International Mathematical Forum, Vol. 12, 2017, no. 3, 97 - 102
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2017.612164

On the Adjoints of Biorthomorphisms

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Abstract

Let X be a vector lattice. We study on the separately orthomorphism mappings from $X \times X$ into X called the biorthomorphism. We extend such mappings on the order duals of X for a semiprime f-algebra X.

Mathematics Subject Classification: 46B42, 46A40, 47B60

Keywords: orthomorphism; biorthomorphism; Arens product; f-algebras; order bounded operators

1. Introduction

Let X be an Archimedean vector lattice(Riesz space) over the set of all real numbers R. Recall that a bilinear map X x X → X is a biorthomorphism if it is separately orthomorphism in each variable. A linear operator T: X → X is called band preserving if $x \perp y$ in X implies $Tx \perp y$, where $x \perp y$ mean $|x| \wedge |y| = 0$.A linear operator T: X → X is called an order bounded operator if it maps an order bounded set in X to an order bounded set in X. An order bounded band preserving operator is said to be an orthomorphism.There a lot of examples of orthomorphisms,[19]. The set of all orthomorphisms on X is denoted by Orth(X).It is well-known that if X is an Archimedean vector lattice then Orth(X) is an Archimedean f-algebra under multiplication by composition with the identity operator I on X as a multiplicative unit,[1], [18]. A separately band preserving bilinear operator which is also separately order bounded is called a biorthomorphism,[4], [9]. The set of all biorthomorphisms of X x X into X is denoted by Orth(X,X). Biorthomorphisms were introduced in by Yilmaz, Rowlans in [17] and Boulabair and Brahmi [5]. It is proven that if X is an Archimedean vector lattice, then Orth(X,X) is a vector lattice with respect to the pointwise addition and ordering, [4].

It is known that if X is a semiprime f-algebra then Orth(X) is a vector sublattice of Orth(X,X) and if X is a semiprime Dedekind complete f-algevbra then Orth(X) is an order ideal in Orth(X,X),[4]. For unexplained terminology and notations we refer to the books, [1][2], [16], [17], [18].

If X is an Archimedean vector lattice, then Orth(X) is an Archimedean vector lattice. The lattice operations in Orth(X) are given by pointwise. That is, if T, S ϵ Orth(X) and $0 \leq x \in X$, then

$$(T \wedge S)(x)=T(x) \wedge S(x) \text{ and } (T \vee S)(x)=T(x) \vee S(x).$$

Orthomorphisms on X are commutative. Let $T : X \times X \to X$ be a bilinear mapping and the operators $T(x, .) : X \to X$ and $T(.,x) : X \to X$ be defined by

$T(x,.)(y)=T(x,y)$ and $T(.,x)(y)=T(x,y)$ for all $y \in X$,[6], [14], [15].

We say that a bilinear mapping T: : $X \times X \to X$ is a biorthomorphism on X if $T(x,.) \in$ Orth(X) and $T(.,x) \in$ Orth(X for every $x \in X$. The lattice operations in Orth(X,X) are given by the following relations:

If A, B $\in$ Orth(X,X) and $0 \leq x$ and $0 \leq y \in X$. then

$$(A \wedge B)(x,y)=A(x,y) \wedge B(x,y) \text{ and } (A \vee B)(x,y)=A(x,y) \vee B(x,y).$$

A biorthomorphism on X is an orthosymmetric mapping and also it is symmetric.Recall that a bilinear mapping $T : X \times X \to X$ is called orthosymmetric if $x \perp y$ implies T(x,y)=0.And T is called symmetric if T(x,y)=T(y,x) for every x,y $\in$ X. T is called separately disjointness preserving if for every $x \in X$, the operators T(x,.) and T(.,x) are disjointness preserving.

By using the commutativity of orthomorphisms, the following equalities can be deduced

For every A, B $\in$ Orth(X,X) and $x,y,z \in X$, the identities

A(B(x,y),z)=A(x,B(y,z))=B(x,A(y,z)) hold, [4], [11], [12], [13].

Let us recall the definition of an f-algebra . Let X be an associative algebra over the set of all real numbers R. X is said to be a lattice ordered algebra if X is simultaneously a vector lattice such that the ordering and the multiplication are compatible. A lattice ordered algebra X is said to be an f-algebra if X satisfies the condition:

$x \wedge y = 0$ and $0 \leq z \in X$ imply $(x z) \wedge y = (z x) \wedge y = 0$.

The set of all nilpotent elements of an Archimedean f-algebra X is denoted by N(X). That is,

$N(X)=\{x \in X: x^2= 0 \}$.

We say that an Archimedean f-algebra X is semiprime if N(X) ={ 0},[12]. That is, X
is semiprime if and only if X has no non zero nilpotent elements. By X^+, we denote the set of all positive elements in X.

Theorem 1.[4]. Let X be an Archimedean vector lattice. If e ∈ X^+ , Orth(X,X) is an | Archimedean f-algebra with respect to the multiplication defined by

$$(A*_e B)(x,y)=A(x,B(e,y)) \text{ for all } A,B \in \text{Orth}(X,X) \text{ and } x,y \in X.$$

Assume that X is an Archimedean vector lattice and it has separating order dual $X^\sim$. Denote the order bidual of X by $X^{\sim\sim}$. The order continous order bidual of X is denoted by $X_n^{\sim\sim}$. It is wellknown that the order bidual and order continuous bidual of X are equal [13].

Theorem 2. Let X be a vector lattice with separating order dual $X^\sim$. Then, Orth($X^{\sim\sim}, X^{\sim\sim}$) is a Dedekind complete vector lattice.

Proof. Let T ∈ Orth($X^{\sim\sim}, X^{\sim\sim}$). Then,|T| and T^+, T^- belong to Orth($X^{\sim\sim}, X^{\sim\sim}$). Dedekind completeness of the second order dual of X implies the Dedekind comlpleteness of Orth($X^{\sim\sim}, X^{\sim\sim}$), [4].

Let X be a unital f-algebra and A ∈ Orth(X , X).Then, the mapping A : X× X→ X , A(x,y), is a bilinear separately orthomorphism. We establish the following bilinear mappings by using Arens product, [3], [7], [8].

$A^\sim : X^\sim \times X \to X^\sim$, $A^\sim(f,x)(y)=f(A(x,y))$ for every x,y ∈ X, f ∈ $X^\sim$, (1)

$A^{\sim\sim} : X^{\sim\sim} \times X^\sim \to X^\sim$, $A^{\sim\sim}(G,f)(x)=G(A^\sim(f,x))$, for every x ∈X, (2)

$A^{\sim\sim\sim} : X^{\sim\sim} \times X^{\sim\sim} \to X^{\sim\sim}$, $A^{\sim\sim\sim}(G,F)(f)=G(A^{\sim\sim}(F,f))$, for every f ∈ $X^\sim$, (3)

We know that if X is an f-algebra with unit , then the second order dual $X^{\sim\sim}$ is also a Dedekind complete f-algebra with unit. Let us define the set K(T)={ F ∈ $X^{\sim\sim}$: T(F,F)=0 }.

Proposition 3[4]. Let X be an f-algebra with separating order dual $X^\sim$ and T ∈ Orth($X^{\sim\sim}$,$X^{\sim\sim}$). Then, the set

K(T)={ F ∈ $X^{\sim\sim}$: T(F,G)=0 for all G ∈ $X^{\sim\sim}$}.

Also, K(T) is an order ideal.

Proof. The proof of the first part of this proposition is straightforward, so we omitted.For the second part of it , we use the Dedekind completeness of $X^{\sim\sim}$. By using the similar idea in [4], it can be proved.

Proposition 4[4]. Let X be an f-algebra with separating order dual $X^\sim$ and A ∈ Orth($X^{\sim\sim}$,$X^{\sim\sim}$) and F ∈ $X^{\sim\sim}$. A(F,F)=0 if and only if A(F,F) ∈ K(A).

An f-algebra X is called trivial if xy=0 for all x,y ∈ X. Otherwise we say that X is a non trivial f-algebra, [18].

Proposition 5[4] Let X be an f-algebra with separating order dual $X^{\sim}$.Then, the following assertions are equivalent:

(i) Orth($X^{\sim\sim}$,$X^{\sim\sim}$) $\neq\{0\}$.
(ii) $X^{\sim\sim}$ is a non trivial f-algebra.
(iii) Orth($X^{\sim\sim}$,$X^{\sim\sim}$) is a non trivial f-algebra.

Theorem 6.[4] Let X be an Archimedean unital (e)f-algebra with the separating order dual $X^{\sim}$.Then the vector lattice Orth($X^{\sim\sim}$, $X^{\sim\sim}$) is a semiprime f-algebra with respect to the multiplication $*_e$.
Let X be an Archimedean unital f-algebra with separating order dual $X^{\sim}$, and its second order dual $X^{\sim\sim}$. then we can define a mapping J: X →Orth(X) and a mapping
K: Orth(X)→Orth($X^{\sim\sim}$), K(f)=f''. where f'' is the second adjoint of f. If T ∈ Orth($X^{\sim\sim}$),
then we can define the mapping T^*: $X^{\sim\sim}\times X^{\sim\sim}\rightarrow X^{\sim\sim}$ by T^*(G,F)=T(GF) for every
G,F ∈ $X^{\sim\sim}$. Here, T^* is a biorthomorphism on $X^{\sim\sim}$.So, we introduce a mapping
H: Orth($X^{\sim\sim}$) → Orth($X^{\sim\sim},X^{\sim\sim}$) by H(T)=$T^*$ for all T ∈ Orth($X^{\sim\sim}$). Here, the mapping H is a one to one lattice homomorphism. Therefore, Orth($X^{\sim\sim}$) is embedded in Orth($X^{\sim\sim},X^{\sim\sim}$) as a vector sublattice. Let $X^{\blacksquare}$={ F.G : F,G ∈ $X^{\sim\sim}$}.

It is a vector sublattice of $X^{\sim\sim}$. Its positive cone is $\{F^2$: F ∈ $X^{\sim\sim}\}$, [10].
Let X be an Archimedean f-algebra with separating order dual $X^{\sim}$. Let
A : X × X →X be a mapping. Then, the second adjoint of A, $A^{\sim\sim}$: $X^{\sim\sim}\times X^{\sim\sim}\rightarrow X^{\sim\sim}$ is a biorthomorphism if and only if there is an orthomorphism $A^{\blacksquare}$: $X^{\blacksquare}\rightarrow X^{\sim\sim}$ such that
$A^{\sim\sim}$(F,G)=$A^{\blacksquare}$(F.G) for all F,G ∈ $X^{\sim\sim}$.

Theorem 7[4]. Let X be an Archimedean unital f-algebra with separating order dual $X^{\sim}$ and the second order dual $X^{\sim\sim}$.
Then , the following claims are true:

i. Orth($X^{\sim\sim}$) is a vector sublattice of Orth($X^{\sim\sim}$, $X^{\sim\sim}$).
ii. Orth($X^{\sim\sim}$) is an order ideal of Orth($X^{\sim\sim}$, $X^{\sim\sim}$).

Proof.(ii) It is clear that Orth($X^{\sim\sim}$,$X^{\sim\sim}$) is a Dedekind complete semiprime unital f-algebra. Let e ∈ $X^{\sim\sim}$ be an order unit in $X^{\sim\sim}$. So, Orth($X^{\sim\sim}$,$X^{\sim\sim}$) is a semiprime f-algebra with the multiplication e_*.It is obvious that Orth($X^{\sim\sim}$) is a ring ideal in Orth($X^{\sim\sim}$,$X^{\sim\sim}$). Since Orth($X^{\sim\sim}$) is a Dedekind complete square root closed ring ideal, it is idempotent. Hence, it is an order ideal in Orth($X^{\sim\sim}$,$X^{\sim\sim}$).

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Received: December 21, 2016; Published: January 18, 2017