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Marshall-Olkin Kumaraswamy Distribution

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Abstract

In this paper, we introduce a new family of continuous distributions called Marshall-Olkin Kumaraswamy (MOKS) distribution.We study some mathematical properties and derive distribution of order statistics, record value properties, two types of entropies, Lorenz,Bonferroni and Zenga curves.We construct an autoregressive model with minification structure.The model parameters are estimated by the maximum likelihood method. An application to a real data set is discussed.

Keywords: Record values; Autoregressive processes; Marshall-Olkin Kumaraswamy distribution

1 Introduction

One of the preferred area of research in the field of the probability distributions is that of generating new distributions starting with a base line distribution by adding one or more additional parameters. While the additional parameters bring in more flexibility at the same time they also complicate the mathematical form of the resulting distribution. But with the advent of sophisticated powerful mathematical and statistical softwares unlike in past nowadays more and more complicate distributions are getting accepted as viable models for data analysis.

The Kumaraswamy's distribution was introduced by Kumaraswamy (1980) as a new probability distribution for double bounded random process with hydrological applications. This distribution is applicable to many natural phenomena whose outcomes have lower and upper bounds such as height of individuals, scores obtained on a test, atmospheric temperature, hydrological data such as daily rainfall, daily stream flow etc. Many papers in hydrology used Kumaraswamy's distribution as a better alternative to the beta distribution. Kumaraswamy (1980) and Ponnambalam et.al ; (2001) have pointed out that depending on the choice of the parameter a and b, Kumaraswamy's distribution can be used to approximate many distribution such as uniform ,triangular or almost any single model distribution and can also reproduce results of beta distribution. The Kumaraswamy's distribution doesn't seem to be very familiar to the Statisticians. Kumaraswamy introduced a distribution with two shape parameters $a > 0$ and $b > 0$ (real). The cumulative distribution function $F(x)$ and the probability density function $f(x)$ are given by,

$$F(x) = 1 - (1 - x^a)^b; 0 < x < 1 \tag{1}$$

and

$$f(x) = f(x; a, b) = abx^{a-1}(1 - x^a)^{b-1}; 0 < x < 1 \tag{2}$$

The most convenient feature of this distribution is that its CDF has a simple form and it is easy to convert. Kumaraswamy's densities are unimodal, uniantimodal, increasing, decreasing or constant which is same as the beta distribution.
$a > 1, b > 1 \Rightarrow$ unimodal; $a < 1, b < 1 \Rightarrow$ uniantimodal
$a > 1, b \leq 1 \Rightarrow$ increasing; $a \leq 1, b > 1 \Rightarrow$ decreasing
$a = b = 1 \Rightarrow$ constant.
Kumaraswamy's distribution has its genesis in terms of uniform order statistics, and has particularly straightforward distribution and quantile function which do not depend on special functions. In probability and Statistics, the Kumaraswamy's double bounded distribution is a family of continuous probability distribution defined on the interval $[0, 1]$ differing in the values of their two non negative shape parameters a and b. In reliability and life testing experiments many times the data are modeled by finite range distributions.

The Kumaraswamy's distribution is similar to the Beta distribution but much simpler to use especially in simulation studies due to the simple closed form of both its probability density function and cumulative distribution function. This distribution was originally proposed by Kumaraswamy for variables that are lower and upper bounded. Boundary behaviour and the main special cases are also common to both Beta and Kumaraswamy's distribution . This distribution could be appropriate in situations where scientists use probability distribution which have infinite lower or upper bounds to fit data, when in reality the bounds are finite.

The Kumaraswamy's distribution is closely related to Beta distribution. Assume that $X_{(a,b)}$ is a Kumaraswamy distributed random variable with parameters a and b. Then $X_{(a,b)}$ is the a^{th} root of a suitably defined Beta distributed random variable. Let $Y_{(1,b)}$ denote the Beta distributed random variable with $\alpha = 1$ and $\beta = b$. Then $X_{(a,b)} = Y_{(1,b)}^{(\frac{1}{a})}$ with equality in distribution.

$$\begin{aligned} P[X_{a,b} \leq x] &= \int_0^x abt^{(a-1)}(1-t^a)^{(b-1)}dt \\ &= \int_0^x b(1-t)^{(b-1)}dt \\ &= P\{Y_{1,b} \leq x^a\} = P\{Y_{1,b}^{(1/a)} \leq x\}. \end{aligned}$$

The Beta and Kumaraswamy's distribution have special cases. Beta$(1,1)$ and Kumaraswamy$(1,1)$ are both uniform$(0,1)$.

If $X \sim B(1,b)$ then $X \sim KS(1,b)$

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$X \sim KS(a,b)$ then $X \sim GB_1(a,1,1,b)$

KS$(a,1)$ distribution is the power function distribution and KS$(1,a)$ distribution is the distribution of one minus that power function random variable.

One of the simplest and widely used time series models is the autoregressive models and it is well known that autoregressive process of appropriate orders is extensively used for modeling time serie and $h(x;a,bk+1)$ denotes the Kumaraswamy's density with parameters a and $b(k+1)$. Thus the MOKS density function can be expressed as an infinite linear combination of Kumaraswamy's densities. Similarly , we can write the cdf of MOKS as,

$$\begin{aligned} G(x) &= \int_0^x g(x)dx \\ &= \sum_k q_k H(x;a,b(k+1)) \end{aligned}$$

where $H(x;a,b(k+1))$ denotes the Kumaraswamy's cumulative function with parameter a and $b(k+1)$ s data. The p^{th} order autoregressive model is defined by

$$X_n = a_1 X_{n-1} + a_2 X_{n-2} + \ldots + a_p X_{n-p} + \epsilon_n$$

where $\{\epsilon_n\}$ is a sequence of independent and identically distributed random variables and $a_1, a_2, \ldots, a_n$ are autoregressive parameters.

A minification processes of the first order is given by

$$X_n = K\min(X_{n-1}, \epsilon_n), n \geq 1 \tag{3}$$

where $K > 1$ is a constant and $\{X_n\}$ is a stationary Markov process with marginal distribution function $F_X(x)$. Because of the structure of (3), the process $\{X_n\}$ is called a minification process.

2 Theory of Marshall - Olkin Distributions

Marshall and Olkin (1997) proposed a flexible semi - parametric family of distributions and defined a new survival function $\bar{G}(x)$ by introducing and additional parameter α such that $\alpha = 1 - \bar{\alpha}; \alpha > 0$ called the tilt parameter.

$$\bar{G}(x, \alpha) = \frac{\alpha \bar{F}(x)}{1 - \bar{\alpha}\bar{F}(x)}; x \in R, \alpha > 0 \tag{4}$$

is the survival function of Marshall - Olkin family of distributions which is related to the proportional odds (PO) model in survival analysis . See Sankaran and Jayakumar (2006), Bennet (1983).
For $\alpha = 1$, we have $\bar{G}(x) = \bar{F}(x)$. The probability density function and hazard rate function corresponding to (4) is given by

$$g(x; \alpha) = \frac{\alpha f(x)}{[1 - (1 - \alpha)\bar{F}(x)]^2}$$

and

$$h(x; \alpha) = \frac{h_F(x)}{(1 - (1 - \alpha)\bar{F}(x))}$$

where $h_F(x)$ is the hazard rate function of F.

Then, $h_F(x) = \frac{f(x)}{\bar{F}(x)}$ see Alice and Jose (2003, 2004 a,b)

3 Marshall-Olkin Kumaraswamy Distribution

A new family of distributions is proposed by using Kumaraswamy's distribution as the baseline distribution in the Marshall - Olkin construction. Consider the Kumaraswamy's distribution with survival function $\bar{F}(x) = (1-x^a)^b; a, b > 0; x \in (0, 1)$. Substituting in (4) we obtain a new family of continuous distribution called the Marshall -Olkin Kumaraswamy distribution denoted by $MOKS(\alpha, a, b)$ with survival function given by

$$\bar{G}(x) = \frac{\alpha(1 - x^a)^b}{1 - \bar{\alpha}(1 - x^a)^b}; \alpha, a, b > 0; 0 < x < 1 \tag{5}$$

The corresponding density function is given by

$$g(x) = \frac{\alpha abx^{a-1}(1-x^a)^{b-1}}{[1-\bar{\alpha}(1-x^a)^b]^2}; \alpha, a, b > 0; 0 < x < 1 \tag{6}$$

If $\alpha = 1, g(x) = f(x)$. ie, we obtain the Kumaraswamy's distribution with parameter $a, b > 0$. The hazard rate function for Kumaraswamy distribution is

$$h_F(x) = \frac{abx^{a-1}(1-x^a)^{b-1}}{(1-x^a)^b}$$

The hazard rate function for MOKS distribution is

$$h_G(x) = \frac{abx^{a-1}}{(1-x^a)[1-\bar{\alpha}(1-x^a)^b]} \tag{7}$$

Reverse hazard rate function for Kumaraswamy distribution is

$$r_F(x) = \frac{g(x)}{G(x)} = \frac{abx^{a-1}(1-x^a)^{b-1}}{[1-(1-x^a)^b]}$$

Reverse hazard rate function for MOKS distribution is

$$r_G(x) = \frac{\alpha abx^{a-1}(1-x^a)^{b-1}}{[1-\bar{\alpha}(1-x^a)^b][1-(1-x^a)^b]}$$

Cumulative hazard rate function for Kumaraswamy distribution is

$$H_F(x) = -lnG(x) = bln(1-x^a)$$

Cumulative hazard rate function for MOKS distribution is

$$H_G(x) = \int_0^x h(x)dx = ln[1-\bar{\alpha}(1-x^a)^b] - ln[1-(1-x^a)^b]$$

If $B_G(x) = \frac{\bar{G}(x)}{G(x)}$ is known as the odds function of a random variable X. It measures the ratio of the probability that the unit will survive beyond X to the probability that it will fail before X. In Kumaraswamy's distribution,

$$B_G(x) = \frac{(1-x^a)^b}{[1-(1-x^a)^b]} = \frac{1}{[(1-x^a)^{-b}]-1}$$

In MOKS distribution,

$$B_G(x) = \frac{\alpha}{[(1-x^a)^{-b}]-1}$$

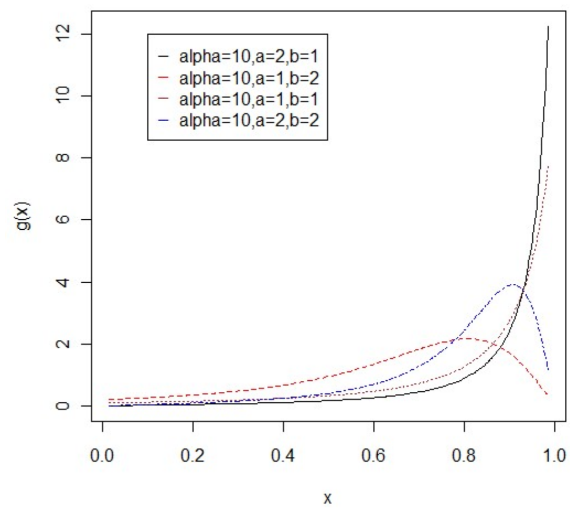

Figure 1: plot of density function for fixed alpha and various a and b

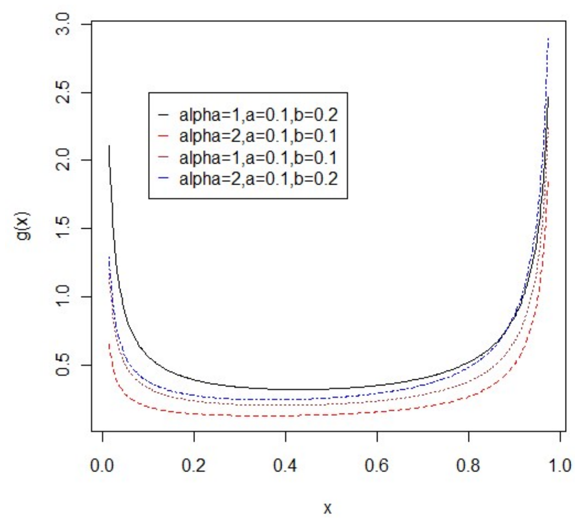

Figure 2: plot of density function for fixed a and various alpha and b

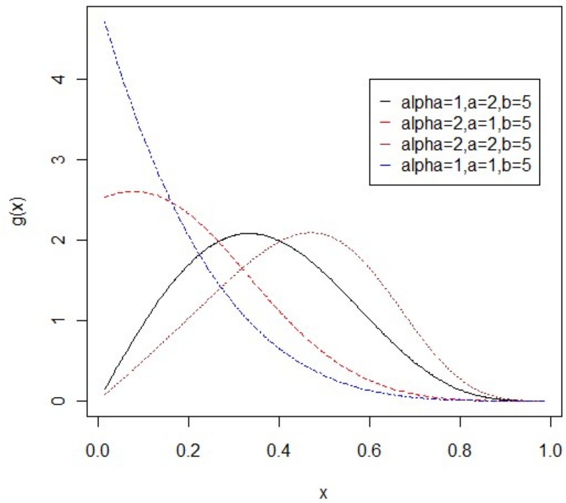

Figure 3: plot of density function for fixed b and various alpha and a

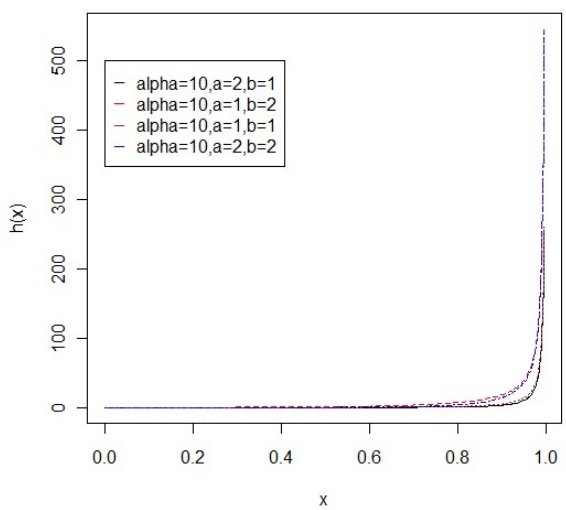

Figure 4: plot of hazard rate function for fixed alpha and various a and b

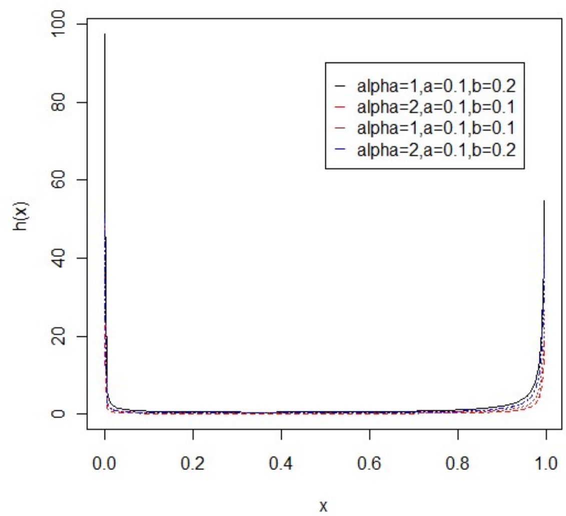

Figure 5: plot of hazard rate function for fixed a and various alpha and b

4 Theoretical properties

4.1 Expansion for the cumulative and density function

Now we give a useful expansion for the MOKS density (6). For any positive real numbers and for $|z| < 1$, a generalized binomial expansion,

$$(1-z)^{-s} = \sum_{k=0}^{\infty} \binom{s+k-1}{k} z^k \tag{8}$$

Using (8) in (6) we have,

$$\begin{aligned} g(x) &= \alpha ab x^{a-1}(1-x^a)^{b-1} \sum_{k=0}^{\infty} \binom{k+1}{k} (1-\alpha)^k (1-x^a)^{bk} \\ &= \sum_{k=0}^{\infty} q_k h(x; a, b(k+1)) \end{aligned}$$

Where

$$q_k = \frac{\alpha(1-\alpha)^k}{k+1} \binom{k+1}{k}$$

and $h(x; a, bk+1)$ denotes the Kumaraswamy's density with parameters a and $b(k+1)$. Thus the MOKS density function can be expressed as an infinite linear combination of Kumaraswamy's densities. Similarly , we can write the cdf of MOKS as,

$$\begin{aligned} G(x) &= \int_0^x g(x)dx \\ &= \sum_k q_k H(x; a, b(k+1)) \end{aligned}$$

where $H(x; a, b(k+1))$ denotes the Kumaraswamy's cumulative function with parameter a and $b(k+1)$

5 Moments and Quantiles

Let X $\sim$ MOKS(α, a, b) for $\alpha = 1, 2, \ldots$, the r^{th} moment is given by

$$E(X^r) = \int_0^1 \frac{x^r \alpha a b x^{a-1}(1-x^a)^{b-1}}{[1-\bar{\alpha}(1-x^a)^b]^2} dx$$

Using the expansion,

$$\frac{1}{[1-\bar{\alpha}(1-x^a)^b]^2} = \sum_{j=0}^{\infty} (j+1)(1-\alpha)^j (1-x^a)^{bj}$$

We obtain,

$$E(X^r) = \alpha b \sum_{j=0}^{\infty} (j+1)(1-\alpha)^j B(1+\frac{r}{a}, b(j+1)) \tag{9}$$

Mgf say$M(t) = E[e^{tx}]$ of MOKS(α, a, b) is

$$M(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r)$$

where $E(X^r)$ follows from (9). The Mean,Variance, Skewness and Kurtosis can be obtained from (9).
Therefore,

$$E(X) = \alpha b \sum_{j=0}^{\infty} (j+1)(1-\alpha)^j B\left(1+\frac{1}{a}, b(j+1)\right)$$

The q^{th} quantile of MOKS distribution is

$$Q(y) = x_q = G^{-1}(q) = \left[1 - \left(\frac{q}{1-\bar{\alpha}(1-q)}\right)^{\frac{1}{b}}\right]^{\frac{1}{a}} ; 0 \leq q \leq 1 \qquad (10)$$

Where $G^{-1}(.)$ is the inverse of the distribution function .
The density of the quantile function is

$$q(y) = \frac{1}{ab}\frac{1-\bar{\alpha}}{(1-y\bar{\alpha})^2}\left(\frac{1-y}{1-y\bar{\alpha}}\right)^{\frac{1}{b}-1}\left[1 - \left(\frac{1-y}{1-y\bar{\alpha}}\right)^{\frac{1}{b}}\right]^{\frac{1}{a}-1} ; \alpha, a, b > 0; 0 < y < 1$$

The Median of the distribution is

$$\text{Median}(X) = \left[1 - \left(\frac{1}{1+\alpha}\right)^{\frac{1}{b}}\right]^{\frac{1}{a}}$$

The Mode for this distribution can be found by solving the first derivative of the function $\log g(x) = 0$.
ie,

$$\frac{d\log g}{dx} = \frac{a-1}{x} - \frac{a(b-1)x^{a-1}}{1-x^a} - \frac{2\bar{\alpha}abx^{a-1}(1-x^a)^{b-1}}{1-\bar{\alpha}(1-x^a)^b} = 0$$

The probability weighted moments (PWMs), first proposed by Greenwood et al.(1979) are expectations of certain functions of a random variable whose mean exists. A general theory for these moments covers the summarization and description of theoretical probability distributions and observed data samples, non parametric estimation of the underlying distribution of an observed sample, estimation of parameters, quantiles of probability distributions and hypothesis tests. The PWMs method can generally be used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly. The $(p, r)^{th}$ PWM of X is defined by,

$$\tau_{(p,r)} = E\{X^p F(X)^r\} = \int_{-\infty}^{\infty} x^p F(x)^r f(x) dx$$

Hence for $MOKS(\alpha, a, b)$,

$$\tau_{(p,r)} = \alpha b \sum_{j=0}^{r}\sum_{k=0}^{\infty}\binom{r+k+1}{k}\binom{r}{j}(-1)^j \bar{\alpha}^k B(\frac{p}{a}+1, b(j+k+1))$$

6 Lorenz, Bonferroni and Zenga curves

Lorenz and Bonferroni curves are applied in many fields such as Economics, Reliability, Demography, Insurance and Medicine (Kleiber and Kotz 2003). Zenga curve was presented by zenga (2007). According to Oluyede and Rajasooriya(2013), the Lorenz curve $L_F(x)$,Bonferroni $B(F(x))$ and Zenga$A(x)$ are defined as follows.

$$L_F(x) = \frac{\int_0^x tf(t)dt}{E(X)}$$

$$B(F(x)) = \frac{\int_0^x tf(t)dt}{F(x)E(X)} = \frac{L_F(x)}{F(x)}$$

$$A(x) = 1 - \frac{[1-F(x)][\int_0^x tf(t)dt]}{F(x)\int_x^\infty tf(t)dt}$$

Using the MOKS distribution , we have
Lorenz curve,

$$L_F(x) = \frac{\sum_{j=0}^\infty (j+1)(1-\alpha)^j B(x^c; 1+\frac{1}{a}, b(j+1))}{\sum_{j=0}^\infty (j+1)(1-\alpha)^j B(1+\frac{1}{a}, b(j+1))}$$

Bonferroni curve,

$$B(F(x)) = \frac{\sum_{j=0}^\infty (j+1)(1-\alpha)^j B(x^c; 1+\frac{1}{a}, b(j+1))}{\sum_{j=0}^\infty (j+1)(1-\alpha)^j B(1+\frac{1}{a}, b(j+1)) q_j H(x; a, b(j+1))}$$

Zenga curve,

$$A(x) = 1 - \frac{[1-\sum_{k=0}^\infty q_k H(x; a, b(k+1))][\sum_{j=0}^\infty (j+1)(1-\alpha)^j B(x^c; 1+\frac{1}{a}, b(j+1))]}{[\sum_{j=0}^\infty q_k H(x; a, b(k+1))]\sum_{j=0}^\infty (j+1)[B(1+\frac{1}{a}, b(j+1)) - B(t^c; 1+\frac{1}{a}, b(j+1))]}$$

7 Order Statistics

Let $X_1, X_2, \ldots X_n$ be a random sample of size n from MOKS (α, a, b) distribution. Let $X_{i:n}$ denotes the i^{th} order statistics. Then the probability density function of $X_{i:n}$ is

$$g_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} g(x) G^{i-1}(x)(1-G(x))^{n-i}$$

where $g(.)$ and $G(.)$ are the pdf and cdf of the MOKS distribution.
Hence

$$\begin{aligned}
g_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} \frac{\alpha a b x^{a-1}(1-x^a)^{b-1}}{[1-\bar{\alpha}(1-x^a)^b]^2} \left\{ \frac{1-(1-x^a)^b}{1-\bar{\alpha}(1-x^a)^b} \right\}^{i-1} \left\{ \frac{\alpha(1-x^a)^b}{1-\bar{\alpha}(1-x^a)^b} \right\}^{n-i} \\
&= \frac{n!}{(i-1)!(n-i)!} \alpha^{n-i+1} \\
&\quad \left\{ \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^j \binom{n+k}{k} \binom{i-1}{j} (1-\alpha)^k a b x^{a-1} (1-x^a)^{b(n-i+j+k+1)-1} \right\} \\
&= \frac{n!}{(i-1)!(n-i)!} \alpha^{n-i+1} \\
&\quad \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^j \binom{n+k}{k} \binom{i-1}{j} (1-\alpha)^k \\
&\quad \frac{1}{(n-i+j+k+1)} KS(x; a, b(n-i+j+k+1))
\end{aligned}$$

Where KS$(x; a, b(n-i+j+k+1))$ denoted the Kumaraswamy's density function with parameters a and $b(n-i+j+k+1)$. So the density function of the order statistics is simply an infinite linear combination of Kumaraswamy's density.

8 Record Values

Chandler (1952) introduced the concept of records. Record values and associated statistics are of greater importance in many real life situations involving data relating to sports, weather, Economics, life testing etc. Galambos and Kotz (1987), Sultan et al (2003) etc have made significant contributions to the theory of records . Let $X_1, X_2, \ldots$ be an infinite sequence of iid random variables having an absolutely continuous cdf $F(x)$ and pdf $f(x)$. An observation X_j will be called an upper record value or a record if its value exceeds that of all previous observations. Then X_j is a record if $X_j > X_i \ \forall i < j$. Let $f_{Rn}(x)$ denotes the pdf of the n^{th} record then

$$f_{Rn}(x) = \frac{f(x)[-\log(1-F(x))]^n}{n!} \tag{11}$$

If $g_{Rn}(x)$ be the density function of the n^{th} record value from MOKS (α, a, b) then

$$g_{Rn}(x) \quad = \quad \frac{\alpha abx^{a-1}(1-x^a)^{b-1}}{[1-\bar{\alpha}(1-x^a)^b]^2 n!}\left[-\log\frac{\alpha(1-x^a)^b}{1-\bar{\alpha}(1-x^a)^b}\right]^n ;\alpha, a, b > 0; 0 < x < 1$$

8.1 Recurrence relation for moments of record values

The recurrence relation can be used to compute all the single moments of record values which is useful for the inference.For the MOKS(α, a, b) distribution with pdf $g(x)$ and cdf $G(x)$ we obtain ,

$$\frac{1-x^a}{abx^{a-1}}[1-\bar{\alpha}(1-x^a)^b]g(x) = 1-G(x)$$

For a record R_n we obtain,

$$\frac{1-x^a}{abx^{a-1}}[1-\bar{\alpha}(1-x^a)^b]g_{Rn}(x) = \frac{[1-G(x)][-log(1-G(x))]^n]}{n!}$$

The above relation will be used to derive a recurrence relation for the moments of record value.

9 Entropies

The entropy of a random variable is a measure of uncertainity variation and has been used in various situations in Science and Engineering.Two popular entropy measures are the Renyi and Shannon entropies.
The Renyi entropy is defined as

$$I_R(\gamma) = \frac{1}{1-\gamma}\log\int_{-\infty}^{\infty} g^{\gamma}(x)dx;\ \gamma > 0, \gamma \neq 1.$$

Then

$$g^{\gamma}(x) = \left[\frac{\alpha abx^{a-1}(1-x^a)^{b-1}}{[1-\bar{\alpha}(1-x^a)^b]^2}\right]^{\gamma}.$$

Using

$$\left[1-\bar{\alpha}(1-x^a)^b\right]^{-2\gamma} = \sum_{k=0}^{\infty}\frac{\Gamma(2\gamma+k)}{\Gamma(2\gamma)k!}(1-\alpha)^k(1-x^a)^{bk}$$

we have

$$\begin{aligned}\int_0^\infty g^\gamma(x)dx &= \int_0^1 \alpha^\gamma a^\gamma b^\gamma x^{\gamma(a-1)}(1-x^a)^{\gamma(b-1)} \\ &\sum_{k=0}^\infty \frac{\Gamma(2\gamma+k)}{\Gamma(2\gamma)k!}(1-\alpha)^k(1-x^a)^{bk}dx \\ &= \alpha^\gamma a^{\gamma-1}b^\gamma \sum_{k=0}^\infty \frac{\Gamma(2\gamma+k)}{\Gamma(2\gamma)k!}(1-\alpha)^k \\ &\frac{\Gamma\frac{\gamma(a-1)+1}{a}\Gamma(bk+\gamma(b-1))}{\Gamma\left(\frac{\gamma(a-1)+1}{a}+bk+\gamma(b-1)\right)}\end{aligned}$$

Therefore,

$$I_R(\gamma) = \frac{1}{1-\gamma}\log\left\{\alpha^\gamma a^{\gamma-1}b^\gamma \sum_{k=0}^\infty \frac{\Gamma(2\gamma+k)}{\Gamma(2\gamma)k!}(1-\alpha)^k \frac{\Gamma\frac{\gamma(a-1)+1}{a}\Gamma(bk+\gamma(b-1)+1)}{\Gamma\left(\frac{\gamma(a-1)+1}{a}+bk+\gamma(b-1)\right)}\right\}$$

The Shannon entropy of a random variable X is defined by $E[-\log g(X)]$. It is the special case of the Renyi entropy when $\gamma \uparrow 1$.

$$E[-\log g(X)] = -\log \alpha ab + 2E[\log(1-\bar{\alpha}(1-x^a)^b)] - (a-1)E(\log X) - (b-1)E[\log(1-x^a)]$$

where,

$$\begin{aligned}E\left[\log(1-\bar{\alpha}(1-x^a)^b)\right] &= \alpha ab\int_0^1 \log\left[1-\bar{\alpha}(1-x^a)^b\right]\frac{x^{a-1}(1-x^a)^{b-1}}{\left[1-\bar{\alpha}(1-x^a)^b\right]^2}dx \\ &= \frac{1}{\alpha}[\log\alpha-\alpha+1] \\ E[\log X] &= \alpha ab\int_0^1 \log x\frac{x^{a-1}(1-x^a)^{b-1}}{[1-\bar{\alpha}(1-x^a)^b]^2}dx \\ &= \frac{-\alpha b}{a(J+1)^2}\sum_{k=0}^\infty \binom{k+1}{k}\alpha^{-k}\sum_{j=0}^\infty(-1)^j\binom{b(k+1)-1}{j} \\ E[\log(1-x^a)] &= \alpha b\sum_{k=0}^\infty \binom{k+1}{k}\alpha^{-k}\sum_{j=0}^\infty \frac{(-1)^j}{j+1}\binom{b(k+1)-1}{j}\end{aligned}$$

Therefore the entropy of X is given by,

$$E[-\log g(X)] = -\log\alpha ab + \frac{2}{\bar{\alpha}}[\log\alpha-\alpha+1] + \alpha b\beta_{jk}\left[\frac{a-1}{a(j+1)}-(b-1)\right],$$

where

$$\beta_{jk} = \sum_{k=0}^{\infty} \binom{k+1}{k} \alpha^{-k} \sum_{j=0}^{\infty} \frac{(-1)^j}{j+1} \binom{b(k+1)-1}{j}.$$

10 Mean residual life and mean waiting time

The mean residual life function (MRL) or life expectancy denoted by $m(t)$ at a time t measures the expected remaining lifetime of an individual of age t. If $S(t)$ is the survival function at t then,

$$\begin{aligned} m(t) &= \frac{1}{S(t)}\left[E(t) - \int_0^t tf(t)dt\right] - t \\ &= \frac{\sum_{j=0}^{\infty} q_j \left[B(1+\frac{1}{a}, b(j+1)) - B(t^c; 1+\frac{1}{a}, b(j+1))\right]}{1-\sum_{j=0}^{\infty} q_j H(x; a, b(j+1))} - t. \end{aligned}$$

The mean waiting time (MWT) of an item failed in an interval $[0, t]$ is given by

$$\begin{aligned} \bar{\mu}(t, \theta) &= t - \frac{1}{F(t)} \int_0^t tf(t)dt \\ &= t - \frac{\sum_{j=0}^{\infty} q_j B(t^c; 1+\frac{1}{a}, b(j+1))}{\sum_{j=0}^{\infty} q_j H(x; a, b(j+1))} \end{aligned}$$

11 Stress - Strength Analysis

Gupta et al (2010) showed that for two independent random variables represent strength (Y) and stress (X) follow the Marshall-Olkin extended distribution with tilt parameter α_1 and α_2.The system fails if stress exceeds the strength. So the reliability of the system is $R = P(X < Y)$. This measure of reliability is widely used in engineering problems. Hence,

$$\begin{aligned} R &= \int_{-\infty}^{\infty} P(Y > X \mid X = x) g(x) dx \\ &= \frac{\frac{\alpha_1}{\alpha_2}}{\left(\frac{\alpha_1}{\alpha_2} - 1\right)^2}\left[-log\frac{\alpha_1}{\alpha_2} + \frac{\alpha_1}{\alpha_2} - 1\right] \end{aligned}$$

To estimate R, it is enough to estimate α_1 and α_2 by the method of mle. Let $(x_1, x_2, \ldots, x_m)$ and $(y_1, y_2, \ldots, y_n)$ be two independent random samples of size m and n from MOKS distribution with tilt parameter α_1 and α_2 and unknown parameters a and b. Then the log-likelihood equation is

$$\begin{aligned} L(\alpha_1, \alpha_2, a, b) &= \sum_{i=1}^{m} \log g(x_i; \alpha_1, a, b) + \sum_{i=1}^{n} \log g(y_i; \alpha_2, a, b) \\ &= m \log \alpha_1 + (m+n) \log a + (m+n) \log b + n \log \alpha_2 \\ &\quad + (a-1) \left[\sum_{i=1}^{m} \log x_i + \sum_{i=1}^{n} \log y_i \right] + (b-1) \left[\sum_{i=1}^{m} \log(1 - {x_i}^a) + \sum_{i=}^{n} \log(1 - {y_i}^a) \right] \\ &\quad -2 \sum_{i=1}^{m} \log \left[1 - (1-\alpha_1)(1 - {x_i}^a)^b \right] - 2 \sum_{i=1}^{n} \log \left[1 - (1-\alpha_2)(1 - {y_i}^a)^b \right] \end{aligned}$$

The maximum likelihood estimates of α_1 and α_2 are the solutions of the non linear equations,

$$\begin{aligned} \frac{\partial L}{\partial \alpha_1} &= \frac{m}{\alpha_1} - 2 \sum_{i=1}^{m} \frac{(1 - {x_i}^a)^b}{[1 - (1-\alpha_1)(1 - {x_i}^a)^b]} = 0 \\ \frac{\partial L}{\partial \alpha_2} &= \frac{n}{\alpha_2} - 2 \sum_{i=1}^{n} \frac{(1 - {y_i}^a)^b}{[1 - (1-\alpha_2)(1 - {y_i}^a)^b]} = 0 \end{aligned}$$

The information matrix has the elements,

$$\begin{aligned} I_{11} &= -E\left(\frac{\partial^2 L}{\partial {\alpha_1}^2} \right) \\ &= \frac{m}{\alpha_1^2} - 2m \int_0^1 \frac{(1 - x_i^a)^{2b} \alpha_1 a b x_i^{a-1} (1 - x_i^a)^{b-1}}{[1 - (1-\alpha_1)(1 - x_i^a)^b]^4} dx \\ &= \frac{m}{3\alpha_1^2} \end{aligned}$$

Similarly,

$$\begin{aligned} I_{22} &= -E\left(\frac{\partial^2 L}{\partial \alpha_2^2} \right) = \frac{n}{3\alpha_2^2} \\ I_{12} &= I_{21} = -E\left(\frac{\partial^2 L}{\partial \alpha_1 \partial \alpha_2} \right) = 0 \end{aligned}$$

Using the property of mle for $m \to \infty, n \to \infty$,

$$(\sqrt{m}(\hat{\alpha}_1 - \alpha_1), \sqrt{n}(\hat{\alpha}_2 - \alpha_2)) \xrightarrow{d} N_2(0, diag(a_{11}^{-1}, a_{22}^{-1})),$$

where,

$$\begin{aligned} a_{11} &= \lim_{m,n \to \infty} \frac{1}{m} I_{11} = \frac{1}{3\alpha_1^2} \\ a_{22} &= \lim_{m,n \to \infty} \frac{1}{n} I_{22} = \frac{1}{3\alpha_2^2} \end{aligned}$$

Now the 95% confidence interval for R is

$$\hat{R} \pm 1.96 \left(\hat{\alpha_1} b_1(\hat{\alpha_1}, \hat{\alpha_2}) \sqrt{\frac{3}{m} + \frac{3}{n}} \right),$$

where $\hat{R} = R(\hat{\alpha_1}, \hat{\alpha_2})$ is the estimate of R and

$$\begin{aligned} b_1(\alpha_1, \alpha_2) &= \frac{\partial R}{\partial \alpha_1} \\ &= \frac{\alpha_2}{(\alpha_1 - \alpha_2)^3} \left[2(\alpha_1 - \alpha_2) + (\alpha_1 + \alpha_2) \log \frac{\alpha_2}{\alpha_1} \right] \end{aligned}$$

12 Marshall-Olkin Kumaraswamy Minification process

We construct a first order autoregressive minification process with the structure as follows:

$$X_n = \begin{cases} \epsilon_n & \text{with probability} \quad p \\ \min(X_{n-1}, \epsilon_n) & \text{with probability} \quad 1-p \end{cases} \tag{12}$$

where $0 \leq p \leq 1$ and $\{\epsilon_n, n \geq 1\}$ is a sequence of iid random variables with $KS(a, b)$ and is independent of $\{X_n\}$.

Theorem 12.1 *Consider the minification process given by* (12) *with* X_0 *distributed as* $MOKS(p, a, b)$ *distribution,* $\{X_n, n \geq 0\}$ *is a stationary Markovian autoregressive model with the marginal as* $MOKS(p, a, b)$ *iff* $\{\epsilon_n\}$ *has a* $KS(a, b)$.

Proof: From (12), we have

$$\bar{F}_{X_n}(x) = p\bar{F}_{\epsilon_n}(x) + (1-p)\bar{F}_{X_{n-1}}(x)\bar{F}_{\epsilon_n}(x) \tag{13}$$

If X_0 has $MOKS(p, a, b)$ and ϵ_1 has $KS(a, b)$ distribution for $n = 1$ we have,

$$\bar{F}_{X_1}(x) = [p + (1-p)\bar{F}_{X_0}(x)]\bar{F}_{\epsilon_1}(x) = \frac{p(1-x^a)^b}{1 - \bar{p}(1-x^a)^b}$$

Therefore, X_1 has $MOKU(p, a, b)$ distribution . Using induction method, we can show that X_n has $MOKS(p, a, b)$ distribution. Hence $\{X_n\}$ is a stationary Markovian autoregressive model and its marginal are $MOKS(p, a, b)$. Conversly, let $\{X_n\}$ be a stationary Markovian autoregressive model with marginal as MOKS distribution ,then the stationary equilibrium is

$$\bar{F}_\epsilon(x) = \frac{\bar{F}_X(x)}{p + (1-p)\bar{F}_X(x)} = (1-x^a)^b$$

which means ϵ_n has $KS(a,b)$ distribution. Hence the converse of the theorem is true.

Remark: Even if X_0 is arbitrary, it is easy to prove that $\{X_n\}$ is stationary and is asymptotically marginally distributed as $MOKS(p,a,b)$

In order to study the behaviour of the process we simulate the sample paths of MOKS $AR(1)$ for various values of a, b and p. Now we consider some properties of MOKS minification process. The joint survival function of X_n and X_{n-1} is given by,

$$\begin{aligned}
\overline{F}(x,y) &= P(X_n > x, X_{n-1} > y) = \big(p\overline{F}_X(y) + (1-p)\overline{F}_X(\max(x,y))\big) \cdot \overline{F}_\varepsilon(x) \\
&= \begin{cases} \big(p\overline{F}_X(y) + (1-p)\overline{F}_X(x)\big) \cdot \overline{F}_\varepsilon(x), & 0 < y < x, \\ \overline{F}_X(y) \cdot \overline{F}_\varepsilon(x), & 0 < x < y. \end{cases}
\end{aligned}$$

$$= \begin{cases} \frac{(1-x^a)^b\{p^2(1-y^a)^b + p(1-p)(1-x^a)^b[1-(1-y^a)^b]\}}{[1-\overline{p}(1-y^a)^b][1-\overline{p}(1-x^a)^b]}, & 0 < y < x, \\ \\ \frac{p(1-y^a)^b(1-x^a)^b}{1-\overline{p}(1-y^a)^b}, & 0 < x < y. \end{cases}$$

The joint survival function is not an absolutely continuous since,

$$\begin{aligned}
P(X_n = X_{n-1}) &= (1-p)P(\epsilon_n \geq X_{n-1}) \\
&= (1-p)\int_0^1 P(\epsilon_n \geq x) f_{X_{n-1}}(x)dx \\
&= (1-p)\int_0^1 \bar{F}_\epsilon(x) f_{X_{n-1}}(x)dx \\
&= \frac{(1-p+p\log p)}{1-p} \in (0,1)
\end{aligned}$$

We have,

$$\begin{aligned}
P(X_n > X_{n-1}) &= pP(\epsilon_n > X_{n-1}) \\
&= p\int_0^1 P(\epsilon_n > x) f_{X_{n-1}}(x)dx \\
&= \frac{p(1-p+p\log p)}{(1-p)^2} \in (0, 1/2)
\end{aligned}$$

Let$E(X_n^k) = \mu_k$ and $E(\epsilon_n) = bB(1+\frac{1}{a}, b) = r$
Consider the autocovariance,

$$\begin{aligned}
Cov(X_n, X_{n-1}) &= E(X_n X_{n-1}) - \mu_1^2 \\
&= (1-p)E[Min(X_{n-1}, \epsilon_n)X_{n-1}] + pr\mu_1
\end{aligned}$$

$$\begin{aligned} E[Min(X_{n-1},\epsilon_n)X_{n-1}] &= E[I(X_{n-1}<\epsilon_n)X_{n-1}^2]+E[I(X_{n-1}>\epsilon_n)X_{n-1}\epsilon^n] \\ &= \mu_2 - E\{F_\epsilon(X_{n-1})X_{n-1}^2\} + E\{F_\epsilon(X_{n-1})X_{n-1}\} \\ &= \mu_2 - \int_0^1 x^2 F_\epsilon(x)g(x)dx + \int_0^1 xF_\epsilon(x)g(x)dx \end{aligned}$$

Thus

$$\begin{aligned} Cov(X_n,X_{n-1}) &= (1-p)\{\mu_2 - \alpha b \Sigma_{j=0}^{s}\Sigma_{k=0}^{\infty}\binom{k+1}{k}\alpha^{-k} \\ & (-1)^j[B(\frac{2}{a}+1,b(j+k+1))+B(\frac{1}{a}+1,b(j+k+1))+pr\mu_1]\} \end{aligned}$$

The first order process can be easily extended to high order process and the corresponding results can be derived.

13 Estimation of parameters

Here we estimate the parameters of MOKS distribution using the maximum likelihood estimation method. Let $X = x_1, x_2, \ldots, x_n$ be a random sample of size n from MOKS (α, a, b), then the log-likelihood function is

$$\begin{aligned} \log L &= n\log\alpha + n\log a + n\log b + (a-1)\sum_{i=1}^{n}\log x_i \\ & +(b-1)\sum_{i=1}^{n}\log(1-x_i^a) - 2\sum_{i=1}^{n}\log[1-\bar{\alpha}(1-x_i^a)^b] \end{aligned}$$

The normal equations are ,

$$\begin{aligned} \frac{\partial \log L}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n}\log x_i - (b-1)\sum_{i=1}^{n}\frac{x_i^a\log x_i}{1-x_i^a} \\ -2b\bar{\alpha}\sum_{i=1}^{n}\frac{x_i^a(1-x_i^a)^{b-1}\log x_i}{[1-\bar{\alpha}(1-x_i^a)^b]} = 0 \end{aligned}$$

$$\frac{\partial \log L}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\log(1-x_i^a) + 2\bar{\alpha}\sum_{i=1}^{n}\frac{(1-x_i^a)^b\log(1-x_i^a)}{[1-\bar{\alpha}(1-x_i^a)^b]} = 0$$

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} - 2\sum_{i=1}^{n}\frac{(1-x_i^a)^b}{[1-\bar{\alpha}(1-x_i^a)^b]} = 0$$

The above system of non linear equations does not have an analytic solutions in α, a and b. So the maximum likelihood estimates can be easily obtained using nlm function in R program.

14 Application

In this section, a data set is fitted to the MOKS distribution. The data set in Table 1 represents Floyd river flood rates for the years 1935-1973 in Iowa,USA. The maximum likelihood estimates, the Akaike Information Criterion (AIC), consistent Akaike Information Criterion (CAIC), the Bayesian Information Criterion(BIC), Hannan-Quinn Information Criterion (HQIC) and the Kolmogrov- Smirnov(K-S) test statistic and the p-value for the K-S statistics for the fitted distributions are reported in Tables 2. The MOKS distribution is fitted to the data set and compared the result with Kumaraswamy. The result shows that MOKS distribution provide good fit to the data. All the computations were done using R programme.

Table 1: Annual flood discharge rates of the Floyd river data(1935-1973)

1460	4050	3570	2060	1300	1390	1720	6280	1360	7440
5320	1400	3240	2710	4520	4840	8320	13900	71500	6250
2260	318	1330	970	1920	15100	2870	20600	3810	726
7500	170	2000	829	17300	4740	13400	2940	5660	

15 Conclusion

In this paper,we propose the new Marshall-Olkin Kumaraswamy distribution. We study some of its structural properties like moments, quantile function,

Table 2: Parameter estimates for annual flood discharge rates

Distribution	Estimates			AIC	CAIC	BIC	HQIC	K-S	p-value
MOKS(α, a, b)	0.00362	1.53452	0.69249	-140.208	-139.522	-135.217	-138.417	0.0681	0.9879
KS(a, b)		0.72762	6.77254	-126.622	-126.289	-123.295	-125.423	0.1487	0.3253

order statistics, record values, entropies, Lorenz, Bonferroni and Zenga curves. Autoregressive model with minification structure is also discussed. The maximum likelihood method is used for estimating the model parameters. An application to a real data set shows that the fit of the new model is superior to the other model. We hope that the proposed model will attract wider applications in several areas like engineering, survival and lifetime data, hydrology etc.

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