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Some Results about the Bruhat Ordering[1]

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Abstract

In this paper, we consider the Bruhat ordering in a Coxeter group, and we get some results about it.

Keywords: Coxeter system; Bruhat ordering; the length function

1. INTRODUCTION

Let $W = (W, S)$ be a Coxeter system. We can define $\leq$ on W as following. $y \leq w$ for $y,\ w \in W$ if and only if y is a subexpression of any reduced expression of w. Clearly $\leq$ is a partial order on W which is called the Bruhat (or Bruhat-Chevally) ordering on W(See [4]). In particular, let W be the dihedral group D_m, for any $y,\ w \in W$, we can get that $y < w$ if and only if $l(y) < l(w)$.
In [1], Shi have that if $s \notin \Re(X) \bigcup \pounds(Y)$. Then $XY < XsY$ for $X, Y \in W$ and $s \in S$. Enlightened by Shi in [1], we generalize this result as follows.
Let $X, Y \in W$, $s \in S$. Then $XY < XsY$ if and only if either $s \notin \Re(X) \bigcup \pounds(Y)$ or $s \in \Re(X) \bigcap \pounds(Y)$. We get this result in Section 3.
And in section 4, we also consider the question about the Bruhat ordering : Let $X, Y, Z \in W$, $s, t \in S$, when dose $XYZ < XsYtZ$ hold ?

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2. PREPARATION

Let $W = (W, S)$ be a Coxeter system with S the set of its Coxeter generators, subject only to relations of the form

$$(ss^{'})^{m(s,s^{'})} = 1,$$

where $m(s,s)$=1, $m(s,s^{'}) \geq 2$ for $s \neq s^{'}$ in S.

For $w \in W$, let $l(w)$ be smallest integer $q \geq 0$ such that $w = s_1 s_2 \cdots w_q$ with $s_1, s_2, \cdots, s_q$ in S. At the same time we say that $s_1 s_2 \cdots s_q$ is a reduced expression of w and $l(w)$ is the length of w.

Let "$\leq$" be the Bruhat ordering on W and $w = s_1 s_2 \cdots s_r$ be reduced, $s_i \in S$. We say that the form $s_{i_1} s_{i_2} \cdots s_{i_q}$ $(1 \leq i_1 < i_2 < \cdots < i_q \leq r)$ is a subexpression of w, and we write that $s_{i_1} s_{i_2} \cdots s_{i_q} \leq s_1 s_2 \cdots s_r$.

Now let $w \in W, s \in S$ and

$$\pounds(w) = \{s \in S \mid sw < w\} \qquad \Re(w) = \{s \in S \mid ws < w\}.$$

Let $x_i \in S$, $y_j \in S$. Then $x_1 x_2 \cdots x_a \equiv y_1 y_2 \cdots y_b$, if $a = b$, x_i=y_i, for each i. Let $X = x_1 x_2 \cdots\ x_a$, $Y = y_1 y_2 \cdots\ y_b$ and $X = Y$. Then there exist $(A), (B), (C)$ Coxeter transformations, such that X can be passed to Y.
(A) If there exist some $s, t \in S$, with $s \neq t$ and $1 \leq i < j \leq a$ such that

$$x_i x_{i+1} \cdots x_j \equiv stst \cdots, \qquad j - i + 1 = m_{s,t}.$$

Where $m_{s,t}$ is the order of st and i, j are integer.
Then we can define a transformation

$$x_1 x_2 \cdots\ x_a \mapsto\ x_1 x_2 \cdots\ x_{i-1} \underbrace{(tsts \cdots)}_{m_{s,t}\ factors} x_{j+1} \cdots\ x_a.$$

(B) If there exist some $i \in Z, 1 \leq i < a$ such that $s_i = s_{i+1}$. Then we define transformation

$$x_1 x_2 \cdots\ x_a \mapsto\ x_1 x_2 \cdots\ x_{i-1} x_{i+2} \cdots\ x_a.$$

(C) For any $i \in Z, s \in S$ and $0 \leq i \leq a$. Then we define transformation

$$x_1 x_2 \cdots x_a \mapsto x_1 x_2 \cdots x_i (ss) x_{i+1} \cdots x_a.$$

Thus if X and Y are reduced. Then X can be only passed to Y by (a).
For $X, Y, Z \in W, s, t \in S$, Let
$P(X, s, Y) = l(X) + l(Y) + 1 - l(XsY)$,
$P(X, s, Y, t, Z) = l(X) + l(Y) + l(Z) + 2 - l(XsYtZ)$.

3. SOME GENERALIZED CONCLUSIONS ABOUT $XY < XsY$

Lemma 3.1. *(See* [1]*) Let $X, Y \in W$, $s \in S$, $X \equiv x_1 x_2 \cdots x_a, Y \equiv y_1 y_2 \cdots y_b$ and they are reduced. Let $g \equiv x_1 x_2 \cdots x_a s y_1 y_2 \cdots y_b$, $s \notin \Re(X) \bigcup \pounds(Y)$ and $P(X, s, Y) > 0$. Then there exists a sequence of expressions $g \equiv g_0, g_1, \cdots, g_{u_1} \cdots, g_{u_r}$ of XsY for some $h, u_1, \cdots, u_r$ such that for each $i, 1 \leq i \leq u_r$, g_i is obtained from g_{i-1} by coxeter transform of kind*

$\neq (C)$ *and they satisfy that*
(i) $g_i \equiv x(i,1)\cdots x(i,k_i)s(i)y(i,1)\cdots y(i,m_i)$ *for* $0 \leq i \leq u_r$.
(ii) There exists some integer $1 \leq h < u_1$ *such that the expressions* $x(i,1)\cdots x(i,k_i)$ *and* $y(i,1)\cdots y(i,m_i)$ *are reduced for all* i, $0 \leq i < h$.
(iii) Either $x(h,1)\cdots x(h,k_h)$ *or* $y(h,1)\cdots y(h,m_h)$ *is not reduced expression, for* h *in (ii).*
(iv) Let $X(i) \equiv x(i,1)\cdots x(i,k_i), Y(i) \equiv y(i,1)\cdots y(i,m_i)$, *for* $0 \leq i \leq u_r$. *Then* $XY = X(i)Y(i)$, $XsY = X(i)s(i)Y(i)$, $s(i) \notin \Re(X(i))\bigcup \pounds(Y(i))$, *for* $0 \leq i \leq u_r$ *and* $P(X(i),s(i),Y(i))$
$= P(X,s,Y)$ *with* $0 \leq i < h$, $P(X(i),s(i),Y(i)) < P(X,s,Y)$ *with* $h \leq i \leq u_r$.
(v)g_{u_1-1} *is not reduced.* $g_{u_1},\cdots,g_{u_r}$ *are reduced and they contain all reduced expressions of* XsY. *Then* $P(X(i),s(i),Y(i)) = 0$, *for* $u_1 \leq i \leq u_r$.

Theorem 3.2. *(See [1]) Let* $X,Y \in W$ *and* $s \in S$. *If* $s \notin \Re(X)\bigcup \pounds(Y)$. *Then* $XY < XsY$.

Theorem 3.3. *Let* $X,Y \in W$, $s \in S$, $s \in \Re(X)\bigcap \pounds(Y)$ *and* $P(X,s,Y) > 2$. *Then there exist* $g(h) \equiv x(h.1)\cdots x(h,k_h)s(h)y(h,1)\cdots y(h,m_h)$ *for some integer* h, *where* $X(h) \equiv x(h.1)\cdots x(h,k_h), Y(h) \equiv y(h,1)\cdots y(h,m_h)$ *which satisfy that*
(i) $XY = X(h)Y(h)$,
(ii) $XsY = X(h)s(h)Y(h)$,
(iii) $s(h) \in \Re(X(h))\bigcap \pounds(Y(h))$,
(iiii) $P(X(h),s(h),Y(h)) < P(X,s,Y)$.

Proof. Let $X' = Xs, Y' = sY$. Then $XsY = X'sY'$, and $s \notin \Re(X')\bigcup \pounds(Y')$. By Lemma 1, we have that $XsY = X'sY' = X'(h)s(h)Y'(h)$=$(X'(h)s(h))s(h)(s(h)Y'(h))$=$X(h)s(h)$ $Y(h)$, where $X(h) = X'(h)s(h)$, $Y(h) = s(h)Y'(h)$.
Then we have that $XY = X'Y' = X'(h)Y'(h) = (X'(h)s(h))(s(h)Y'(h)) = X(h)Y(h)$. Since $s(h) \notin \Re(X'(h))\bigcup \pounds(Y'(h))$, then $s(h) \in \Re(X(h))\bigcap \pounds(Y(h))$. $P(X(h),s(h),Y(h)) = P(X'(h)s(h),s(h),s(h)Y'(h)) = P(X'(h),s(h),Y'(h))+2 < P(X',s,Y')+2 = P(Xs,s,sY)+2 = P(X,s,Y)$.

□

Lemma 3.4. *Let* $X,Y \in W$, $s \in S$. *Then*
(i)$XY < XsY$ *if and only if* $l(XsY) > l(XY)$.
(ii)$XsY < XY$ *if and only if* $l(XY) > l(XsY)$.

Proof. Since $XY = XsY(Y^{-1}sY)$, $XsY = XY(Y^{-1}sY)$. Then the result is clear. □

Proposition 3.5. *Let* $X,Y \in W$, $s \in S$. *If* $l(XsY) > l(XY)$ *and* $s \notin \Re(X)$. *Then* $s \notin \pounds(Y)$.

Proof. If $s \in \pounds(Y)$. Let $Y' = sY$, then $s \notin \pounds(Y')$. By Lemma 3.4 and Theorem3. 2, we know $XsY = XY' > XsY' = XY$. Since $s \notin \Re(X)\bigcup \pounds(Y')$. This is a contradiction. □

Proposition 3.6. *Let* $X,Y \in W$, $s \in S$. *If* $l(XsY) > l(XY)$ *and* $s \notin \pounds(Y)$. *Then* $s \notin \Re(X)$.

Proof. The proof is similar to proof of Proposition 3.5. □

Corollary 3.7. *Let $X, Y \in W$, $s \in S$. If $l(XsY) > l(XY)$. Then either $s \notin \Re(X) \bigcup \pounds(Y)$ or $s \in \Re(X) \bigcap \pounds(Y)$.*

Proof. We can get the result easily by Proposition 3.5 and Proposition 3.6. □

Theorem 3.8. *Let $X, Y \in W$, $s \in S$. If $s \in \Re(X) \bigcap \pounds(Y)$. Then $XY < XsY$.*

Proof. Since $s \in \Re(X) \bigcap \pounds(Y)$, then $XsY = (Xs)s(sY) = X^{'}sY^{'}$. Thus $s \notin \Re(X^{'}) \bigcup \pounds(Y^{'})$. Hence $XsY = X^{'}sY^{'} > X^{'}Y^{'} = (X^{'}s)(sY^{'}) = XY$. □

Corollary 3.9. *Let $X, Y \in W$, $s \in S$. Then $XY < XsY$ if and only if either $s \notin \Re(X) \bigcup \pounds(Y)$ or $s \in \Re(X) \bigcap \pounds(Y)$.*

Proof. It is easy from Theorem 3.8, Theorem 3.2 and Corollary 3.7. □

Lemma 3.10. *Given $Y \in W$, let $w = s_1 s_2 \cdots\ s_r$ be reduced, $s_i \in S$. Then $l(s_1 s_2 \cdots s_r Y) = l(Y) - r$ if and only if $s_i \in \pounds(s_{i+1} \cdots s_r Y)$ for each $1 \leq i \leq r$.*

Proof. If $s_i \in \pounds(s_{i+1} \cdots s_r Y)$, for each $1 \leq i \leq r$, then we obtain easily the result. Assume that $l(s_1 s_2 \cdots s_r Y) = l(Y) - r$. We can apply induction on r. If $r = 1$, then it is trivial. Now suppose that $r > 1$. By the inductive hypothesis, then $s_i \in \pounds(s_{i+1} \cdots s_r Y)$ for each $2 \leq i \leq r$, and $l(s_2 \cdots s_r Y) = l(Y) - r + 1$, since $l(s_2 \cdots s_r) < r$. If $s_1 \notin \pounds(s_2 \cdots s_r Y)$, then $l(s_1 s_2 \cdots s_r Y) = l(s_2 \cdots s_r Y) + 1 = l(Y) - r + 1 + 1 = l(Y) - r + 2$. This is a contradiction. □

Theorem 3.11. *Let $X, Y \in W$, $J = \Re(x)$ and $w \in W_J$, $l(wY) = l(Y) - l(w)$. Where (W_J, J) is a Coxeter system. Then $XY < XwY$.*

Proof. Let $w = s_1 s_2 \cdots s_r$ be reduced, $s_i \in S$. We can apply induction on r. If $r = 1$, it is trivial. Now suppose that $r > 1$. By the inductive hypothesis, then $XY < Xs_2 \cdots s_r Y$ by Lemma 3.10, since $l(s_2 \cdots s_r) < r$. We know that $s_1 \in \pounds(s_2 \cdots s_r Y)$ and $s_1 \in \Re(X)$, hence $Xs_2 \cdots s_r Y < XwY$. Then $XY < XwY$. □

4. RESULTS ABOUT $XYZ < XsYtZ$

Theorem 4.1. *Let $X, Y, Z \in W$, $s, t \in S$. If $t \notin \Re(XsY) \bigcup \pounds(Z)$, $s \notin \Re(X) \bigcup \pounds(YZ)$. Then $XYZ < XsYtZ$.*

Proof. If $t \notin \Re(XsY) \bigcup \pounds(Z)$, then $XsYZ < XsYtZ$. If $s \notin \Re(X) \bigcup \pounds(YZ)$, then $XYZ < XsYZ$ from Corollary 3.9.
Thus $XYZ < XsYtZ$.
Similarly, if $t \in \Re(XsY) \bigcap \pounds(Z)$, $s \notin \Re(X) \bigcup \pounds(YZ)$, then $XYZ < XsYtZ$.
If $t \notin \Re(XsY) \bigcup \pounds(Z)$, $s \in \Re(X) \bigcap \pounds(YZ)$,then $XYZ < XsYtZ$.
If $t \in \Re(XsY) \bigcap \pounds(Z)$, $s \in \Re(X) \bigcap \pounds(YZ)$, then $XYZ < XsYtZ$. □

Theorem 4.2. *Let $X, Y, Z \in W$, $s, t \in S$. If $s \notin \Re(X) \bigcup \pounds(YtZ)$, $t \notin \Re(XY) \bigcup \pounds(Z)$. Then $XYZ < XsYtZ$.*

Proof. We know that $s \notin \Re(X) \bigcup \pounds(YtZ)$. Then $XYtZ < XsYtZ$. If $t \notin \Re(XY) \bigcup \pounds(Z)$. Then $XYZ < XYtZ$ by Corollary 3.9. Thus $XYZ < XsYtZ$.
Similarly, if $s \in \Re(X) \bigcap \pounds(YtZ)$, $t \notin \Re(XY) \bigcup \pounds(Z)$, then $XYZ < XsYtZ$.
If $s \notin \Re(X) \bigcup \pounds(YtZ)$, $t \in \Re(XY) \bigcap \pounds(Z)$, then $XYZ < XsYtZ$.
If $s \in \Re(X) \bigcap \pounds(YtZ)$, $t \in \Re(XY) \bigcap \pounds(Z)$, then $XYZ < XsYtZ$. □

Let $W = D_{10} = \langle s, t\rangle$ be the with $m_{s,t} = 10$, $X = tst$, $Y = sts$, $Z = ststs$. Then $XYZ = t$, $XsYtZ = ststs$. Thus $XYZ < XsYtZ$. Clearly they do not satisfy these conditions above.

Theorem 4.3. *Let $X, Y, Z \in W$, $s, t \in S$. If $s \notin \Re(X) \bigcup \pounds(Y)$ and $P(X, s, Y) = P(X, s, Y, t, Z)$. Then $XYZ < XsYtZ$.*

Proof. We can apply induction on $P(X, s, Y, t, Z)$. Since $l(XsYtZ) \equiv l(X)+l(Y)+l(Z)+2 \mod 2$, $l(XsY) \equiv l(X)+l(Y)+1 \mod 2$, therefore $P(X, s, Y)$ and $P(X, s, Y, t, Z)$ are even. Now if $P(X, s, Y, t, Z) = 0$, then it is trivial. Now suppose that $P(X, s, Y, t, Z) > 0$, hence $P(X, s, Y) > 0$. We know that there exist $g(h) \equiv x(h.1) \cdots x(h, k_h)s(h)y(h, 1) \cdots y(h, m_h) \equiv X(h)s(h)Y(h)$ by Lemma 1, where $X(h) \equiv x(h.1) \cdots x(h, k_h)$, $Y(h) \equiv y(h, 1) \cdots y(h, m_h)$. They satisfies that $s(h) \notin \Re(X(h)) \bigcup \pounds(Y(h))$, $P(X(h), s(h), Y(h)) < P(X, s, Y)$ and $XsY = X(h)s(h)Y(h)$. Let $P(X(h), s(h), Y(h)) = P(X, s, Y) - 2m$, then $l(X(h)) + l(Y(h)) = l(X) + l(Y) - 2m$ and $P(X, s, Y, t, Z) = l(X) + l(Y) + l(Z) + 2 - l(XsYtZ) = l(X(h)) + l(Y(h)) + l(Z) + 2 - l(X(h)s(h)Y(h)tZ) + 2m = P(X(h), s(h), Y(h), t, Z) + 2m$. Hence $P(X(h), s(h), Y(h), t, Z) = P(X, s, Y, t, Z) - 2m = P(X, s, Y) - 2m = P(X(h), s(h), Y(h))$. $s(h) \notin \Re(X(h)) \bigcup \pounds(Y(h))$.
By induction hypothesis, we have $XYZ = X(h)Y(h)Z < X(h)s(h)Y(h)tZ = XsYtZ$, since $XY = X(h)Y(h)$. □

Corollary 4.4. *Let $X, Y, Z \in W$, $s, t \in S$. If $t \notin \Re(Y) \bigcup \pounds(Z)$ and $P(Y, t, Z) = P(X, s, Y, t, Z)$. Then $XYZ < XsYtZ$.*

Proof. The proof is similar to proof of Theorem 4.3. □

Theorem 4.5. *Let $X, Y, Z \in W$, $s, t \in S$, $s \in \Re(X) \bigcap \pounds(Y)$. If $P(X, s, Y) = P(X, s, Y, t, Z)$. Then $XYZ < XsYtZ$.*

Proof. Since $s \in \Re(X) \bigcap \pounds(Y)$, hence $P(Xs, s, sY) = P(Xs, s, sY, t, Z)$ and $s \notin \Re(Xs) \bigcup \pounds(sY)$. We have $(Xs)(sY)Z < (Xs)s(sY)tZ$ by Theorem 3.4. Thus $XYZ < XsYtZ$. □

Corollary 4.6. *Let $X, Y, Z \in W$, $s, t \in S$, $t \in \Re(Y) \bigcap \pounds(Z)$, If $P(Y, t, Z) = P(X, s, Y, t, Z)$. Then $XYZ < XsYtZ$.*

Proof. The proof is similar to proof of Theorem 4.5.

Let $X, Y, Z \in W$, $s, t \in S$. Let $X(r)s(r)Y(r)t(r)Z(r)$ be an expression obtained from the expression $XsYtZ$ by some Coxeter transformations of kind $\neq (C)$. Namely $XsYtZ \mapsto X(1)s(1)Y(1)t(1)Z(1) \mapsto \cdots \mapsto X(r)s(r)Y(r)t(r)Z(r)$. Where we suppose that these Coxeter transformations do not involve s and t, (if some Coxeter transformation involves s or t, then it must be Coxeter transformation of kind (A).)
Clearly, $XYZ = X(1)Y(1)Z(1) = \cdots = X(r)Y(r)Z(r)$,
$XsYtZ = X(1)s(1)Y(1)t(1)Z(1) = \cdots = X(r)s(r)Y(r)t(r)Z(r)$. □

Theorem 4.7. *Let $X, Y, Z \in W$, $s, t \in S$. If there exist $X(r)s(r)Y(r)t(r)Z(r)$ obtained as above. Let $Y(r) = y_1 y_2 \cdots y_b$ be a reduce expression. If they satisfy either $s(r) \notin \Re(X(r)) \bigcup \pounds(Y_k)$ or $s(r) \in \Re(X(r)) \bigcap \pounds(Y_k)$ and satisfy $t(r) \notin \Re(Y'_k) \bigcup \pounds(Z(k))$ or $t(r) \in \Re(Y'_k) \bigcap \pounds(Z(k))$ and $l(X(r)s(r)Y(r)t(r)Z(r)) = l(X(r)s(r)Y_k) + l(Y_k^{'}t(r)Z(r))$ for some $0 \le k \le b$, where $Y_k = y_1 \cdots y_k$, $Y_k^{'} = y_{k+1} \cdots y_b$. (When $k = 0$ or b, then $Y_k = e$ or Y, e is identity of W.) Then $XYZ < XsYtZ$.*

Proof. We can apply Induction on $P(X(r), s(r), Y(r), t(r), Z(r))$. We know that $P(X(r), s(r), Y(r), t(r), Z(r)) = l(X(r)) + l(Y_k) + l(Y_k^{'}) + l(Z(r)) + 2 - l(X(r)s(r)Y(r)t(r)Z(r))$, $P(X(r), s(r), Y_k) = l(X(r)) + l(Y_k) + 1 - l(X(r)s(r)Y_k)$, $P(Y_k^{'}, t(r), Z(r)) = l((Y_k^{'}) + l(t(r)) + 1 - l(Y_k^{'}t(r)Z(r))$.
Then we have $P(X(r), s(r), Y(r), t(r), Z(r)) = P(X(r), s(r), Y_k) + P(Y_k^{'}, t(r), Z(r))$ if and only if $l(X(r)s(r)Y(r)t(r)Z(r)) = l(X(r)s(r)Y_k) + l(Y_k^{'}t(r)Z(r))$.
Now If $P(X(r), s(r), Y(r), t(r), Z(r)) = 0$,
then $XYZ = X(r)Y(r)Z(r) < X(r)s(r)Y(r)t(r)Z(r) = XsYtZ$.

In case of $P(X(r), s(r), Y(r), t(r), Z(r)) > 0$. We have that either $P(X(r), s(r), Y_k) > 0$ or $P(Y_k^{'}, t(r), Z(r)) > 0$ (or both). We assume that $P(X(r), s(r), Y_k) > 0$ and $s(r) \notin \Re(X(r)) \bigcup \pounds(Y_k)$, then there exist that $X(r,h)s(r,h)Y_k(h)$ obtained from the expression $X(r)s(r)Y_k$ by coxeter transformation of kind (A) (B) by Lemma 1 which satisfy $P(X(r,h), s(r,h), Y_k(h)) < P(X(r), s(r), Y_k)$.
Thus $P(X(r,h), s(r,h), Y_k(h)Y_k^{'}, t(r), Z(r)) = P(X(r,h), s(r,h), Y_k(h)) + P(Y_k^{'}, t(r), Z(r)) < P(X(r), s(r), Y(r), t(r), Z(r))$, $s(r,h) \notin \Re(X(r,h)) \bigcup \pounds(Y_k(h))$, $X(r,h)s(r,h)Y_k(h)Y_k^{'}t(r)Z(r) = X(r)s(r)Y(r)t(r)Z(r) = XsYtZ$
and $X(r,h)Y_k(h)Y_k^{'}Z(r) = X(r)Y(r)Z(r) = XYZ$.
So By induction hypothesis, we have $XYZ = X(r,h)Y_k(h)Y_k^{'}Z(r) < X(r,h)s(r,h)Y_k(h)Y_k^{'}t(r)Z(r) = XsYtZ$.
If $s(r) \in \Re(X(r)) \bigcap \pounds(Y_k)$, then $s(r) \notin \Re(X(r)s(r)) \bigcup \pounds(s(r)Y_k)$,

$$P(X(r)s(r), s(r), s(r)Y(r), t(r), Z(r)) = P(X(r)s(r), s(r), s(r)Y_k) + P(Y_k^{'}, t(r), Z(r)), XsYtZ$$

$$= (X(r)s(r))s(r)(s(r)Y(r))t(r)Z(r)$$

and

$$XYZ = (X(r)s(r))(s(r)Y(r))Z(r).$$

Therefore

$$XYZ = (X(r)s(r))(s(r)Y(r))t(r)Z(r) < (X(r)s(r))s(r)(s(r)Y(r))t(r)Z(r) = XsYtZ.$$

Similarly for $P(Y_k^{'}, t(r), Z(r)) > 0$. □

Let $W = H_4 = \langle s_1, s_2, s_3, s_4 \rangle$ be the with $m_{s_1,s_2} = 5$, $m_{s_2,s_3} = 3$, $m_{s_3,s_4} = 3$, $m_{s_1,s_3} = 2$, $m_{s_1,s_4} = 2$, $m_{s_2,s_4} = 2$. Let $X = s_4 s_2 s_1 s_2$, $Y = s_2 s_1 s_2 s_1$, $Z = s_2 s_4 s_3 s_4$, $s = s_1$, $t = s_1$. Then $XYZ < XsYtZ$. Clearly, they do not satisfy the conditions of Theorem 4.1 and Theorem 4.2. But we can get $XYZ < XsYtZ$ by Theorem 4.7. We will provide the Coxeter transformation's process that $X(r)s(r)Y(r)t(r)Z(r)$ was obtained from $XsYtZ$ by certain Coxeter transformations as following

$$\underbrace{(s_4s_2)s_1s_2}_{X}\underbrace{(s_1)}_{s}\underbrace{s_2s_1s_2s_1}_{Y}\underbrace{(s_1)}_{t}\underbrace{s_2s_4s_3s_4}_{Z}\mapsto\underbrace{(s_2s_4)s_1s_2}_{X(1)}\underbrace{(s_1)}_{s(1)}\underbrace{s_2s_1s_2s_1}_{Y(1)}\underbrace{(s_1)}_{t(1)}\underbrace{s_2s_4s_3s_4}_{Z(1)}$$
$$\mapsto\underbrace{s_2(s_1s_4)s_2}_{X(2)}\underbrace{(s_1)}_{s(2)}\underbrace{s_2s_1s_2s_1}_{Y(2)}\underbrace{(s_1)}_{t(2)}\underbrace{s_2s_4s_3s_4}_{Z(2)}\mapsto\underbrace{s_2s_1(s_2s_4)}_{X(3)}\underbrace{(s_1)}_{s(3)}\underbrace{s_2s_1s_2s_1}_{Y(3)}\underbrace{(s_1)}_{t(3)}\underbrace{s_2s_4s_3s_4}_{Z(3)}$$
$$\mapsto\underbrace{s_2s_1s_2}_{X(4)}\underbrace{(s_1)}_{s(4)}\underbrace{s_4s_2s_1s_2s_1}_{Y(4)}\underbrace{(s_1)}_{t(4)}\underbrace{s_2s_4s_3s_4}_{Z(4)}\mapsto\underbrace{s_2s_1s_2}_{X(5)}\underbrace{(s_1)}_{s(5)}\underbrace{(s_2s_4)s_1s_2s_1}_{Y(5)}\underbrace{(s_1)}_{t(5)}\underbrace{s_2s_4s_3s_4}_{Z(5)}$$
$$\mapsto\underbrace{s_2s_1s_2}_{X(6)}\underbrace{(s_1)}_{s(6)}\underbrace{s_2(s_1s_4)s_2s_1}_{Y(6)}\underbrace{(s_1)}_{t(6)}\underbrace{s_2s_4s_3s_4}_{Z(6)}\mapsto\underbrace{s_2s_1s_2}_{X(7)}\underbrace{(s_1)}_{s(7)}\underbrace{s_2s_1(s_2s_4)s_1}_{Y(7)}\underbrace{(s_1)}_{t(7)}\underbrace{s_2s_4s_3s_4}_{Z(7)}$$
$$\mapsto\underbrace{s_2s_1s_2}_{X(8)}\underbrace{(s_1)}_{s(8)}\underbrace{s_2s_1s_2s_1s_4}_{Y(8)}\underbrace{(s_1)}_{t(8)}\underbrace{s_2s_4s_3s_4}_{Z(8)}$$

We can see $s(8)\notin\Re(X(8))\bigcup\pounds(Y_4(8))$, $t(8)\notin\Re(Y_4'(8))\bigcup\pounds(Z(8))$ and
$l(X(8)s(8)Y(8)t(8)Z(8))$
$=l(X(8)s(8)Y_4)+l(Y_4'(8)t(8)Z(8))$. Thus $XYZ<XsYtZ$.

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