mathcal{L}(Z)$. Then $XYZ < XsYtZ$.

**Proof.** The proof is similar to proof of Theorem 4.5.

Let $X, Y, Z \in W$, $s, t \in S$. Let $X(r)s(r)Y(r)t(r)Z(r)$ be an expression obtained from the expression $XsYtZ$ by some Coxeter transformations of kind $\neq (C)$. Namely $XsYtZ \mapsto X(1)s(1)Y(1)t(1)Z(1) \mapsto \cdots \mapsto X(r)s(r)Y(r)t(r)Z(r)$. Where we suppose that these Coxeter transformations do not involve $s$ and $t$, (if some Coxeter transformation involves $s$ or $t$, then it must be Coxeter transformation of kind $(A)$.)

Clearly, $XYZ = X(1)Y(1)Z(1) = \cdots = X(r)Y(r)Z(r)$,

$XsYtZ = X(1)s(1)Y(1)t(1)Z(1) = \cdots = X(r)s(r)Y(r)t(r)Z(r)$. □
Theorem 4.7. Let \( X, Y, Z \in W, t \in S \). If there exist \( X(r)s(r)Y(r)t(r)Z(r) \) obtained as above. Let \( Y(r) = y_1y_2 \cdots y_b \) be a reduce expression. If they satisfy either \( s(r) \notin \mathcal{R}(X(r)) \cup \mathcal{L}(Y_k) \) or \( s(r) \notin \mathcal{R}(X(r)) \cap \mathcal{L}(Y_k) \) and satisfy \( t(r) \notin \mathcal{R}(Y_k') \cup \mathcal{L}(Z(k)) \) or \( t(r) \in \mathcal{R}(Y_k') \cap \mathcal{L}(Z(k)) \) and \( l(X(r)s(r)Y(r)t(r)Z(r)) = l(\mathcal{L}(X(r)s(r)Y_k) + l(Y_k't(r)Z(r)) \) for some \( 0 \leq k \leq b \), where \( Y_k = y_1 \cdots y_k, Y_k' = y_{k+1} \cdots y_b \). (When \( k = 0 \) or \( b \), then \( Y_k = e \) or \( Y \), \( e \) is identity of \( W \).) Then \( XYZ < XsY tZ \).

Proof. We can apply Induction on \( P(X(r), s(r), Y(r), t(r), Z(r)) \). We know that

\[
\begin{align*}
P(X(r), s(r), Y(r), t(r), Z(r)) &= l(X(r)) + l(Y_k) + l(Y_k') + l(Z(r)) + 2l(X(r)s(r)Y(r)t(r)Z(r)), \\
P(X(r), s(r), Y_k) &= l(X(r)) + l(Y_k) + 1 - l(X(r)s(r)Y_k), \\
P(Y_k', t(r), Z(r)) &= l(Y_k') + l(t(r)) + 1 - l(Y_k't(r)Z(r)).
\end{align*}
\]

Then we have

\[
P(X(r), s(r), Y(r), t(r), Z(r)) = P(X(r), s(r), Y_k) + P(Y_k', t(r), Z(r)) \text{ if and only if } l(X(r)s(r)Y(r)t(r)Z(r)) = l(X(r)s(r)Y_k) + l(Y_k't(r)Z(r)).
\]

Now if \( P(X(r), s(r), Y(r), t(r), Z(r)) = 0 \),
then \( XYZ = X(r)Y(r)Z(r) < X(r)s(r)Y(r)t(r)Z(r) = XsY tZ \).

In case of \( P(X(r), s(r), Y(r), t(r), Z(r)) > 0 \). We have that either \( P(X(r), s(r), Y_k) > 0 \) or \( P(Y_k', t(r), Z(r)) > 0 \) (or both). We assume that \( P(X(r), s(r), Y_k) > 0 \) and

\[
s(r) \notin \mathcal{R}(X(r)) \cup \mathcal{L}(Y_k) \text{, then there exist that } X(r, h)s(r, h)Y_k(h) \text{ obtained from the expression } X(r)s(r)Y_k \text{ by coxeter transformation of kind (A) (B) by Lemma 1 which satisfy } P(X(r, h), s(r, h), Y_k(h)) < P(X(r), s(r), Y_k) \text{.}
\]

Thus

\[
P(X(r, h), s(r, h), Y_k(h)Y_k', t(r), Z(r)) = P(X(r, h), s(r, h), Y_k(h)) + P(Y_k', t(r), Z(r)) < P(X(r), s(r), Y(r), t(r), Z(r)), \text{ if and only if } s(r) \notin \mathcal{R}(X(r, h)) \cup \mathcal{L}(Y_k(h)) \text{, then } X(r, h)s(r, h)Y_k(h)Y_k't(r)Z(r) = X(r)s(r)Y(r)t(r)Z(r) = XsY tZ
\]

and

\[
X(r, h)Y_k(h)Y_k'Z(r) = X(r)Y(r)Z(r) = XYZ.
\]

So By induction hypothesis, we have \( XYZ = X(r, h)Y_k(h)Y_k'Z(r) < X(r, h)s(r, h)Y_k(h)Y_k't(r)Z(r) = XsY tZ \).

If \( s(r) \in \mathcal{R}(X(r)) \cap \mathcal{L}(Y_k) \), then \( s(r) \notin \mathcal{R}(X(r)s(r)) \cup \mathcal{L}(s(r)Y_k) \),

\[
P(X(r)s(r), s(r), Y(r)t(r), Z(r)) = P(X(r)s(r), s(r), s(r)Y_k) + P(Y_k', t(r), Z(r)), XsY tZ
\]

and

\[
XYZ = (X(r)s(r))(s(r)Y(r)t(r))Z(r).
\]

Therefore

\[
XYZ = (X(r)s(r))(s(r)Y(r)t(r)Z(r)) = (X(r)s(r))s(r)(s(r)Y(r))t(r)Z(r) = XsY tZ.
\]

Similarly for \( P(Y_k', t(r), Z(r)) > 0 \). \( \square \)

Let \( W = H_4 = \{s_1, s_2, s_3, s_4\} \) be the with \( m_{s_1, s_2} = 5, m_{s_2, s_3} = 3, m_{s_3, s_4} = 3, m_{s_1, s_3} = 2, m_{s_1, s_4} = 2, m_{s_2, s_4} = 2 \). Let \( X = s_4s_2s_1s_2, Y = s_2s_1s_2s_1, Z = s_2s_4s_3s_4, s = s_1, t = s_1 \). Then \( XYZ < XsY tZ \). Clearly, they do not satisfy the conditions of Theorem 4.1 and Theorem 4.2. But we can get \( XYZ < XsY tZ \) by Theorem 4.7. We will provide the coxeter transformation’s process that \( X(r)s(r)Y(r)t(r)Z(r) \) was obtained from \( XsY tZ \) by certain coxeter transformations as following.
We can see $s(8) \notin R(X(8)) \cup L(Y_4(8))$, $t(8) \notin R(Y_4'(8)) \cup L(Z(8))$ and $l(X(8)s(8)Y(8)t(8)Z(8)) = l(X(8)s(8)Y(8)) + l(Y_4(8)t(8)Z(8))$. Thus $XYZ < XsYtZ$.

**References**


Received: November 9, 2016; Published: January 4, 2017