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Magnetic Pseudo Null and Magnetic Null Curves in Minkowski 3-Space

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Abstract

In this paper, firstly we define the notions of T-magnetic, N-magnetic and B-magnetic pseudo null curves in Minkowski 3-space, obtain the magnetic vector field V when the pseudo null curve is a T-magnetic, N-magnetic and B-magnetic trajectory of V and give an example for these magnetic curves. After, we define the notions of ξ-magnetic, N-magnetic and W-magnetic null curves in Minkowski 3-space. Also, by obtaining the Lorentz force according to the Cartan frame of these curves, we investigate the existence of a magnetic vector field V of a curve α to be α is a ξ-magnetic, N-magnetic or W-magnetic null trajectory of V.

Mathematics Subject Classification: 53C50, 53C80

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1 Introduction

Recently, magnetic curves that have been proposed for computer graphics purposes are a particle tracing technique that generates a wide variety of curves and spirals under the influence of a magnetic field. In a uniform magnetic field, the motion of a particle of charge q and mass m, travelling with velocity $\vec{v}$ under magnetic induction $\vec{B}$ is the result of Lorentz force, $F = q(\vec{v} \times \vec{B})$, which can be written as $m\frac{d\vec{v}}{dt} = q(\vec{v} \times \vec{B})$, where $\times$ represents the cross product operation. It describes the motion of charged particles experiencing Lorentz force. In [19], the authors have obtained the components of magnetic curves and investigated the magnetic curves with constant logarithmic curvature graph (LCG) and logarithmic torsion graph (LTG) gradient.

Also, the magnetic curves on a Riemannian manifold (M, g) are trajectories of charged particles moving on M under the action of a magnetic field F. A *magnetic field* is a closed 2-form F on M and the *Lorentz force* of the magnetic field F on (M, g) is a (1,1)-tensor field Φ given by $g(\Phi(X), Y) = F(X, Y)$, for any vector fields $X, Y \in \chi(M)$. In dimension 3, the magnetic fields may be defined using divergence-free vector fields. As Killing vector fields have zero divergence, one may define a special class of magnetic fields called *Killing magnetic fields*.

Different approaches in the study of magnetic curves for a certain magnetic field and on the fixed energy level have been rewieved by Munteanu in [14]. He has emphasized them in the case when the magnetic trajectory corresponds to a Killing vector field associated to a screw motion in the Euclidean 3-space. In [15], the authors have investigated the trajectories of charged particles moving in a space modeled by the homogeneous 3-space $S^2 \times \mathbb{R}$ under the action of the Killing magnetic fields.

In [18], the authors have classified all magnetic curves in the 3-dimensional Minkowski space corresponding to the Killing magnetic field $V = a\partial_x + b\partial_y + c\partial_z$, with $a, b, c \in \mathbb{R}$. They have found that, they are helices in E^3_1 and draw the most relevant of them. In 3D semi-Riemannian manifolds, Özdemir et al. have determined the notions of T-magnetic, N-magnetic and B-magnetic curves and give some characterizations for them, where T, N an B are the tangent, normal and binormal vectors of a curve α, respectively [16].

In any 3D Riemannian manifold (M, g), magnetic fields of nonzero constant length are one to one correspondence to almost contact structure compatible to the metric g. From this fact, many authors have motivated to study magnetic curves with closed fundamental 2-form in almost contact metric 3-manifolds, Sasakian manifolds, quasi-para-Sasakian manifolds and etc (see [4], [10], [11], [17]).

On the other hand, recently the geometry of null curves and pseudo null curves have been studied by many mathematicians according to Cartan frame and Frenet frame, respectively (see [1], [6], [7], [8], [9], [12]).

In this paper, firstly we define the notions of T-magnetic, N-magnetic and B-magnetic pseudo null curves in Minkowski 3-space, obtain the magnetic vector field V when the pseudo null curve is a T-magnetic, N-magnetic and

B-magnetic trajectory of V and give an example for these magnetic curves. After, we define the notions of ξ-magnetic null curve, N-magnetic null curve and W-magnetic null curve in Minkowski 3-space. Later by obtaining the Lorentz force according to the Cartan frame of these curves, we prove that, if α is a null curve in a Minkowski 3-space, then there isn't a magnetic vector field V of a curve α to be α is a ξ-magnetic, N-magnetic or W-magnetic null trajectory of V. Also, we see that, the magnetic vector field V of the curve α to be α is a ξ-magnetic, N-magnetic or W-magnetic null trajectory of V can only be the zero vector.

2 Preliminaries

Firstly, we will recall the Frenet equations of a pseudo null curve and Cartan equations of a null curve in Minkowski 3-space.

Let R_1^3 be a 3-dimensional Minkowski space defined as a space to be usual 3-dimensional vector space consisting of vectors $\{(x^0, x^1, x^2) : x^0, x^1, x^2 \in \mathbb{R}\}$, but with a linear connection ∇ corresponding to its Minkowski metric g given by

$$g(x, y) = x^0y^0 + x^1y^1 - x^2y^2.$$

Here, there are three categories of vector fields, namely,

spacelike if $g(X, X) > 0$ or $X = 0$,

timelike if $g(X, X) < 0$,

lightlike if $g(X, X) = 0$, $X \neq 0$. In general, the type into which a given vector field X falls is called the *causal character* of X [8].

Now, let we assume that the curve α is a spacelike curve with the tangent vector T. Let the vector $T'(s)$ is null for any s such that $T'(s) \neq 0$ and it is not proportional to $T(s)$. We define the principal normal vector as $N(s) = T'(s)$ which is independent linear with $T(s)$. Let B be the unique null vector such that $g(N, B) = -1$ and orthogonal to T. The vector $B(s)$ is called the binormal vector of α at s. Then, the Frenet equations are

$$\begin{aligned} T' &= \nabla_T T = N, \\ N' &= \nabla_T N = -\tau N, \\ B' &= \nabla_T B = T + \tau B, \end{aligned} \tag{1}$$

where the following equations are satisfy:

$$\begin{aligned} g(T, T) &= 1, \;\; g(N, N) = g(B, B) = 0, \\ g(N, B) &= -1, \; g(T, N) = g(T, B) = 0 \end{aligned} \tag{2}$$

with the vector product $\times$ given by

$$B \times N = T, \quad T \times N = N, \quad B \times T = B. \tag{3}$$

Here, the function τ is called the torsion of α at s and the first curvature of α is equal to 1 (we know that, the first curvature of α is equal to 0 if α is a straight line and it is equal to 1 in the all other cases and here we will assume that the curve α is not a straight line). This spacelike curve α with null principal normal N is called a *pseudo null curve* [9].

Now, let the curve α be a null curve which preserves its causal character. Then, all its tangent vectors are null. To derive the Frenet type equations of a null curve α, defined by $\alpha : [a, b] \longrightarrow \mathbb{R}^3_1$, Cartan [5] has shown that with respect to an affine parameter, say p, and a positively oriented set $\{\alpha'(p), \alpha''(p), \alpha'''(p)\}$, $\forall p \in [a, b]$, there exist a local frame $F = \{\xi = \alpha', N, W\}$, called *Cartan frame* satisfying

$$\begin{aligned} g(\xi,\xi) &= g(N,N) = 0, \quad g(W,W) = 1, \\ g(W,\xi) &= g(W,N) = 0, \quad g(\xi,N) = 1, \end{aligned} \tag{4}$$

with the vector product $\times$ given by

$$\xi \times W = -\xi, \quad \xi \times N = -W, \quad W \times N = -N. \tag{5}$$

The Cartan equations are given by

$$\begin{aligned} \xi' &= \nabla_\xi \xi = \kappa W, \\ N' &= \nabla_\xi N = -\tau W, \\ W' &= \nabla_\xi W = -\tau\xi + \kappa N, \end{aligned} \tag{6}$$

where κ and τ are the curvature and torsion functions of α with respect to F, respectively. Here, p is called *distinguished parameter* of α. Also, the vector fields N and W define the line bundles $ntr(\alpha)$ and $S(T\alpha^\perp)$ over α, respectively. The line bundle $S(T\alpha^\perp)$ is called the *screen vector bundle* and $ntr(\alpha)$ the *null transversal vector bundle* of α with respect to $S(T\alpha^\perp)$ (for detail, see [1], [6], [7] and [8]).

Now, we will give some informations about the magnetic curves in 3-dimensional semi-Riemannian manifolds.

A divergence-free vector field defines a magnetic field in a three-dimensional semi-Riemannian manifold M. It is known that, $V \in \chi(M^n)$ is a Killing vector field if and only if $L_V g = 0$ or, equivalently, $\nabla V(p)$ is a skew-symmetric operator in $T_p(M^n)$, at each point $p \in M^n$. It is clear that, any Killing vector field on (M^n, g) is divergence-free. In particular, if $n = 3$, then every Killing vector field defines a magnetic field that will be called a *Killing magnetic field* [2].

Let (M, g) be an n-dimensional semi-Riemannian manifold. A *magnetic field* is a closed 2-form F on M and the *Lorentz force* Φ of the magnetic field F on (M, g) is defined to be a skew-symmetric operator given by

$$g(\Phi(X), Y) = F(X, Y), \quad \forall X, Y \in \chi(M). \tag{7}$$

The *magnetic trajectories* of F are curves α on M that satisfy the *Lorentz equation* (sometimes called the *Newton equation*)

$$\nabla_{\alpha'}\alpha' = \Phi(\alpha'). \tag{8}$$

The Lorentz equation generalizes the equation satisfied by the geodesics of M, namely $\nabla_{\alpha'}\alpha' = 0$.

Note that, one can define on M the cross product of two vectors $X, Y \in \chi(M)$ as follows

$$g(X \times Y, Z) = dv_g(X, Y, Z), \quad \forall Z \in \chi(M).$$

If V is a Killing vector field on M, let $F_V = \imath_V dv_g$ be the corresponding Killing magnetic field. By $\imath$ we denote the inner product. Then, the Lorentz force of F_V is

$$\Phi(X) = V \times X.$$

Consequently, the Lorentz force equation (8) can be written as

$$\nabla_{\alpha'}\alpha' = V \times \alpha' \tag{9}$$

(for detail see [14], [16]).

3 Magnetic Pseudo Null Curves in Minkowski 3-Space

In this section, we will investigate the T-magnetic, N-magnetic and B-magnetic pseudo null curves in Minkowski 3-space. Also, we obtain the magnetic vector field V when the pseudo null curve is a T-magnetic, N-magnetic and B-magnetic trajectory of V and give an example for these magnetic curves.

3.1 T-Magnetic Pseudo Null Curves in Minkowski 3-Space

Definition 3.1. *Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3_1$ be a pseudo null curve in a Minkowski 3-space and F_V be a magnetic field in $\mathbb{R}^3_1$. If the tangent vector field of the curve satisfies the Lorentz force equation $\nabla_{\alpha'}\alpha' = \Phi(\alpha') = V \times \alpha'$, then the curve α is called a T-**magnetic pseudo null curve**.*

Proposition 3.2. *Let α be a unit speed T-magnetic pseudo null curve in Minkowski 3-space. Then, the Lorentz force according to the Frenet frame is obtained as*

$$\begin{bmatrix} \Phi(T) \\ \Phi(N) \\ \Phi(B) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\rho & 0 \\ 1 & 0 & \rho \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{10}$$

where ρ is a certain function defined by $\rho = g(\Phi N, B)$.

Proof. Let α be a T-magnetic pseudo null curve in Minkowski 3-space with the Frenet apparatus $\{T, N, B, \kappa = 1, \tau\}$. From the definition of the T-magnetic pseudo null curve, we know that $\Phi(T) = N$. On the other hand, since $\Phi(N) \in Sp\{T, N, B\}$, we have $\Phi(N) = a_1T + a_2N + a_3B$. So, we get from (2)

$$\begin{aligned} a_1 &= g(\Phi N, T) = -g(N, \Phi T) = -g(N, N) = 0, \\ a_2 &= -g(\Phi N, B) = -\rho, \\ a_3 &= -g(\Phi N, N) = 0 \end{aligned}$$

and hence we obtain that, $\Phi(N) = -\rho N$.

Furthermore, from $\Phi(B) = b_1T + b_2N + b_3B$, we have

$$\begin{aligned} b_1 &= g(\Phi B, T) = -g(B, \Phi T) = -g(B, N) = 1, \\ b_2 &= -g(\Phi B, B) = 0, \\ b_3 &= -g(\Phi B, N) = g(B, \Phi N) = \rho \end{aligned}$$

and so, we can write $\Phi(B) = T + \rho B$, which completes the proof. □

Proposition 3.3. *Let α be a unit speed T-magnetic pseudo null curve in Minkowski 3-space. Then, the pseudo null curve α is a T-magnetic trajectory of a Killing magnetic vector field V if and only if the Killing magnetic vector field V is*

$$V = -\rho T - N \tag{11}$$

along the curve α.

Proof. Let α be a T-magnetic pseudo null trajectory of a Killing magnetic vector field V. Using Proposition 3.2 and taking $V = aT + bN + cB$; from $\Phi(T) = V \times T$, we get

$$b = -1, \ c = 0;$$

from $\Phi(N) = V \times N$, we get

$$a = -\rho, \ c = 0$$

and from $\Phi(B) = V \times B$, we get

$$a = -\rho, \ b = -1$$

and so the Killing magnetic vector field V can be written by (11). Conversely, if the magnetic vector field V is the form of (11), then one can easily see that $V \times T = \Phi(T)$ holds. So, the pseudo null curve α is a T-magnetic projectory of the magnetic vector field V. □

Corollary 3.4. *Let α be a unit speed T-magnetic pseudo null curve in Minkowski 3-space. Then, the Killing magnetic vector field V whose T-magnetic trajectory is the pseudo null curve α is a spacelike vector.*

Proof. The proof is obvious from (2) and (11). □

3.2 N-Magnetic Pseudo Null Curves in Minkowski 3-Space

Definition 3.5. *Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3_1$ be a pseudo null curve in Minkowski 3-space and F_V be a magnetic field in $\mathbb{R}^3_1$. If the principal normal vector field of the curve satisfies the Lorentz force equation $\nabla_{\alpha'} N = \Phi(N) = V \times N$, then the curve α is called an N-**magnetic pseudo null curve**.*

Proposition 3.6. *Let α be a unit speed N-magnetic pseudo null curve in Minkowski 3-space. Then, the Lorentz force according to the Frenet frame is obtained as*

$$\begin{bmatrix} \Phi(T) \\ \Phi(N) \\ \Phi(B) \end{bmatrix} = \begin{bmatrix} 0 & -\mu & 0 \\ 0 & -\tau & 0 \\ -\mu & 0 & \tau \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{12}$$

where μ is a certain function defined by $\mu = g(\Phi T, B)$.

Proof. The proof is similar with the proof of Proposition 3.2. □

Proposition 3.7. *Let α be a unit speed N-magnetic pseudo null curve in Minkowski 3-space. Then, the pseudo null curve α is an N-magnetic trajectory of a Killing magnetic vector field V if and only if the Killing magnetic vector field V is*

$$V = -\tau T + \mu N \tag{13}$$

along the curve α.

Proof. The proof is similar with the proof of Proposition 3.3. □

Corollary 3.8. *Let α be a unit speed N-magnetic pseudo null curve in Minkowski 3-space. Then, the Killing magnetic vector field V whose N-magnetic trajectory is the pseudo null curve α is a spacelike vector.*

Proof. The proof is obvious from (2) and (13). □

3.3 B-Magnetic Pseudo Null Curves in Minkowski 3-Space

Definition 3.9. *Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3_1$ be a pseudo null curve in Minkowski 3-space and F_V be a magnetic field in $\mathbb{R}^3_1$. If the binormal vector field of the curve satisfies the Lorentz force equation $\nabla_{\alpha'} B = \Phi(B) = V \times B$, then the curve α is called a* ***B-magnetic pseudo null curve****.*

Proposition 3.10. *Let α be a unit speed B-magnetic pseudo null curve in Minkowski 3-space. Then, the Lorentz force according to the Frenet frame is obtained as*

$$\begin{bmatrix} \Phi(T) \\ \Phi(N) \\ \Phi(B) \end{bmatrix} = \begin{bmatrix} 0 & 1 & -\gamma \\ -\gamma & -\tau & 0 \\ 1 & 0 & \tau \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{14}$$

where α is a certain function defined by $\gamma = g(\Phi T, N)$.

Proof. The proof is similar with the proof of Proposition 3.2. □

Proposition 3.11. *Let α be a unit speed B-magnetic pseudo null curve in Minkowski 3-space. Then, the pseudo null curve α is a B-magnetic trajectory of a Killing magnetic vector field V if and only if the Killing magnetic vector field V is*

$$V = -\tau T - N - \gamma B \tag{15}$$

along the curve α.

Proof. The proof is similar with the proof of Proposition 3.3. □

Corollary 3.12. *Let α be a unit speed B-magnetic pseudo null curve in Minkowski 3-space. Then, the Killing magnetic vector field V whose B-magnetic trajectory is the pseudo null curve α is*

i) spacelike, if $\tau^2 > 2\gamma$;
ii) timelike, if $\tau^2 < 2\gamma$, $\gamma > 0$;
iii) null, if $\tau^2 = 2\gamma$, $\gamma \geq 0$.

Proof. The proof is obvious from (2) and (15). □

Example 3.13. *Let us consider the curve*

$$\alpha(s) = (e^s, s, e^s) \tag{16}$$

in Minkowski 3-space. Then, the tangent vector T and normal vector N of this curve are

$$T(s) = \alpha'(s) = (e^s, 1, e^s) \tag{17}$$

and

$$N(s) = T'(s) = (e^s, 0, e^s), \tag{18}$$

respectively. Here, one can easily see that, the tangent vector T is spacelike and the normal vector N is null. Now, the null binormal vector B of the curve α satisfying the conditions

$$\begin{aligned} g(N,B) &= -1,\ g(T,B)=0, \\ B\times N &= T,\ B\times T = B \end{aligned} \tag{19}$$

is found as

$$B(s)=\left(\frac{e^{2s}-1}{2e^{s}},1,\frac{e^{2s}+1}{2e^{s}}\right). \tag{20}$$

Thus, the curve α is a pseudo null curve in Minkowski 3-space. Also, one can easily see that, the torsion of α is $\tau=-1$.

Now, let us find the magnetic vector field V when the pseudo null curve α is a T-magnetic, N-magnetic and B-magnetic trajectory of the magnetic vector field V, respectively:

i) *If the pseudo null curve α is T-magnetic, then from Proposition 3.2, taking*

$$\Phi=\begin{bmatrix} 0 & 1 & 0 \\ 0 & -\rho & 0 \\ 1 & 0 & \rho \end{bmatrix}$$

and using (18) and (20), from $\rho=g(\Phi N,B)$, we have

$$\rho=-\frac{e^{2s}+1}{e^{2s}+3}. \tag{21}$$

Using (21) in (11), we obtain the magnetic vector field V as

$$V=\frac{1}{e^{2s}+3}\left(-2e^{s},e^{2s}+1,-2e^{s}\right). \tag{22}$$

Here, it can be seen that, from (17) and (22), $\nabla_{\alpha'}\alpha'=V\times\alpha'$ satisfies. So, the pseudo null curve α is a T-magnetic pseudo null curve with the magnetic vector field (22).

ii) *If the pseudo null curve α is N-magnetic, then from Proposition 3.6, taking (since $\tau=-1$)*

$$\Phi=\begin{bmatrix} 0 & -\mu & 0 \\ 0 & 1 & 0 \\ -\mu & 0 & -1 \end{bmatrix}$$

and using (17) and (20), from $\mu=g(\Phi T,B)$, we have

$$\mu=\frac{3e^{s}+e^{3s}}{e^{s}+e^{2s}-e^{3s}-1}. \tag{23}$$

Using (23) in (13), we obtain the magnetic vector field V as

$$V=\left(\frac{e^{3s}+4e^{2s}-e^{s}}{-e^{3s}+e^{2s}+e^{s}-1},1,\frac{e^{3s}+4e^{2s}-e^{s}}{-e^{3s}+e^{2s}+e^{s}-1}\right),\ s\neq 0. \tag{24}$$

Here, it can be seen that, from (18) and (24), $\nabla_{\alpha'}N=V\times N$ satisfies. So, the pseudo null curve α is an N-magnetic pseudo null curve with the magnetic vector field (24).

iii) *If the pseudo null curve α is B-magnetic, then from Proposition 3.10, taking (since $\tau=-1$)*

$$\Phi=\begin{bmatrix}0 & 1 & -\gamma\\ -\gamma & 1 & 0\\ 1 & 0 & -1\end{bmatrix}$$

and using (17) and (18), from $\gamma=g(\Phi T,N)$, we have

$$\gamma=\frac{e^{s}}{e^{2s}+1}. \tag{25}$$

Using (25) in (15), we obtain the magnetic vector field V as

$$V=\left(\frac{1-e^{2s}}{2(1+e^{2s})},\frac{1+e^{2s}-e^{s}}{1+e^{2s}},-\frac{1}{2}\right). \tag{26}$$

Here, it can be seen that, from (20) and (26), $\nabla_{\alpha'}B=V\times B$ satisfies. So, the pseudo null curve α is a B-magnetic pseudo null curve with the magnetic vector field (26).

When the pseudo null curve α is T-magnetic, the figure of α and V can be drawn as following:

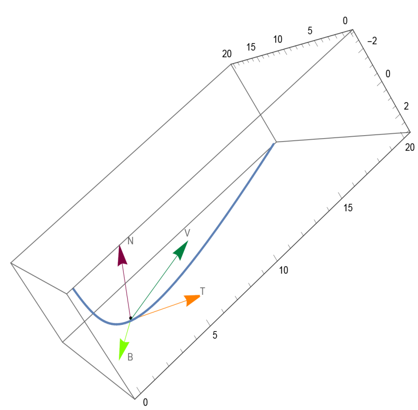

Figure 1: T-magnetic curve α and the T-magnetic vector field V

Similarly, if the pseudo null curve α is N-magnetic and B-magnetic, one can draw the figure of α and V as above.

4 Magnetic Null Curves in Minkowski 3-Space

In [13], the authors have investigated the motion of charged particles near a magnetic null curve using classical trajectory Monte Carlo (CTMC) simulations. They say that, a magnetic null curve is a one-dimensional curve in three-dimensional space where the magnetic field is zero along the curve and non-zero elsewhere. They have investigated the charged particle motion along magnetic null curves and discussed possible applications. Also, they have seen that, the magnetic null curve is formed from the superposition of the magnetic fields of two infinite, straight, parallel wires carrying identical current and a magnetic null line exists directly between the wires.

Here, firstly we will define the notions of ξ-magnetic null curve, N-magnetic null curve and W-magnetic null curve in Minkowski 3-space. Later we will investigate the existence of a magnetic vector field V of a curve α to be α is a ξ-magnetic, N-magnetic or W-magnetic null trajectory of V by obtaining the Lorentz force according to the Cartan frame of these curves.

Definition 4.1. *Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3_1$ be a null curve in Minkowski 3-space and F_V be a magnetic field in $\mathbb{R}^3_1$.*

i) *If the tangent vector field of the curve satisfies the Lorentz force equation $\nabla_{\alpha'}\alpha' = \Phi(\alpha') = V \times \alpha'$, then the curve α is called a* ***ξ-magnetic null curve****;*

ii) *if the normal vector field of the curve satisfies the Lorentz force equation $\nabla_{\alpha'}N = \Phi(N) = V \times N$, then the curve α is called a* ***N-magnetic null curve;***

iii) *if the binormal vector field of the curve satisfies the Lorentz force equation $\nabla_{\alpha'}W = \Phi(W) = V \times W$, then the curve α is called a* ***W-magnetic null curve.***

Proposition 4.2. ***i)*** *Let α be a ξ-magnetic null curve in Minkowski 3-space with the Frenet apparatus $\{\xi, N, W, \kappa, \tau\}$. Then, the Lorentz force according to the Cartan frame is obtained as*

$$\begin{bmatrix} \Phi(\xi) \\ \Phi(N) \\ \Phi(W) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \kappa \\ 0 & 0 & \rho \\ -\rho & -\kappa & 0 \end{bmatrix} \begin{bmatrix} \xi \\ N \\ W \end{bmatrix}, \tag{27}$$

where ρ is a certain function defined by $\rho = g(\Phi N, W)$;

ii) *Let α be an N-magnetic null curve in Minkowski 3-space with the Frenet apparatus $\{\xi, N, W, \kappa, \tau\}$. Then, the Lorentz force according to the Cartan frame is obtained as*

$$\begin{bmatrix} \Phi(\xi) \\ \Phi(N) \\ \Phi(W) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \mu \\ 0 & 0 & -\tau \\ \tau & -\mu & 0 \end{bmatrix} \begin{bmatrix} \xi \\ N \\ W \end{bmatrix}, \tag{28}$$

where μ is a certain function defined by $\mu = g(\Phi\xi, W)$;

iii) *Let α be a W-magnetic null curve in Minkowski 3-space with the Frenet apparatus $\{\xi, N, W, \kappa, \tau\}$. Then, the Lorentz force according to the Cartan frame is obtained as*

$$\begin{bmatrix} \Phi(\xi) \\ \Phi(N) \\ \Phi(W) \end{bmatrix} = \begin{bmatrix} \beta & 0 & -\kappa \\ 0 & -\beta & \tau \\ -\tau & \kappa & 0 \end{bmatrix} \begin{bmatrix} \xi \\ N \\ W \end{bmatrix}, \tag{29}$$

where β is a certain function defined by $\beta = g(\Phi\xi, N)$.

Proof. **i)** Let α be a ξ-magnetic null curve in Minkowski 3-space with the Frenet apparatus $\{\xi, N, W, \kappa, \tau\}$. From the definition of the ξ-magnetic null curve, we know that $\Phi(\xi) = \kappa W$. Since $\Phi(N) \in Sp\{\xi, N, W\}$, we have $\Phi(N) = a_1\xi + a_2N + a_3W$. So, we get

$$\begin{aligned} a_1 &= g(\Phi N, N) = 0, \\ a_2 &= g(\Phi N, \xi) = -g(N, \Phi\xi) = -g(N, \kappa W) = 0, \\ a_3 &= g(\Phi N, W) = \rho \end{aligned}$$

and hence we obtain that, $\Phi(N) = \rho W$.

Furthermore, from $\Phi(W) = b_1\xi + b_2N + b_3W$, we have

$$\begin{aligned} b_1 &= g(\Phi W, N) = -g(W, \Phi N) = -\rho, \\ b_2 &= g(\Phi W, \xi) = -g(W, \Phi\xi) = -g(W, \kappa W) = -\kappa, \\ b_3 &= g(\Phi W, W) = 0 \end{aligned}$$

and so, we can write $\Phi(W) = -\rho\xi - \kappa N$.

ii) and iii) can be proven with the similar procedure in i). □

Theorem 4.3. *Let α be a null curve in Minkowski 3-space. Then, there isn't a nonzero Killing magnetic vector field V of the curve α to be α is a ξ-magnetic, N-magnetic or W-magnetic null trajectory of V.*

Proof. Let α be a ξ-magnetic null trajectory of a magnetic vector field V. Using the Proposition 4.2 and taking $V = a\xi + bN + cW$; from $\Phi(\xi) = V \times \xi$, we get

$$b = \kappa, \ c = 0; \tag{30}$$

from $\Phi(N) = V \times N$, we get

$$a = -\rho, \ c = 0 \tag{31}$$

and from $\Phi(W) = V \times W$, we get

$$a = \rho, \ b = -\kappa. \tag{32}$$

Thus, from (30)-(32), we have a contradiction. If α is an N-magnetic null trajectory or a W-magnetic null trajectory of a magnetic vector field V, then we reach similar contradictions and this completes the proof. □

Corollary 4.4. *Let α be a null curve in Minkowski 3-space. Then, the magnetic vector field V of the curve α to be α is a ξ-magnetic, N-magnetic or W-magnetic null trajectory of V is a zero vector.*

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