On the Distribution of Certain Subsets of Quadratfrei Numbers

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Abstract

In this article we study the distribution of certain subsets of quadratfrei numbers.

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1 Introduction and Preliminary Results

A positive integer $n$ is quadratfrei if it is either a product of different primes or 1. For example, $n = 2$ and $n = 5.7.23$ are quadratfrei. Let $Q$ be the set of quadratfrei numbers, it is well-known this set has positive density $\frac{6}{\pi^2}$. That is, if $Q(x)$ is the number of quadratfrei numbers not exceeding $x$ we have

$$\lim_{x \to \infty} \frac{Q(x)}{x} = \frac{6}{\pi^2}.$$ 

More precisely we have the following Theorem.

**Theorem 1.1** The following formula holds

$$Q(x) = \frac{6}{\pi^2}x + f(x)\sqrt{x},$$

where $f(x)$ is a bounded function for $x \geq 1$. 
Proof. See, for example, [2, Chapter XVIII, Theorem 333].

We now establish some theorems and formulae that we need in the next sections.

The following theorem is sometimes called either the principle of inclusion-exclusion or the principle of cross-classification. We now enunciate the principle.

**Theorem 1.2** Let $S$ be a set of $N$ distinct elements, and let $S_1, \ldots, S_r$ be arbitrary subsets of $S$ containing $N_1, \ldots, N_r$ elements, respectively. For $1 \leq i < j < \ldots < l \leq r$, let $S_{ij\ldots l}$ be the intersection of $S_i, S_j, \ldots, S_l$ and let $N_{ij\ldots l}$ be the number of elements of $S_{ij\ldots l}$. Then the number $K$ of elements of $S$ not in any of $S_1, \ldots, S_r$ is

$$K = N - \sum_{1 \leq i \leq r} N_i + \sum_{1 \leq i < j \leq r} N_{ij} - \sum_{1 \leq i < j < k \leq r} N_{ijk} + \ldots + (-1)^r N_{12\ldots r}.$$  

Proof. See, for example, either [2, page 233] or [3, page 84].

In this article (as usual) $\lfloor . \rfloor$ denotes the integer-part function. Note that

$$0 \leq x - \lfloor x \rfloor < 1. \quad (1)$$

The function $\phi(n)$ shall denote the number of positive integers less than or equal to $m$ that are relatively prime to $m$. This well-known function is called the Euler $\phi(n)$—function. We have the following Theorem

**Theorem 1.3** The following formula holds

$$\phi(n) = n \prod_{p | n} \left(1 - \frac{1}{p}\right)$$

with $p$ taking as values the distinct prime divisors of $n$.

Proof. See, for example, [2, Chapter XVI, Theorem 261].

In this article, the sum $\sum_{n \leq x}$ is interpreted as $\sum_{n=1}^{[x]}$.

**Theorem 1.4** (The second Mobius inversion formula) Let $f(x)$ and $g(x)$ be functions defined for $x \geq 1$. If

$$g(x) = \sum_{n \leq x} f \left(\frac{x}{n}\right) \quad (x \geq 1)$$

then

$$f(x) = \sum_{n \leq x} \mu(n) g \left(\frac{x}{n}\right) \quad (x \geq 1)$$

where $\mu(n)$ is the Mobius function.
Proof. See, for example, either [1, Chapter VI, Theorem 19] or [2, Chapter XVI, Theorem 268].

**Theorem 1.5** The following formula holds

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2}$$

Proof. See, for example, [2, Chapter XVII, Theorem 287 and page 245].

The following formula is well-known

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \quad (x \neq 1) \tag{2}$$

## 2 Quadratfrei Multiple of a Set of Primes

Let $q_1, q_2, \ldots, q_s$ be $s \geq 1$ different primes fixed. Let $A_{q_1, \ldots, q_s}$ be the set of positive integers such that in their prime factorization $q_1, q_2, \ldots, q_s$ have odd exponents. Let $A_{q_1, \ldots, q_s}(x)$ be the number of these numbers not exceeding $x$. We have the following lemma.

**Lemma 2.1** The following asymptotic formula holds

$$A_{q_1, \ldots, q_s}(x) = \frac{1}{(q_1 + 1)(q_2 + 1) \cdots (q_s + 1)} x + g_{q_1, \ldots, q_s}(x) \log^s x \tag{3}$$

where the function $g_{q_1, \ldots, q_s}(x)$ is bounded on the interval $\left(\frac{3}{2}, \infty\right)$. That is $|g_{q_1, \ldots, q_s}(x)| < K_1$ on the interval $\left(\frac{3}{2}, \infty\right)$ and $K_1$ is a positive number. If $1 \leq x \leq \frac{3}{2}$ then $A_{q_1, \ldots, q_s}(x) = 0$.

Proof. Suppose that the $k_i$ ($1 \leq i \leq s$) are fixed odd numbers and suppose that $q_1 \cdots q_s$ and $r_i$ ($1 \leq r_i \leq q_1 \cdots q_s - 1$) are relatively prime. Note that the number of $r_i$ is (see Theorem 1.3) $(q_1 - 1) \cdots (q_s - 1)$. Consider the inequality

$$q_1^{k_1} \cdots q_s^{k_s} (q_1 \cdots q_s n - r_i) \leq x \quad (4)$$

Equation (4) gives

$$n \leq \frac{x}{q_1^{k_1+1} \cdots q_s^{k_s+1}} + \frac{r_i}{q_1 \cdots q_s}$$

That is, either

$$n = 1, 2, \ldots, \left\lfloor \frac{x}{q_1^{k_1+1} \cdots q_s^{k_s+1}} + \frac{r_i}{q_1 \cdots q_s} \right\rfloor \tag{5}$$
or
\[
\left| \frac{x}{q_1^{k_1+1} \cdots q_s^{k_s+1}} + \frac{r_i}{q_1 \cdots q_s} \right| = 0 \quad (6)
\]

Now, consider the inequality
\[
\frac{x}{q_1 \cdots q_s^{2h_i} \cdots q_s} + \frac{r_i}{q_1 \cdots q_s} < 1
\]

We search the positive integers \(h_i\) that satisfy this inequality.

This inequality will be true for all \(r_i\) if it is true for the greater \(r_i\), namely \(q_1 \cdots q_s - 1\). Therefore we consider the inequality
\[
\frac{x}{q_1 \cdots q_s^{2h_i} \cdots q_s} + \frac{q_1 \cdots q_s - 1}{q_1 \cdots q_s} < 1
\]

That is,
\[
\frac{x}{q_1 \cdots q_s^{2h_i} \cdots q_s} < \frac{1}{q_1 \cdots q_s}
\]

That is,
\[
\frac{x}{q_i^{2h_i}} < \frac{1}{q_i}
\]

That is,
\[
q_i^{2h_i - 1} > x
\]

That is
\[
h_i > \frac{\log_{q_i} x + 1}{2}
\]

Therefore we choose
\[
h_i = \left\lfloor \frac{\log_{q_i} x + 1}{2} \right\rfloor + 1 = \frac{1}{2} \log_{q_i} x + \frac{3}{2} - \epsilon_i(x) = \frac{1}{2} \log x + \frac{3}{2} - \epsilon_i(x) \quad (7)
\]

where \(0 \leq \epsilon_i(x) < 1\). Consequently we have (see (5) and (6))

\[
A_{q_1, \ldots, q_s}(x) = \sum_{k_i + 1 \in S_i} \sum_{r_i} \left| \frac{x}{q_1^{k_1+1} \cdots q_s^{k_s+1}} + \frac{r_i}{q_1 \cdots q_s} \right| \quad (8)
\]

where
\[
S_i = \{2, 4, \ldots, 2h_i\} \quad (i = 1, \ldots, s)
\]

If we eliminate \([.]\) then equation (8) gives
\[
A_{q_1, \ldots, q_s}(x) = \sum_{k_i + 1 \in S_i} \sum_{r_i} \frac{x}{q_1^{k_1+1} \cdots q_s^{k_s+1}} - F_1(x) + F_2(x)
\]
Subsets of quadratfrei numbers

\[ (q_1 - 1) \cdots (q_s - 1) x \sum_{k_{i+1} \in S_i} \frac{1}{q_1^{k_1+1} \cdots q_s^{k_s+1}} - F_1(x) + F_2(x) \]

\[ = (q_1 - 1) \cdots (q_s - 1) x \left( \frac{1}{q_1^2} + \cdots + \frac{1}{q_1^{2h_1}} \right) \cdots \left( \frac{1}{q_s^2} + \cdots + \frac{1}{q_s^{2h_s}} \right) \]

\[ - F_1(x) + F_2(x) \quad (9) \]

We have (see (1))

\[ 0 \leq F_1(x) = \sum_{k_{i+1} \in S_i} \sum_{r_i} \left( \frac{x}{q_1^{k_1+1} \cdots q_s^{k_s+1}} + \frac{r_i}{q_1 \cdots q_s} \right) \]

\[ - \sum_{k_{i+1} \in S_i} \sum_{r_i} \left( \frac{x}{q_1^{k_1+1} \cdots q_s^{k_s+1}} + \frac{r_i}{q_1 \cdots q_s} \right) \]

\[ \leq (q_1 - 1) \cdots (q_s - 1) 2^{s} h_1 \cdots h_s \quad (10) \]

That is

\[ F_1(x) = f_1(x)(q_1 - 1) \cdots (q_s - 1) 2^{s} h_1 \cdots h_s \]

\[ = f_1(x)(q_1 - 1) \cdots (q_s - 1) 2^{s} \frac{h_1}{\log x} \cdots \frac{h_s}{\log x} \log^s x \quad (11) \]

where

\[ 0 \leq f_1(x) \leq 1. \quad (12) \]

Now, we have (see (7))

\[ \frac{h_i}{\log x} = \frac{1}{2 \log q_i} + \left( \frac{3}{2} - \epsilon_i(x) \right) \frac{1}{\log x} \]

If \( x > \frac{3}{2} \) then

\[ 0 < \frac{1}{\log x} < \frac{1}{\log \frac{3}{2}} \]

\[ \frac{1}{2} < \frac{3}{2} - \epsilon_i(x) \leq \frac{3}{2} \]

and consequently

\[ \frac{1}{2 \log q_i} < \frac{h_i}{\log x} < \frac{1}{2 \log q_i} + \frac{3}{2} \frac{1}{\log \frac{3}{2}} \quad (13) \]

Equations (11), (12) and (13) give

\[ F_1(x) = g_1(x) \log^s x \quad (14) \]
where \( g_1(x) \) is bounded in the interval \((3/2, \infty)\).

We have (compare with (10))

\[
0 \leq F_2(x) = \sum_{k_{i+1} \in S_i} \sum_{r_i} \frac{r_i}{q_1 \cdots q_s} \leq (q_1 - 1) \cdots (q_s - 1)2^s h_1 \cdots h_s
\]

Therefore, in the same way we obtain

\[
F_2(x) = f_2(x)(q_1 - 1) \cdots (q_s - 1)2^s h_1 \cdots h_s = g_2(x) \log^s x
\]  

where \( g_2(x) \) is bounded in the interval \((3/2, \infty)\).

Note that (see (2))

\[
\frac{1}{q_i^2} + \frac{1}{q_i^3} + \cdots + \frac{1}{q_i^{2h_i}} = \frac{1}{q_i^2} \left(1 + \frac{1}{q_i} + \cdots + \frac{1}{q_i^{2(h_i-1)}}\right)
\]

\[
= \frac{1}{q_i^2} \left(1 - \left(\frac{1}{q_i^2}\right)^{h_i}\right) = \frac{1}{q_i^2 - 1} \left(1 - \frac{1}{q_i^{2h_i}}\right) \quad (i = 1, \ldots, s) \tag{16}
\]

Substituting (16) into (9) we find that

\[
A_{q_1, \ldots, q_s}(x) = \frac{1}{(q_1 + 1) \cdots (q_s + 1)} x \prod_{i=1}^s \left(1 - \frac{1}{q_i^{2h_i}}\right) - F_1(x) + F_2(x) \tag{17}
\]

Note that (see (7))

\[
-\frac{1}{q_i^{2h_i}} = \frac{1}{x} (-q_i^{2\epsilon_i(x)-3}) \tag{18}
\]

Now, we have

\[-3 \leq 2\epsilon_i(x) - 3 < -1\]

Therefore

\[q_i^{-3} \leq q_i^{2\epsilon_i(x)-3} < q_i^{-1}\]

and finally

\[-q_i^{-1} < -q_i^{2\epsilon_i(x)-3} \leq -q_i^{-3} \tag{19}\]

On the other hand if \( x > \frac{3}{2} \) we have

\[0 < \frac{1}{x^r} < \frac{2^r}{3^r} \quad (r \geq 1) \tag{20}\]

and

\[0 < \frac{1}{\log^s x} < \frac{1}{\log^s \frac{3}{2}} \tag{21}\]
Equations (17), (18), (19), (20) and (21) give

\[ A_{q_1,\ldots,q_s}(x) = \frac{1}{(q_1 + 1)\cdots(q_s + 1)} x + g_3(x) \log^s x - F_1(x) + F_2(x) \]  

(22)

where \( g_3(x) \) is bounded in the interval \((3/2, \infty)\).

Finally equations (22), (14) and (15) give (3). The lemma is proved.

Let \( Q_{q_1,\ldots,q_s}(x) \) be the number of quadratfrei numbers multiple of the primes \( q_1, \ldots, q_s \) not exceeding \( x \). We have the following theorem.

**Theorem 2.2** The following asymptotic formula holds

\[ Q_{q_1,\ldots,q_s}(x) = \frac{1}{(q_1 + 1)\cdots(q_s + 1)} \frac{6}{\pi^2} x + h_{q_1,\ldots,q_s}(x) \sqrt{x} \log^s x \]  

(23)

where the function \( h_{q_1,\ldots,q_s}(x) \) is bounded on the interval \( (\frac{3}{2}, \infty) \).

Proof. We have (see the definitions of \( A_{q_1,\ldots,q_s}(x) \) and \( Q_{q_1,\ldots,q_s}(x) \))

\[ A_{q_1,\ldots,q_s}(y^2) = \sum_{d \leq y} Q_{q_1,\ldots,q_s} \left( \frac{y^2}{d^2} \right) \]  

(24)

Equation (24) and Theorem 1.4 give

\[ Q_{q_1,\ldots,q_s}(y^2) = \sum_{d \leq y} \mu(d) A_{q_1,\ldots,q_s} \left( \frac{y^2}{d^2} \right) \]  

(25)

Suppose \( y \geq y_0 \) where \( y_0 \) is large. Note that

\[ 1 \leq \frac{y^2}{d^2} \leq \frac{3}{2} \]

if and only if

\[ \sqrt{\frac{2}{3}} y \leq d \leq y \]  

(26)

Consequently, in this case, we have (see Lemma 2.1)

\[ A_{q_1,\ldots,q_s} \left( \frac{y^2}{d^2} \right) = 0 \]  

(27)

Therefore we have (see (25), (26), (27) and Lemma 2.1)

\[ Q_{q_1,\ldots,q_s}(y^2) = \sum_{d \leq y} \mu(d) A_{q_1,\ldots,q_s} \left( \frac{y^2}{d^2} \right) = \sum_{d < \sqrt{\frac{2}{3}} y} \mu(d) A_{q_1,\ldots,q_s} \left( \frac{y^2}{d^2} \right) + \sum_{\sqrt{\frac{2}{3}} y \leq d \leq y} 0 \]
Now, we have (see Theorem 1.5)
\[
\sum_{d<\sqrt{\frac{2}{3}}y} \mu(d) = \sum_{d=1}^{\infty} \mu(d) - \sum_{d \geq \sqrt{\frac{2}{3}}y} \mu(d) = 6 - \sum_{d \geq \sqrt{\frac{2}{3}}y} \mu(d)
\]  
(29)

On the other hand, we have
\[
\left| \sum_{d \geq \sqrt{\frac{2}{3}}y} \frac{\mu(d)}{d^2} \right| \leq \sum_{d \geq \sqrt{\frac{2}{3}}y} \frac{1}{d^2} \leq \frac{1}{\sqrt{\frac{2}{3}}y - 1} \left( \int_{\sqrt{\frac{2}{3}}}^{\frac{1}{\sqrt{\frac{2}{3}}y - 1}} \frac{1}{x^2} dx \right) = \frac{1}{\sqrt{\frac{2}{3}}y - 1}
\]

That is
\[
\sum_{d \geq \sqrt{\frac{2}{3}}y} \frac{\mu(d)}{d^2} = f_1(y) \frac{1}{\sqrt{\frac{2}{3}}y - 1}
\]  
(30)

where \(-1 \leq f_1(y) \leq 1.

Equations (29) and (30) give
\[
\sum_{d \leq \sqrt{\frac{2}{3}}y} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2} - f_1(y) \frac{1}{\sqrt{\frac{2}{3}}y - 1}
\]

Consequently
\[
\frac{1}{(q_1 + 1) \cdots (q_s + 1)} y^2 \sum_{d < \sqrt{\frac{2}{3}}y} \mu(d) = \frac{1}{(q_1 + 1) \cdots (q_s + 1)} \frac{6}{\pi^2} y^2 - \frac{1}{(q_1 + 1) \cdots (q_s + 1)} f_1(y) \frac{y}{\sqrt{\frac{2}{3}}y - 1} y = \frac{1}{(q_1 + 1) \cdots (q_s + 1)} \frac{6}{\pi^2} y^2 + f_2(y) y
\]  
(31)

where \(f_2(y)\) is bounded for \(y \geq y_0\).

We have (see Lemma 2.1)
\[
\left| \sum_{d < \sqrt{\frac{2}{3}}y} \mu(d) g_{q_1, \ldots, q_s} \left( \frac{y^2}{d^2} \right) \log^s \left( \frac{y^2}{d^2} \right) \right| \leq \sum_{d < \sqrt{\frac{2}{3}}y} |\mu(d)| \left| g_{q_1, \ldots, q_s} \left( \frac{y^2}{d^2} \right) \right| \log^s \left( \frac{y^2}{d^2} \right)
\]
\[
\leq K_1 \sum_{d < \sqrt{\frac{2}{3}}y} \log^s \left( \frac{y^2}{d^2} \right) \leq K_1 \log^s y^2 \sum_{d < \sqrt{\frac{2}{3}}y} 1 \leq K_1 \log^s y^2 \sum_{d \leq y} 1 \leq K_1 y \log^s y^2
\]
That is
\[
\sum_{d < \sqrt{\frac{y}{2}}} \mu(d) g_{q_1, \ldots, q_s} \left( \frac{y^2}{d^2} \right) \log^s \left( \frac{y^2}{d^2} \right) = f_3(y) y \log^s y^2 \tag{32}
\]
where \(-K_1 \leq f_3(y) \leq K_1\) for \(y \geq y_0\).

Equations (28), (31) and (32) give \((y \geq y_0)\)

\[
Q_{q_1, \ldots, q_s}(y^2) = \frac{1}{(q_1 + 1) \cdots (q_s + 1)} \frac{6}{\pi^2} y^2 + f_2(y) y + f_3(y) y \log^s y^2
\]

\[
= \frac{1}{(q_1 + 1) \cdots (q_s + 1)} \frac{6}{\pi^2} y^2 + f_4(y) y \log^s y^2
\]

where \(f_4(y)\) is bounded for \(y \geq y_0\).

Replacing \(y^2\) by \(x\), we obtain \((x \geq y_0^2)\)

\[
Q_{q_1, \ldots, q_s}(x) = \frac{1}{(q_1 + 1) \cdots (q_s + 1)} \frac{6}{\pi^2} x + f_4(\sqrt{x}) \sqrt{x} \log^s x \tag{33}
\]

where \(f_4(\sqrt{x})\) is bounded for \(x \geq y_0^2\).

If \(3/2 < x < y_0^2\) we can write the equality

\[
Q_{q_1, \ldots, q_s}(x) = \frac{1}{(q_1 + 1) \cdots (q_s + 1)} \frac{6}{\pi^2} x + f_5(x) \sqrt{x} \log^s x \tag{34}
\]

That is

\[
f_5(x) = \frac{1}{\sqrt{x} \log^s x} \left( Q_{q_1, \ldots, q_s}(x) - \frac{1}{(q_1 + 1) \cdots (q_s + 1)} \frac{6}{\pi^2} x \right)
\]

Consequently \(f_5(x)\) is bounded in the interval \(3/2 < x < y_0^2\).

If we put

\[
h_{q_1, \ldots, q_s}(x) = f_4(\sqrt{x}) \quad (x \geq y_0^2)
\]

and

\[
h_{q_1, \ldots, q_s}(x) = f_5(x) \quad (3/2 < x < y_0^2)
\]

then (33) and (34) give (23). The theorem is proved.

Let \(N_{q_1, \ldots, q_s}(x)\) be the number of quadratfrei numbers not divisible by \(q_1 \cdots q_s\) not exceeding \(x\). We have the following Corollary.

**Corollary 2.3** The following asymptotic formula holds

\[
N_{q_1, \ldots, q_s}(x) = \frac{q_1 \cdots q_s}{(q_1 + 1) \cdots (q_s + 1)} \frac{6}{\pi^2} x + f_{q_1, \ldots, q_s}(x) \sqrt{x} \log^s x
\]

where the function \(f_{q_1, \ldots, q_s}(x)\) is bounded on the interval \(\left( \frac{3}{2}, \infty \right)\).
Proof. If \( x > 3/2 \) we have (see Theorem 1.2, Theorem 1.1 and Theorem 2.2)

\[
N_{q_1, \ldots, q_s}(x) = Q(x) - \sum_{1 \leq i \leq s} Q_{q_i}(x) + \sum_{1 \leq i < j \leq s} Q_{q_i, q_j}(x) - \sum_{1 \leq i < j < k \leq s} Q_{q_i, q_j, q_k}(x) + \cdots + (-1)^s Q_{q_1, \ldots, q_s}(x) = \frac{6}{\pi^2} x - \sum_{1 \leq i \leq s} \frac{6}{\pi^2} \frac{1}{q_i + 1} x + \left( -1 \right)^s \frac{6}{\pi^2} \frac{1}{(q_1 + 1)(q_2 + 1)} x + f_{q_1, \ldots, q_s}(x) \sqrt{x} \log^s x
\]

The corollary is proved.

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References


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