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Remotality in Topological Vector Spaces

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Abstract

In this paper we introduce the concept of remotality in topological vector spaces. Some results are proved.

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1 Introduction

Let X be a Banach space and E a bounded set in X. For $x \in X$, set

$$D(x,E) = \sup\{\|x - y\| : y \in E\}$$

The set E is called remotal if for any $x \in X$ there exists $z \in E$ such that $D(x,E) = \|x - z\|$. The point z is called a farthest point of E from x. The concept of remotal sets in Banach spaces goes back to the sixties, [7]. The study of remotal sets is little more difficult and less developed than that of proximinal sets. While best approximation has applications in many branches of mathematics,[4], remotal sets and farthest points have applications in the study of geometry of Banach spaces, [3], [6]. For further results on remotal sets we refer to [1], [2], and [5]. The concept of remotal sets out the scope of normed and metric spaces never been introduced or studied. It is the object of this paper to introduce the concept of remotality in topological vector spaces where no metric or norm is available.

2 Basic Definitions

Definition 1.1: Let X be a vector space. A subset E of X is called **balanced** if $\forall \alpha \in \mathbb{R}, |\alpha| \leq 1$ we have $\alpha E \subseteq E$. Where $\alpha E = \{\alpha e : e \in E\}$.**E** is called **absolutely convex** if it is convex and

balanced. The set E is called **absorbing** if for each $x \in X$, there exists $r \in \mathbb{R}$ such that $x \in \alpha E, \forall \alpha > r$, where $\alpha E = \{\alpha e : e \in E\}$.

Let (X, τ) be a topological vector space. X is called **locally convex** if there is a local basis at $\{0\}$ consisting of open, balanced convex sets.

Observe that, if (X, τ) is a topological vector space then the translation function $f : X \to X : x \to x + a$, for a fixed $a \in X$, is a continuos (1-1) function. Hence every neighborhood base of $\{0\}$

can be translated to be a neighborhood base for any given x in X.

Through out this paper, we assume that $0 \in (X, \tau)$ has a **balanced convex absorbing** neighborhood, which we denote by $U(0)$.

Notation1.2

(i) $tU(0) = \{ty : y \in U(0)\}$. Notice that if $0 < t \leq 1$ then $tU(0) \subseteq U(0)$, and if $1 < t < \infty$, then $U(0) \subseteq tU(0)$.

(1) The open neighborhood of zero $U(0)$ will be written as $U(0, 1)$. This means that $U(0)$ is multiplied by 1, and $tU(0)$ will be written as $U(0, t)$

(ii) $tU(x)$ denotes $tU(0) + x$. Which means we enlarged $U(x)$ in all directions by t, keeping x fixed. We will call x the center of $U(x)$.

Remark 1.3. (**i**). $U(a) = U(0) + a$ is a balanced convex absorbing neighborhood of $\{a\}$.

(ii). $tU(0)$ is a balanced convex absorbing neighborhood of $\{0\}$, $\forall t > 0$.

(iii).A subset E of X is called **bounded** if there exists some $t \geq 0$, such that $E \subseteq tU(0)$.

Definition 1.4. For any $x, y \in (X, \tau)$ we define:

(i) $[x, y] = \{tx + (1 - t)y : 0 \leq t \leq 1\}$, and we will call it the segment joining x to y.

(ii) $[x, y, -] = [x, y] \cup \{z \in X : y \in [x, z]\}$.

(iii) $[-, x, y] = [x, y] \cup \{z \in X : x \in [z, y]\}$.

$(iv)U(x, [x, y])$ denotes the open neighborhood centered at x, with $y \in \partial U(x, [x, y])$. So the whole segment $[x, y]$ is in $U(x, [x, y])$.

Since $U(0) = U(0, 1)$ is assumed to be absorbing, then for any $x \in X$, $x \notin \overline{U(0, 1)}$, there exists $t > 0$ such that $tx \in \partial \overline{U(0, 1)}$. Now, $-tx \in \partial U(x, [x, -tx])$, by definition. Thus $U(0, [0, tx]) = U(0, 1) \subseteq U(x, [x, -tx])$

Lemma 1.5. For every $x, y \in X$, there exists $t \in (0, \infty)$ such that $y \in \partial(tU(x))$, the **boundary** of $tU(x)$.

Proof. Let us prove the result for $x = 0$. Let y be any element in X. If $y \notin \partial U(0)$, then $[0, y, -] \cap U(0) \neq \varphi$, since $U(0)$ is assumed absorbing. More precisely, $[0, y, -] \cap \partial U(0) = \{w\}$. Then $y = tw$, and $y \in \partial tU(0)$.Since the translation map is continuos, then this result is still true for any $x \in X$.

3 Remotality in topological vector spaces

Here we introduce a definition of farthest points, and remotal sets. The new definition can be applied in a topological vector space that is not necessarily a normed space.

Definition 2.1. Let E be a bounded set in the topological vector space (X, τ). Let $x \in X$. The point $e \in E$ is called a **farthest point** in E from x, if there exists a $t \in (0, \infty)$ such that

$e \in \partial E \cap \partial(tU(x))$ and $E \subseteq tU(x)$. Observe that, as E is assumed to be

bounded we can guarantee that $E \subseteq \overline{tU(x)}$ (the closure of $tU(x)$) for some t. A set E in a topological vector space X is called **remotal** if every $x \in X$ has a farthest point in E.

Theorem 4.2.1: Every compact set in a topological vector space is remotal.

Proof . Let $E \subseteq X$ be a compact set. Since the translation is a continuos open map, it is enough to show that 0 has a farthest point in E. So, let $U(0)$ be the balanced symmetric convex

absorbing neighborhood of zero. Let

$$J = \{tU(0) : 0 < t < \infty\}$$

Clearly J is an open cover for E. But the set E is compact. So there is a finite subcover for E, say

$$J_1 = \{t_1U(0), ...t_nU(0)\}, t_1 < t_2 < .... < t_n$$

Define

$$J_2 = \{t_1\overline{U(0)}, ...t_n\overline{U(0)}\}$$

J_2 is still a cover for E. Since

$$t_1\overline{U(0)} \subset t_2\overline{U(0)}....... \subset t_n\overline{U(0)}$$

then $E \subseteq t_n\overline{U(0)}$.

Let t_n be the least possible real number such that $E \subseteq t_n\overline{U(0)}$,and let

$$s = \inf\{t : E \subseteq t\overline{U(0)}\}$$

If $E \cap \partial_n\overline{U(0)} = \varphi$, then we have two cases :

(i) There exists $s < t_n$ such that $E \subseteq s\overline{U(0)}$, but this contradicts the previous assumption about t_n.

(ii) There exists a sequence $x_n \in E$, such that $x_n \in \partial s_k U(0)$. Where $s_k U(0) \cap E \neq \varphi$, and

$$\bigcup_k s_k U(0) = t_n U(0)$$

Since E is compact, (x_n) has a convergent subsequence say $x_{n_k} \to x \in E$.

$$(x_{n_k}) \in E \cap \partial s_k U(0) \subseteq E \subseteq \bigcup_k s_k U(0).$$

Thus, $x \in t_n\overline{U(0)}$. But $\bigcup_k \overline{s_k U(0)} = t_n\overline{U(0)}$, and $x \notin \left(\overline{s_k U(0)}\right)^o, \forall k$.

Thus, $x \in t_n\overline{U(0)}$ and $x \notin \left(t_n\overline{U(0)}\right)^o$. So, $x \in \partial\left(t_n\overline{U(0)}\right)$. Hence, $x \in E \cap \partial\left(t_n\overline{U(0)}\right)$, and E is remotal. This ends the proof

Theorem 4.2.2 : In a topological space (X, τ), the set $\overline{U(0,1)}$ is remotal.

Proof: Let

$$E = \overline{U(0,1)}, x \in X.$$

If $x = 0$, then for any

$$y \in \partial\overline{U(0,1)}, y \in F(x, E).$$

If $x \neq 0$, then there exists $t > 0$ such that $tx \in \partial\overline{U(0,1)}$ as$\overline{U(0,1)}$ is absorbing . But since $U(0,1)$ is symmetric then $-tx \in \partial\overline{U(0,1)}$. Now, consider the set $\overline{U(x,[x,-tx])}$, by the previous

assumption,

$$-tx \in \partial U(x, [x, -tx])$$

and $-tx \in \partial\overline{U(0,1)}$.

Further,

$$U(0,1) \subseteq U(x, [x, -tx]).$$

Hence, $-tx \in F(x, E)$,and the set $\overline{U(0,1)}$ is remotal.

Corollary 4.2.1: The set $\overline{U(x,[x,y])}$ is remotal for any x, y in a topological vector space X.

Proof: This follows directly by the continuity of the translation mapping on a topological vector space, and using Theorem 4.2.2.

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