International Mathematical Forum, Vol. 12, 2017, no. 3, 145 - 150
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2017.6564

On a Paper of H.H. Abu-Zinadah and A.S. Aloufi

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Abstract

Abu-Zinadah and Aloufi (2014) consider the exponentiated Gompertz distribution and study certain properties of this distribution. The title of their paper is "Some characterizations of the exponentiated Gompertz distribution", but no characterizations are presented in the paper. The title should have been "Some properties of the exponentiated Gompertz distribution". In the present short note we establish certain characterizations of this distribution in four directions.

1 Introduction

Characterizations of distributions is an important research area which has recently attracted the attention of many researchers. This short note deals with various characterizations of exponentiated Gompertz (EG_{pz}) distribution to complete, in some way, the work of Abu-Zinadah and Aloufi (2014). These characterizations are based on: (i) a simple relationship between two truncated moments; (ii) the hazard function; (iii) the reverse hazard function and (iv) a single function of the random variable. It should be mentioned that for characterization (i) the *cdf* (cumulative distribution function) need not have a closed form.

Abu-Zinadah and Aloufi (2014) considered EG_{pz} distribution with *cdf* and *pdf* (probability density function) given, respectively, by

$$F(x;\theta,\alpha,\lambda) = \left(1-\exp\left[-\lambda\left(e^{\alpha x}-1\right)\right]\right)^{\theta}, \quad x \geq 0, \tag{1.1}$$

and

$$f\left(x;\theta,\alpha,\lambda\right)=\theta\lambda\alpha e^{\alpha x}\exp\left[-\lambda\left(e^{\alpha x}-1\right)\right]\left(1-\exp\left[-\lambda\left(e^{\alpha x}-1\right)\right]\right)^{\theta-1}, \quad (1.2)$$

$x>0$, where θ,α , λ are all positive parameters.

2 Characterizations of EG$_{pz}$ distribution

We present our characterizations $(i)-(iv)$ in four subsections.

2.1 Characterizations based on two truncated moments

In this subsection we present characterizations of EG$_{pz}$ distribution in terms of a simple relationship between two truncated moments. This characterization result employs a theorem due to Glänzel [2] , see Theorem 2.1.1 below. Note that the result holds also when the interval H is not closed. Moreover, as mentioned above, it could be also applied when the *cdf* F does not have a closed form. As shown in [3], this characterization is stable in the sense of weak convergence.

Theorem 2.1.1. Let $(\Omega,\mathcal{F},\mathbf{P})$ be a given probability space and let $H=[d,e]$ be an interval for some $d<e$ $(d=-\infty,\ e=\infty$ might as well be allowed$)$. Let $X:\Omega\to H$ be a continuous random variable with the distribution function F and let g and h be two real functions defined on H such that

$$\mathbf{E}\left[g\left(X\right)\mid X\geq x\right]=\mathbf{E}\left[h\left(X\right)\mid X\geq x\right]\xi\left(x\right),\quad x\in H,$$

is defined with some real function η. Assume that $g,\ h\in C^{1}\left(H\right)$, $\xi\in C^{2}\left(H\right)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\xi h=g$ has no real solution in the interior of H. Then F is uniquely determined by the functions g, h and ξ , particularly

$$F\left(x\right)=\int_{a}^{x}C\left|\frac{\xi'\left(u\right)}{\xi\left(u\right)h\left(u\right)-g\left(u\right)}\right|\exp\left(-s\left(u\right)\right)\,du\ ,$$

where the function s is a solution of the differential equation $s'=\frac{\xi'\,h}{\xi\,h\,-\,g}$ and C is the normalization constant, such that $\int_{H}dF=1$.

Proposition 2.1.1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $h(x) = (1 - \exp[-\lambda(e^{\alpha x} - 1)])^{1-\theta}$ and $g(x) = h(x) \exp[-\lambda(e^{\alpha x} - 1)]$ for $x \in (0, \infty)$. The random variable X belongs to EG_{pz} family (1.2) if and only if the function ξ defined in Theorem 2.1.1 has the form

$$\xi(x) = \frac{1}{2} \exp[-\lambda(e^{\alpha x} - 1)], \quad x \in (0, \infty). \tag{2.2.1}$$

Proof. Let X be a random variable with *pdf* (1.2), then

$$(1 - F(x)) E[h(x) \mid X \geq x] = \theta \exp[-\lambda(e^{\alpha x} - 1)], \quad x \in (0, \infty),$$

and

$$(1 - F(x)) E[g(x) \mid X \geq x] = \frac{1}{2} \exp[-2\lambda(e^{\alpha x} - 1)], \quad x \in (0, \infty),$$

and finally

$$\xi(x) h(x) - g(x) = -\frac{1}{2} h(x) \exp[-\lambda(e^{\alpha x} - 1)] < 0 \quad for \quad x \in (0, \infty).$$

Conversely, if ξ is given as above, then

$$s'(x) = \frac{\xi'(x) h(x)}{\xi(x) h(x) - g(x)} = \lambda \alpha e^{\alpha x}, \quad x \in (0, \infty),$$

and hence

$$s(x) = \lambda(e^{\alpha x} - 1), \quad x \in (0, \infty).$$

Now, in view of Theorem 2.1.1, X has density (1.2).

Corollary 2.1.1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $h(x)$ be as in Proposition 2.1.1. The pdf of X is (1.2) if and only if there exist functions g and ξ defined in Theorem 2.1.1 satisfying the differential equation

$$\frac{\xi'(x) h(x)}{\xi(x) h(x) - g(x)} = \lambda \alpha e^{\alpha x}, \quad x \in (0, \infty).$$

The general solution of the differential equation in Corollary 2.1.1 is

$$\xi(x) = \exp\left[\lambda\left(e^{\alpha x}-1\right)\right]\left[-\int \lambda\alpha e^{\alpha x}\exp\left[-\lambda\left(e^{\alpha x}-1\right)\right]\left(h(x)\right)^{-1} g(x) + D\right],$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 2.1.1 with $D = 0$. However, it should be also noted that there are other triplets (h, g, ξ) satisfying the conditions of Theorem 2.1.1.

2.2 Characterization based on hazard function

It is known that the hazard function, h_F, of a twice differentiable distribution function, F, satisfies the first order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h_F'(x)}{h_F(x)} - h_F(x). \qquad (2.2.1)$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following characterization establish a non-trivial characterization of EG_{pz} distribution, for $\theta = 1$, in terms of the hazard function, which is not of the trivial form given in (2.2.1).

Proposition 2.2.1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable. For $\theta = 1$,the *pdf* of X is (1.2) if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h_F'(x) - \alpha h_F(x) = 0, \qquad (2.2.2)$$

with the initial condition $h_F(0) = \lambda\alpha$.

Proof. If X has *pdf* (1.2), then clearly (2.2.2) holds. Now, if (2.2.2) holds, then

$$\frac{d}{du}\left\{\log\left(h_F(u)\right)\right\} = \alpha u,$$

or, equivalently,

$$\log\left(h_F\left(x\right)/h_F\left(0\right)\right) = \alpha x,$$

or

$$h_F\left(x\right) = \lambda\alpha e^{\alpha x},$$

which is the hazard function of the EG_{pz} distribution for $\theta = 1$.

2.3 Characterization in terms of the reverse hazard function

The reverse hazard function, r_F, of a twice differentiable distribution function, F , is defined as

$$r_F\left(x\right) = \frac{f\left(x\right)}{F\left(x\right)}, \quad x \in supportofF.$$

Proposition 2.3.1. Let $X : \Omega \rightarrow (0, \infty)$ be a continuous random variable. The *pdf* of X is (1.2) if and only if its reverse hazard function $r_F\left(x\right)$ satisfies the differential equation

$$r'_F\left(x\right) - \alpha r_F\left(x\right) = -\frac{\theta\lambda^2\alpha^2 e^{2\alpha x}\exp\left[\lambda\left(e^{\alpha x}-1\right)\right]}{\left(\exp\left[-\lambda\left(e^{\alpha x}-1\right)\right]-1\right)^2}. \quad (2.3.1)$$

Proof. If X has *pdf* (1.2), then clearly (2.3.1) holds. Now, if (2.3.1) holds, then

$$\frac{d}{du}\left\{e^{-\alpha x} r_F\left(x\right)\right\} = \theta\lambda\alpha\frac{d}{dx}\left\{\frac{\exp\left[-\lambda\left(e^{\alpha x}-1\right)\right]}{1-\exp\left[-\lambda\left(e^{\alpha x}-1\right)\right]}\right\},$$

or, equivalently,

$$\log\left(r_F\left(x\right)/h_F\left(0\right)\right) = \alpha x,$$

or

$$r_F\left(x\right) = \frac{\theta\lambda\alpha e^{\alpha x}\exp\left[-\lambda\left(e^{\alpha x}-1\right)\right]}{1-\exp\left[-\lambda\left(e^{\alpha x}-1\right)\right]},$$

which is the reverse hazard function of the EG_{pz} distribution.

2.4 Characterization based on truncated moment of certain function of the random variable

The following proposition has already appeared in our unpublished work (Hamedani, Technical Report, 2013, MSCS, MU) , so we will just state it here which can be used to characterize EG_{pz} distribution.

Proposition 2.4.1. Let $X : \Omega \to (a,b)$ be a continuous random variable with *cdf* F . Let $\psi(x)$ be a differentiable function on (a,b) with $\lim_{x\to b}\psi(x)=1$. Then for $\delta \neq 1$,

$$E\left[\psi(X) \mid X \le x\right] = \delta\psi(x), \quad x \in (a,b),$$

if and only if

$$\psi(x) = (F(x))^{\frac{1}{\delta}-1}, \quad x \in (a,b).$$

Remark 2.4.1. It is easy to see that for certain functions $\psi(x)$ on $(0,\infty)$, e.g., $\psi(x) = 1-\exp\left[-\lambda\left(e^{\alpha x}-1\right)\right]$ and $\delta = \frac{\theta}{1+\theta}$, Proposition 2.4.1 provides a characterization of EG_{pz} distribution.

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Received: June 12, 2016; Published: January 19, 2017