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On the Distribution of Certain Subsets of Quadratfrei Numbers

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Abstract

In this article we study the distribution of certain subsets of quadratfrei numbers.

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1 Introduction and Preliminary Results

A positive integer n is quadratfrei if it is either a product of different primes or 1. For example, $n = 2$ and $n = 5.7.23$ are quadratfrei. Let Q be the set of quadratfrei numbers, it is well-known this set has positive density $\frac{6}{\pi^2}$. That is, if $Q(x)$ is the number of quadratfrei numbers not exceeding x we have

$$\lim_{x\to\infty} \frac{Q(x)}{x} = \frac{6}{\pi^2}.$$

More precisely we have the following Theorem.

Theorem 1.1 *The following formula holds*

$$Q(x) = \frac{6}{\pi^2}x + f(x)\sqrt{x},$$

where $f(x)$ is a bounded function for $x \geq 1$.

Proof. See, for example, [2, Chapter XVIII, Theorem 333].

We now establish some theorems and formulae that we need in the next sections.

The following theorem is sometimes called either the principle of inclusion-exclusion or the principle of cross-classification. We now enunciate the principle.

Theorem 1.2 *Let S be a set of N distinct elements, and let $S_1, \ldots, S_r$ be arbitrary subsets of S containing $N_1, \ldots, N_r$ elements, respectively. For $1 \leq i < j < \ldots < l \leq r$, let $S_{ij\ldots l}$ be the intersection of $S_i, S_j, \ldots, S_l$ and let $N_{ij\ldots l}$ be the number of elements of $S_{ij\ldots l}$. Then the number K of elements of S not in any of $S_1, \ldots, S_r$ is*

$$K = N - \sum_{1 \leq i \leq r} N_i + \sum_{1 \leq i < j \leq r} N_{ij} - \sum_{1 \leq i < j < k \leq r} N_{ijk} + \ldots + (-1)^r N_{12\ldots r}.$$

Proof. See, for example, either [2, page 233] or [3, page 84].

In this article (as usual) $\lfloor . \rfloor$ denotes the integer-part function. Note that

$$0 \leq x - \lfloor x \rfloor < 1. \tag{1}$$

The function $\phi(n)$ shall denote the number of positive integers less than or equal to m that are relatively prime to m. This well-known function is called the Euler $\phi(n)-$ function. We have the following Theorem

Theorem 1.3 *The following formula holds*

$$\phi(n) = n \prod_{p/n} \left(1 - \frac{1}{p}\right)$$

with p taking as values the distinct prime divisors of n.

Proof. See, for example, [2, Chapter XVI, Theorem 261].

In this article, the sum $\sum_{n \leq x}$ is interpreted as $\sum_{n=1}^{\lfloor x \rfloor}$.

Theorem 1.4 *(The second Mobius inversion formula) Let $f(x)$ and $g(x)$ be functions defined for $x \geq 1$. If*

$$g(x) = \sum_{n \leq x} f\left(\frac{x}{n}\right) \qquad (x \geq 1)$$

then

$$f(x) = \sum_{n \leq x} \mu(n) g\left(\frac{x}{n}\right) \qquad (x \geq 1)$$

where $\mu(n)$ is the Mobius function.

Proof. See, for example, either [1, Chapter VI, Theorem 19] or [2, Chapter XVI, Theorem 268].

Theorem 1.5 *The following formula holds*

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2}$$

Proof. See, for example, [2, Chapter XVII, Theorem 287 and page 245].

The following formula is well-known

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \qquad (x \neq 1) \tag{2}$$

2 Quadratfrei Multiple of a Set of Primes

Let $q_1, q_2, \ldots, q_s$ be $s \geq 1$ different primes fixed. Let $A_{q_1,\ldots,q_s}$ be the set of positive integers such that in their prime factorization $q_1, q_2, \ldots, q_s$ have odd exponents. Let $A_{q_1,\ldots,q_s}(x)$ be the number of these numbers not exceeding x. We have the following lemma.

Lemma 2.1 *The following asymptotic formula holds*

$$A_{q_1,\ldots,q_s}(x) = \frac{1}{(q_1+1)(q_2+1)\cdots(q_s+1)} x + g_{q_1,\ldots,q_s}(x) \log^s x \tag{3}$$

where the function $g_{q_1,\ldots,q_s}(x)$ is bounded on the interval $\left(\frac{3}{2}, \infty\right)$. That is $|g_{q_1,\ldots,q_s}(x)| < K_1$ on the interval $\left(\frac{3}{2}, \infty\right)$ and K_1 is a positive number. If $1 \leq x \leq \frac{3}{2}$ then $A_{q_1,\ldots,q_s}(x) = 0$.

Proof. Suppose that the k_i $(1 \leq i \leq s)$ are fixed odd numbers and suppose that $q_1 \cdots q_s$ and r_i $(1 \leq r_i \leq q_1 \cdots q_s - 1)$ are relatively prime. Note that the number of r_i is (see Theorem 1.3) $(q_1 - 1) \cdots (q_s - 1)$. Consider the inequality

$$q_1^{k_1} \cdots q_s^{k_s} (q_1 \cdots q_s n - r_i) \leq x \tag{4}$$

Equation (4) gives

$$n \leq \frac{x}{q_1^{k_1+1} \cdots q_s^{k_s+1}} + \frac{r_i}{q_1 \cdots q_s}$$

That is, either

$$n = 1, 2, \ldots, \left\lfloor \frac{x}{q_1^{k_1+1} \cdots q_s^{k_s+1}} + \frac{r_i}{q_1 \cdots q_s} \right\rfloor \tag{5}$$

or

$$\left\lfloor \frac{x}{q_1^{k_1+1}\cdots q_s^{k_s+1}} + \frac{r_i}{q_1\cdots q_s} \right\rfloor = 0 \tag{6}$$

Now, consider the inequality

$$\frac{x}{q_1\cdots q_i^{2h_i}\cdots q_s} + \frac{r_i}{q_1\cdots q_s} < 1$$

We search the positive integers h_i that satisfy this inequality.

This inequality will be true for all r_i if it is true for the greater r_i, namely $q_1\cdots q_s - 1$. Therefore we consider the inequality

$$\frac{x}{q_1\cdots q_i^{2h_i}\cdots q_s} + \frac{q_1\cdots q_s - 1}{q_1\cdots q_s} < 1$$

That is,

$$\frac{x}{q_1\cdots q_i^{2h_i}\cdots q_s} < \frac{1}{q_1\cdots q_s}$$

That is,

$$\frac{x}{q_i^{2h_i}} < \frac{1}{q_i}$$

That is,

$$q_i^{2h_i-1} > x$$

That is

$$h_i > \frac{\log_{q_i} x + 1}{2}$$

Therefore we choose

$$h_i = \left\lfloor \frac{\log_{q_i} x + 1}{2} \right\rfloor + 1 = \frac{1}{2}\log_{q_i} x + \frac{3}{2} - \epsilon_i(x) = \frac{1}{2}\frac{\log x}{\log q_i} + \frac{3}{2} - \epsilon_i(x) \tag{7}$$

where $0 \leq \epsilon_i(x) < 1$. Consequently we have (see (5) and (6))

$$A_{q_1,\ldots,q_s}(x) = \sum_{\substack{k_i+1\in S_i\\ i=1,\ldots,s}} \sum_{r_i} \left\lfloor \frac{x}{q_1^{k_1+1}\cdots q_s^{k_s+1}} + \frac{r_i}{q_1\cdots q_s} \right\rfloor \tag{8}$$

where

$$S_i = \{2, 4, \ldots, 2h_i\} \qquad (i = 1, \ldots, s)$$

If we eliminate $\lfloor . \rfloor$ then equation (8) gives

$$A_{q_1,\ldots,q_s}(x) = \sum_{\substack{k_i+1\in S_i\\ i=1,\ldots,s}} \sum_{r_i} \frac{x}{q_1^{k_1+1}\cdots q_s^{k_s+1}} - F_1(x) + F_2(x)$$

$$
\begin{aligned}
&= (q_1-1)\cdots(q_s-1)x\sum_{\substack{k_i+1\in S_i\\ i=1,\dots,s}}\frac{1}{q_1^{k_1+1}\cdots q_s^{k_s+1}}-F_1(x)+F_2(x)\\
&= (q_1-1)\cdots(q_s-1)x\left(\frac{1}{q_1^2}+\cdots+\frac{1}{q_1^{2h_1}}\right)\cdots\left(\frac{1}{q_s^2}+\cdots+\frac{1}{q_s^{2h_s}}\right)\\
&- F_1(x)+F_2(x) \qquad (9)
\end{aligned}
$$

We have (see (1))

$$
\begin{aligned}
0\le F_1(x) &= \sum_{\substack{k_i+1\in S_i\\ i=1,\dots,s}}\sum_{r_i}\left(\frac{x}{q_1^{k_1+1}\cdots q_s^{k_s+1}}+\frac{r_i}{q_1\cdots q_s}\right)\\
&- \sum_{\substack{k_i+1\in S_i\\ i=1,\dots,s}}\sum_{r_i}\left\lfloor\frac{x}{q_1^{k_1+1}\cdots q_s^{k_s+1}}+\frac{r_i}{q_1\cdots q_s}\right\rfloor\\
&\le (q_1-1)\cdots(q_s-1)2^s h_1\cdots h_s \qquad (10)
\end{aligned}
$$

That is

$$
\begin{aligned}
F_1(x) &= f_1(x)(q_1-1)\cdots(q_s-1)2^s h_1\cdots h_s\\
&= f_1(x)(q_1-1)\cdots(q_s-1)2^s\frac{h_1}{\log x}\cdots\frac{h_s}{\log x}\log^s x \qquad (11)
\end{aligned}
$$

where

$$
0\le f_1(x)\le 1. \qquad (12)
$$

Now, we have (see (7))

$$
\frac{h_i}{\log x}=\frac{1}{2\log q_i}+\left(\frac{3}{2}-\epsilon_i(x)\right)\frac{1}{\log x}
$$

If $x>\frac{3}{2}$ then

$$
0<\frac{1}{\log x}<\frac{1}{\log\frac{3}{2}}
$$

$$
\frac{1}{2}<\frac{3}{2}-\epsilon_i(x)\le\frac{3}{2}
$$

and consequently

$$
\frac{1}{2\log q_i}<\frac{h_i}{\log x}<\frac{1}{2\log q_i}+\frac{3}{2}\frac{1}{\log\frac{3}{2}} \qquad (13)
$$

Equations (11), (12) and (13) give

$$
F_1(x)=g_1(x)\log^s x \qquad (14)
$$

where $g_1(x)$ is bounded in the interval $(3/2, \infty)$.

We have (compare with (10))

$$0 \leq F_2(x) = \sum_{\substack{k_i+1 \in S_i \\ i=1,\ldots,s}} \sum_{r_i} \frac{r_i}{q_1 \cdots q_s} \leq (q_1 - 1) \cdots (q_s - 1) 2^s h_1 \cdots h_s$$

Therefore, in the same way we obtain

$$F_2(x) = f_2(x)(q_1 - 1) \cdots (q_s - 1) 2^s h_1 \cdots h_s = g_2(x) \log^s x \tag{15}$$

where $g_2(x)$ is bounded in the interval $(3/2, \infty)$.

Note that (see (2))

$$\frac{1}{q_i^2} + \frac{1}{q_i^4} + \cdots + \frac{1}{q_i^{2h_i}} = \frac{1}{q_i^2} \left(1 + \frac{1}{q_i^2} + \cdots + \frac{1}{q_i^{2(h_i-1)}}\right)$$

$$= \frac{1}{q_i^2} \left(\frac{1 - \left(\frac{1}{p_i^2}\right)^{h_i}}{1 - \frac{1}{q_i^2}} \right) = \frac{1}{q_i^2 - 1} \left(1 - \frac{1}{q_i^{2h_i}}\right) \qquad (i = 1, \ldots, s) \tag{16}$$

Substituting (16) into (9) we find that

$$A_{q_1,\ldots,q_s}(x) = \frac{1}{(q_1 + 1) \cdots (q_s + 1)} x \prod_{i=1}^{s} \left(1 - \frac{1}{q_i^{2h_i}}\right) - F_1(x) + F_2(x) \tag{17}$$

Note that (see (7))

$$-\frac{1}{q_i^{2h_i}} = \frac{1}{x} \left(-q_i^{2\epsilon_i(x)-3}\right) \tag{18}$$

Now, we have

$$-3 \leq 2\epsilon_i(x) - 3 < -1$$

Therefore

$$q_i^{-3} \leq q_i^{2\epsilon_i(x)-3} < q_i^{-1}$$

and finally

$$-q_i^{-1} < -q_i^{2\epsilon_i(x)-3} \leq -q_i^{-3} \tag{19}$$

On the other hand if $x > \frac{3}{2}$ we have

$$0 < \frac{1}{x^r} < \frac{2^r}{3^r} \qquad (r \geq 1) \tag{20}$$

and

$$0 < \frac{1}{\log^s x} < \frac{1}{\log^s \frac{3}{2}} \tag{21}$$

Equations (17), (18), (19), (20) and (21) give

$$A_{q_1,\ldots,q_s}(x) = \frac{1}{(q_1+1)\cdots(q_s+1)}x + g_3(x)\log^s x - F_1(x) + F_2(x) \tag{22}$$

where $g_3(x)$ is bounded in the interval $(3/2, \infty)$.

Finally equations (22), (14) and (15) give (3). The lemma is proved.

Let $Q_{q_1,\ldots,q_s}(x)$ be the number of quadratfrei numbers multiple of the primes $q_1, \ldots, q_s$ not exceeding x. We have the following theorem.

Theorem 2.2 *The following asymptotic formula holds*

$$Q_{q_1,\ldots,q_s}(x) = \frac{1}{(q_1+1)\cdots(q_s+1)}\frac{6}{\pi^2}x + h_{q_1,\ldots,q_s}(x)\sqrt{x}\log^s x \tag{23}$$

where the function $h_{q_1,\ldots,q_s}(x)$ is bounded on the interval $\left(\frac{3}{2}, \infty\right)$.

Proof. We have (see the definitions of $A_{q_1,\ldots,q_s}(x)$ and $Q_{q_1,\ldots,q_s}(x)$)

$$A_{q_1,\ldots,q_s}(y^2) = \sum_{d\le y} Q_{q_1,\ldots,q_s}\left(\frac{y^2}{d^2}\right) \tag{24}$$

Equation (24) and Theorem 1.4 give

$$Q_{q_1,\ldots,q_s}(y^2) = \sum_{d\le y} \mu(d) A_{q_1,\ldots,q_s}\left(\frac{y^2}{d^2}\right) \tag{25}$$

Suppose $y \ge y_o$ where y_0 is large. Note that

$$1 \le \frac{y^2}{d^2} \le \frac{3}{2}$$

if and only if

$$\sqrt{\frac{2}{3}}\; y \le d \le y \tag{26}$$

Consequently, in this case, we have (see Lemma 2.1)

$$A_{q_1,\ldots,q_s}\left(\frac{y^2}{d^2}\right) = 0 \tag{27}$$

Therefore we have (see (25), (26), (27) and Lemma 2.1)

$$Q_{q_1,\ldots,q_s}(y^2) = \sum_{d\le y} \mu(d) A_{q_1,\ldots,q_s}\left(\frac{y^2}{d^2}\right) = \sum_{d<\sqrt{\frac{2}{3}}\; y} \mu(d) A_{q_1,\ldots,q_s}\left(\frac{y^2}{d^2}\right) + \sum_{\sqrt{\frac{2}{3}}\; y\le d\le y} 0$$

$$= \sum_{d<\sqrt{\frac{2}{3}}\ y} \mu(d) A_{q_1,\ldots,q_s}\left(\frac{y^2}{d^2}\right) = \sum_{d<\sqrt{\frac{2}{3}}\ y} \mu(d) \left(\frac{1}{(q_1+1)\cdots(q_s+1)}\frac{y^2}{d^2}\right.$$

$$+ \left. g_{q_1,\ldots,q_s}\left(\frac{y^2}{d^2}\right)\log^s\left(\frac{y^2}{d^2}\right)\right) = \frac{1}{(q_1+1)\cdots(q_s+1)} y^2 \sum_{d<\sqrt{\frac{2}{3}}\ y} \frac{\mu(d)}{d^2}$$

$$+ \sum_{d<\sqrt{\frac{2}{3}}\ y} \mu(d) g_{q_1,\ldots,q_s}\left(\frac{y^2}{d^2}\right)\log^s\left(\frac{y^2}{d^2}\right) \tag{28}$$

Now, we have (see Theorem 1.5)

$$\sum_{d<\sqrt{\frac{2}{3}}\ y} \frac{\mu(d)}{d^2} = \sum_{d=1}^{\infty}\frac{\mu(d)}{d^2} - \sum_{d\geq\sqrt{\frac{2}{3}}\ y} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2} - \sum_{d\geq\sqrt{\frac{2}{3}}\ y} \frac{\mu(d)}{d^2} \tag{29}$$

On the other hand, we have

$$\left|\sum_{d\geq\sqrt{\frac{2}{3}}\ y} \frac{\mu(d)}{d^2}\right| \leq \sum_{d\geq\sqrt{\frac{2}{3}}\ y} \frac{1}{d^2} \leq \int_{\sqrt{\frac{2}{3}}\ y-1}^{\infty} \frac{1}{x^2}\ dx = \frac{1}{\sqrt{\frac{2}{3}}\ y-1}$$

That is

$$\sum_{d\geq\sqrt{\frac{2}{3}}\ y} \frac{\mu(d)}{d^2} = f_1(y)\frac{1}{\sqrt{\frac{2}{3}}\ y-1} \tag{30}$$

where $-1 \leq f_1(y) \leq 1$.

Equations (29) and (30) give

$$\sum_{d<\sqrt{\frac{2}{3}}\ y} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2} - f_1(y)\frac{1}{\sqrt{\frac{2}{3}}\ y-1}$$

Consequently

$$\frac{1}{(q_1+1)\cdots(q_s+1)} y^2 \sum_{d<\sqrt{\frac{2}{3}}\ y} \frac{\mu(d)}{d^2} = \frac{1}{(q_1+1)\cdots(q_s+1)}\frac{6}{\pi^2}y^2$$

$$- \frac{1}{(q_1+1)\cdots(q_s+1)} f_1(y)\frac{y}{\sqrt{\frac{2}{3}}\ y-1} y = \frac{1}{(q_1+1)\cdots(q_s+1)}\frac{6}{\pi^2}y^2$$

$$+ f_2(y) y \tag{31}$$

where $f_2(y)$ is bounded for $y \geq y_0$.

We have (see Lemma 2.1)

$$\left|\sum_{d<\sqrt{\frac{2}{3}}\ y} \mu(d) g_{q_1,\ldots,q_s}\left(\frac{y^2}{d^2}\right)\log^s\left(\frac{y^2}{d^2}\right)\right| \leq \sum_{d<\sqrt{\frac{2}{3}}\ y} |\mu(d)| \left|g_{q_1,\ldots,q_s}\left(\frac{y^2}{d^2}\right)\right| \log^s\left(\frac{y^2}{d^2}\right)$$

$$\leq K_1 \sum_{d<\sqrt{\frac{2}{3}}\ y} \log^s\left(\frac{y^2}{d^2}\right) \leq K_1 \log^s y^2 \sum_{d<\sqrt{\frac{2}{3}}\ y} 1 \leq K_1 \log^s y^2 \sum_{d\leq y} 1 \leq K_1 y \log^s y^2$$

That is

$$\sum_{d<\sqrt{\frac{2}{3}}\ y} \mu(d) g_{q_1,\ldots,q_s}\left(\frac{y^2}{d^2}\right) \log^s\left(\frac{y^2}{d^2}\right) = f_3(y) y \log^s y^2 \qquad (32)$$

where $-K_1 \leq f_3(y) \leq K_1$ for $y \geq y_0$.

Equations (28), (31) and (32) give ($y \geq y_0$)

$$\begin{aligned} Q_{q_1,\ldots,q_s}(y^2) &= \frac{1}{(q_1+1)\cdots(q_s+1)}\frac{6}{\pi^2}y^2 + f_2(y)y + f_3(y)y\log^s y^2 \\ &= \frac{1}{(q_1+1)\cdots(q_s+1)}\frac{6}{\pi^2}y^2 + f_4(y)y\log^s y^2 \end{aligned}$$

where $f_4(y)$ is bounded for $y \geq y_0$.

Replacing y^2 by x, we obtain ($x \geq y_0^2$)

$$Q_{q_1,\ldots,q_s}(x) = \frac{1}{(q_1+1)\cdots(q_s+1)}\frac{6}{\pi^2}x + f_4(\sqrt{x})\sqrt{x}\log^s x \qquad (33)$$

where $f_4(\sqrt{x})$ is bounded for $x \geq y_0^2$.

If $3/2 < x < y_0^2$ we can write the equality

$$Q_{q_1,\ldots,q_s}(x) = \frac{1}{(q_1+1)\cdots(q_s+1)}\frac{6}{\pi^2}x + f_5(x)\sqrt{x}\log^s x \qquad (34)$$

That is

$$f_5(x) = \frac{1}{\sqrt{x}\log^s x}\left(Q_{q_1,\ldots,q_s}(x) - \frac{1}{(q_1+1)\cdots(q_s+1)}\frac{6}{\pi^2}x\right)$$

Consequently $f_5(x)$ is bounded in the interval $3/2 < x < y_0^2$.

If we put

$$h_{q_1,\ldots,q_s}(x) = f_4(\sqrt{x}) \qquad (x \geq y_0^2)$$

and

$$h_{q_1,\ldots,q_s}(x) = f_5(x) \qquad (3/2 < x < y_0^2)$$

then (33) and (34) give (23). The theorem is proved.

Let $N_{q_1,\ldots,q_s}(x)$ be the number of quadratfrei numbers not divisible by $q_1\cdots q_s$ not exceeding x. We have the following Corollary.

Corollary 2.3 *The following asymptotic formula holds*

$$N_{q_1,\ldots,q_s}(x) = \frac{q_1\cdots q_s}{(q_1+1)\cdots(q_s+1)}\frac{6}{\pi^2}x + f_{q_1,\ldots,q_s}(x)\sqrt{x}\log^s x$$

where the function $f_{q_1,\ldots,q_s}(x)$ *is bounded on the interval* $\left(\frac{3}{2},\infty\right)$.

Proof. If $x > 3/2$ we have (see Theorem 1.2, Theorem 1.1 and Theorem 2.2)

$$\begin{aligned}
N_{q_1,\dots,q_s}(x) &= Q(x) - \sum_{1\le i\le s} Q_{q_i}(x) + \sum_{1\le i<j\le s} Q_{q_i,q_j}(x) - \sum_{1\le i<j<k\le s} Q_{q_i,q_j,q_k}(x) \\
&+ \cdots + (-1)^s Q_{q_1,\dots,q_s}(x) = \frac{6}{\pi^2}x - \sum_{1\le i\le s} \frac{6}{\pi^2}\frac{1}{q_i+1}x \\
&+ \sum_{1\le i<j\le s} \frac{6}{\pi^2}\frac{1}{(q_i+1)(q_j+1)}x \\
&- \sum_{1\le i<j<k\le s} \frac{6}{\pi^2}\frac{1}{(q_i+1)(q_j+1)(q_k+1)}x \\
&+ \cdots + (-1)^s \frac{6}{\pi^2}\frac{1}{(q_1+1)\cdots(q_s+1)}x + f_{q_1,\dots,q_s}(x)\sqrt{x}\log^s x \\
&= \frac{6}{\pi^2}x\left(1-\frac{1}{q_1+1}\right)\cdots\left(1-\frac{1}{q_s+1}\right) + f_{q_1,\dots,q_s}(x)\sqrt{x}\log^s x \\
&= \frac{q_1\cdots q_s}{(q_1+1)\cdots(q_s+1)}\frac{6}{\pi^2}x + f_{q_1,\dots,q_s}(x)\sqrt{x}\log^s x
\end{aligned}$$

The corollary is proved.

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