International Mathematical Forum, Vol. 12, 2017, no. 4, 159 - 171
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2017.612165

Convolution and Subordination Properties of Analytic Functions with Bounded Radius Rotations

Khalida Inayat Noor

Department of Mathematics
COMSATS Institute of Information Technology
Islamabad, Pakistan

Bushra Malik

Department of Mathematics
COMSATS Institute of Information Technology
Islamabad, Pakistan

Syed Zakar Hussain Bukhari

Department of Mathematics
Mirpur University of Science and Technology (MUST)
Mirpur-10250(AJK), Pakistan

Abstract

A function analytic and locally univalent in a simply connected domain is of bounded radius rotation if its range has bounded radius rotation which is defined as the total variation of the direction angle of the radial vector to the boundary curve under a complete circuit. In this paper, we introduce some subclasses of analytic functions with bounded radius rotation involving subordination and establish integral and convolution preserving properties. We also determine estimates for the growth and distortion bounds, inclusions, conditions for starlikeness and bounds on coefficients differences. Most of our findings are related with the existing known results and several of their applications are found in literature of the subject.

Mathematics Subject Classification: Primary: 30C45, Secondary: 30C80

Keywords: Bounded radius rotation; Convolution; Schwarz function; subordination

1 Introduction

Let $\mathcal{H}(\mathbb{U})$ represent the class of all analytic functions f defined in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$, and for a positive integer n and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] := \left\{ f \in \mathcal{H}(\mathbb{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots,\ z \in \mathbb{U} \right\}.$$

Also, we denote

$$\mathcal{A} := \{ f \in \mathcal{H}[0,1] : f'(0) = 1 \}$$

The class of univalent functions is represented by $\mathcal{S}$ and it is a subclass of the class $\mathcal{A}$, whereas, $\mathcal{S}^*, \mathcal{C}, \mathcal{K}$ and $\mathcal{Q}$ are the well-known classes of starlike, convex, close-to-convex and quasi-convex functions respectively.

Let $\mathcal{P}$ denote the well-known class of *Carathéodory functions* p such that $p \in \mathcal{H}(\mathbb{U})$, with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$, $z \in \mathbb{U}$. Also $\mathcal{P}(\beta)$ represents the class of *Carathéodory functions* p such that $p \in \mathcal{H}(\mathbb{U})$ with $p(0) = 1$ and $\operatorname{Re} p(z) > \beta, 0 \le \beta < 1, z \in \mathbb{U}$. For details, we refer [4].

For $f, g \in \mathcal{H}(\mathbb{U})$, we say that f *is subordinate to* g and write as $f(z) \prec g(z)$, if there exists a *Schwarz function* w, that is, $w \in \mathcal{H}(\mathbb{U})$, with $w(0) = 0$ and $|w(z)| < 1$, such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{U}).$$

Using subordination defined above, Janowski [6] introduced the class $\mathcal{P}[A,B]$ for $-1 \le B < A \le 1$. A function p analytic in $\mathbb{U}$ such that $p(0) = 1$ belongs to the class $\mathcal{P}[A,B]$, if

$$p(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}) \quad \text{or} \quad p(z) = \frac{1+Aw(z)}{1+Bw(z)} \qquad (z \in \mathbb{U}),$$

where w is a Schwarz function. Geometrically, the image $p(\mathbb{U})$ of $p \in \mathcal{P}[A,B]$ lies inside the open unit disk centered on the real axis with diameter ends at $p(-1)$ and $p(1)$. Clearly, $\mathcal{P}[A,B] \subset \mathcal{P}\left(\frac{1-A}{1-B}\right)$. Using a general bilinear fractional transformation $T(z) = \frac{1+A_1 z}{1+B_1 z}$, $A_1 \in \mathbb{C}$, $B_1 \in [-1,0]$ and $A_1 \neq B_1$, Noor [8] introduced the class $\mathcal{P}[A_1, B_1]$ consisting of functions p with $p(0) = 1$ and $p(z) \prec \frac{1+A_1 z}{1+B_1 z}$. Further extending this idea, Noor [9] introduced the class $\mathcal{P}_k[A,B]$ which is defined in the following.

A function p analytic in $\mathbb{U}$ with $p(0) = 1$, belongs to the class $\mathcal{P}_k[A, B]$, if and only if

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \quad p_{1,}\ p_2 \in \mathcal{P}[A, B].$$

Also $\mathcal{P}_k[1, -1] = \mathcal{P}_k$ and $\mathcal{P}_k[1-2\beta, -1] = \mathcal{P}_k(\beta)$. Pinchuk [18] studied the class $\mathcal{P}_k$, whereas Padmanabhan and Parvatham [16] introduced and investigated the class $\mathcal{P}_k(\beta)$. For $-1 \le B < A \le 1$ and $b \in \mathbb{C}/\{0\}$, $p \in \mathcal{P}_k[b, A,B]$ with $p(0) = 1$, if and only if there exists $p_1 \in \mathcal{P}_k[A, B]$ such that $p(z) = bp_1(z) + (1 - b), z \in \mathbb{U}$. A function f belongs to the class $\mathcal{V}_k[A, B]$, $k \ge 2$, $-1 \le B < A \le 1$, if and only if

$$\frac{(zf'(z))'}{f'(z)} \in \mathcal{P}_k[A, B] \qquad (z \in \mathbb{U}).$$

Similarly, $f \in \mathcal{R}_k[A, B]$, if and only if

$$\frac{zf'(z)}{f(z)} \in \mathcal{P}_k[A, B] \qquad (z \in \mathbb{U}).$$

For detail, we refer [11, 12]. We note that $\mathcal{V}_k[1 - 2\beta, 1] = \mathcal{V}_k(\beta)$ and $\mathcal{R}_k[1 - 2\beta, 1] = \mathcal{R}_k(\beta)$, $0 \le \beta < 1$.

Definition 1.1. Let $f \in \mathcal{A}$. Then $f \in \mathcal{T}_{k,m}[b, A, B; C, D]$, $k, m \ge 2, -1 \le B < A \le 1$, if and only if there exists

$$g \in \mathcal{V}_m[C, D], -1 \le C < D \le 1,$$

such that

$$\left\{1 + \frac{1}{b}\left(\frac{f'(z)}{g'(z)} - 1\right)\right\} \in \mathcal{P}_k[A, B],\ b \in \mathbb{C}/\{0\} \qquad (z \in \mathbb{U}).$$

Definition 1.2. Let $f \in \mathcal{A}$. Then $f \in \mathcal{T}^*_{k,m}[b, A, B; C, D]$ for $k, m \ge 2, -1 \le B < A \le 1$ and $b \in \mathbb{C}/\{0\}$, if and only if there exists $g \in \mathcal{V}_m[C, D], -1 \le D < C \le 1$ such that

$$\frac{(zf'(z))'}{g'(z)} \in \mathcal{P}_k[b, A, B] \qquad (z \in \mathbb{U}).$$

It is clear that $f \in \mathcal{T}^*_{k,m}[b, A, B; C, D]$, if and only if $zf' \in \mathcal{T}_{k,m}[b, A, B; C, D]$.

For special choices, we refer Al-Amiri and Fernando [1], Kaplan [7], Noor [3, 10, 13, 15], Polatoğlu and others [19 − 22] and Silvia [26] with references therein.

Definition 1.3. Let $f \in \mathcal{A}$. Then $f \in \mathcal{M}^{\alpha}_{k,m}[b, A, B; C, D]$ for $k, m \ge$

$2,\ \alpha \geq 0,\ b \in \mathbb{C}/\{0\}$ and $-1 \leq B < A \leq 1$, if and only if there exists $g \in \mathcal{V}_m[C,D], -1 \leq D < C \leq 1$ such that

$$\left\{(1-\alpha)\frac{f'(z)}{g'(z)} + \frac{\alpha\,(zf'(z))'}{g'(z)}\right\} \in \mathcal{P}_k[b,A,B] \qquad (z \in \mathbb{U}).$$

It follows from the above definitions that

$$f \in \mathcal{M}^{\alpha}_{k,m}[b,A,B;C,D] \text{ if and only if } \ (1-\alpha)\,f + \alpha z f' \in \mathcal{T}_{k,m}[b,A,B;C,D]. \tag{1}$$

For special cases, see [3, 10, 13, 15] and others. To avoid repetition of parameters, we assume these parameters within the above specific ranges throughout our discussion. However, any change will obviously be mentioned.

2 Preliminary Notes

Lemma 2.1[24]. *If* $f \in \mathcal{C},\ g \in \mathcal{S}^*$, *then for any analytic function* F *in* $\mathbb{U}$,

$$\frac{(f * Fg)\,(\mathbb{U})}{(f * g)\,(\mathbb{U})} \subset \overline{co}F\,(\mathbb{U}) \qquad (z \in \mathbb{U}),$$

where $\overline{co}\ F(\mathbb{U})$ *denotes the closed convex hull of* $F(\mathbb{U})$.

Lemma 2.2[17]. *Let* N *and* D *be analytic in* $\mathbb{U}$, D *maps* $\mathbb{U}$ *onto a many-sheeted starlike region*

$$N(0) = 0 = D(0),\ N'(0) = D'(0) = 1 \ \text{ and } \ \frac{N'(z)}{D'(z)} \in \mathcal{P}[A,B] \qquad (z \in \mathbb{U}).$$

Then, $\frac{N(z)}{D(z)} \in \mathcal{P}[A,B] \quad (z \in \mathbb{U})$.

Lemma 2.3. *Let* $p \in \mathcal{P}(b),\ b \in \mathbb{C}/\{0\}$ *and* $h \in \mathcal{P}_k(\beta),\ k \geq 2, 0 \leq \beta < 1$, *where* $p(0) = h(0) = 1$. *Then for* $|z| = r < 1$,

$$(i) \qquad \frac{1}{2\pi}\int_0^{2\pi} |\left(h(re^{i\theta}\right)|^2 d\theta \leq \frac{1 + [k^2(1-\beta) - 1)r^2}{1-r^2},$$

$$(ii) \qquad \frac{1}{2\pi}\int_0^{2\pi} |\left(p(re^{i\theta}\right)|^2 d\theta \leq \frac{1 + [4|b|^2 - 1)r^2}{1-r^2}.$$

For the proof of part (i), we refer [14] and the part (ii) is proved in [10].

Lemma 2.4 *Let* $f \in \mathcal{V}_k[A,B],\ k \geq 2,\ -1 \leq B < A \leq 1$. *Then for* $|z| = r < 1$,

$$\begin{cases} \frac{(1-Br)^{\left(\frac{A-B}{B}\right)\left(\frac{k}{4}+\frac{1}{2}\right)}}{(1+Br)^{\left(\frac{A-B}{B}\right)\left(\frac{k}{4}-\frac{1}{2}\right)}}, & B \neq 0 \\ \exp\left(-A\frac{k}{2}r\right), & B = 0 \end{cases} \leq |f'(z)| \leq \begin{cases} \frac{(1+Br)^{\left(\frac{A-B}{B}\right)\left(\frac{k}{4}+\frac{1}{2}\right)}}{(1-Br)^{\left(\frac{A-B}{B}\right)\left(\frac{k}{4}-\frac{1}{2}\right)}}, & B \neq 0 \\ \exp\left(A\frac{k}{2}r\right), & B = 0 \end{cases}.$$

The result is sharp for the extremal function:

$$f_0(z) = \left(\frac{(1+Bz)^{\left(\frac{k}{4}+\frac{1}{2}\right)}}{(1-Bz)^{\left(\frac{k}{4}-\frac{1}{2}\right)}} \right)^{\frac{A-B}{B}}, \; B \neq 0 \; \text{ and } \; f_0(z) = \exp\left(A\frac{k}{2}z \right), \; \text{ for } \; B = 0.$$

Lemma 2.5 [25]. *If* $g \in \mathcal{S}^*[C, D], -1 \leq D < C \leq 1$, *then so does* $\varphi * g$ *for* φ *analytic and convex in* $\mathbb{U}$.

Lemma 2.6 [11]. *Let* $\beta, n \in \mathbb{Z}^+$ *and* $f \in \mathcal{V}_k[A, B]$ *and* F *be defined by*

$$(F(z))^{\beta} = \frac{\beta+n}{z^n} \int_0^z t^{n-1} (f(t))^{\beta} dt \qquad (z \in \mathbb{U}). \tag{2}$$

Then $f \in \mathcal{V}_2(\frac{1-A}{1-B}) = \mathcal{C}\left(\frac{1-A}{1-B}\right)$ *for* $|z| < r_0$, *where* r_0 *is given by*

$$r_0 = \frac{4}{k(1-B) + \sqrt{k^2(1-B)^2 + 16B}}. \tag{3}$$

.

3 Results and Discussion

In the following theorem, we study some integral preserving properties under certain assumption on parameters.

Theorem 3.1. *Let* $f \in \mathcal{M}^{\alpha}_{k,m}[b, A, B; C, D]$. *Then* $F \in \mathcal{T}_{k,m}[b, A, B; C, D]$ *such that for* $\alpha > 0$,

$$f(z) = \frac{1}{\alpha} z^{1-\frac{1}{\alpha}} \int_0^z t^{\frac{1}{\alpha}-2} F(t) dt \qquad (z \in \mathbb{U}). \tag{4}$$

. *Proof*. From (4),we can write

$$(1-\alpha)f(z) + \alpha z f'(z) = F(z) \qquad (z \in \mathbb{U})$$

and the result follows immediately by using (1)

The following theorems deal with the convolution preserving properties under certain assumption on parameters.

Theorem 3.2. *Let* $f \in \mathcal{T}_{k,2}[b, A, B; C, D]$ *and* Ψ *be any convex univalent function in* $\mathbb{U}$. *Then*

$$\Psi * f \in \mathcal{T}_{k,2}[b, A, B; C, D] \qquad (z \in \mathbb{U}).$$

Proof. For $G \in \mathcal{R}_2[C, D] \equiv \mathcal{S}^*[C, D]$, consider

$$\frac{z(\Psi * f)'(z)}{(\Psi * G)(z)} = \frac{\Psi(z) * \frac{zf'(z)}{G(z)} G(z)}{\Psi(z) * \; G(z)} \qquad (z \in \mathbb{U}).$$

From Lemma 2.4, we note that $\Psi * G \in \mathcal{S}^*[C,D]$ and using Lemma 2.1, we have

$$\frac{z(\Psi * f)'(z)}{(\Psi * G)(z)} \in \mathcal{P}_k[b,A,B] \qquad (z \in \mathbb{U}),$$

where $\Psi * G \in \mathcal{S}^*[C,D]$, which proves the required result.

Theorem 3.3. *Let $f \in \mathcal{M}^{\alpha}_{k,2}[b,A,B;C,D]$, $\alpha \geq 0$ and Ψ be any convex univalent function in $\mathbb{U}$. Then*

$$f * \Psi \in \mathcal{M}^{\alpha}_{k,2}[b,A,B;C,D] \qquad (z \in \mathbb{U}).$$

Proof. Using (1), we have

$$(1-\alpha)f(z) + \alpha z f'(z) \in \mathcal{T}_{k,2}[b,A,B;C,D] \qquad (z \in \mathbb{U}).$$

In view of Theorem 3.2, we write

$$\{(1-\alpha)f(z) + \alpha z f'(z)\} * \Psi(z) \in \mathcal{T}_{k,2}[b,A,B;C,D],$$

which implies that

$$(1-\alpha)(f * \Psi)(z) + \alpha z (f * \Psi)(z))' \in \mathcal{T}_{k,2}[b,A,B;C,D] \qquad (z \in \mathbb{U}).$$

Again using relation (1), we have the required result.

Applications of Theorem 3.3

Let $f \in \mathcal{M}^{\alpha}_{k,m}[b,A,B;C,D]$. Then the functions Ψ_i such that

$$\Psi_1(z) = \int_0^z \frac{f(t)}{t} dt, \quad \Psi_2(z) = \frac{2}{z} \int_0^z f(t) dt, \quad \Psi_3(z) = \int_0^z \frac{f(t) - f(xt)}{t - xt} dt, \;\; |x| \leq 1, \; x \neq 1$$

and

$$\Psi_4(z) = \frac{1+c}{z^c} \int_0^z t^{c-1} f(t) dt, \;\; \operatorname{Re} c > 0 \qquad (z \in \mathbb{U})$$

are also belong to $\mathcal{M}^{\alpha}_{k,m}[b,A,B;C,D]$.

The proof is immediatly follows, when we observe that $\Psi_i(z) = (f * \varphi_i)(z), \; i = 1,2,3,4 \qquad (z \in \mathbb{U})$, where φ_i is convex in $\mathbb{U}$ such that

$$\varphi_1(z) = -\log(1-z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n, \qquad \varphi_2(z) = \frac{-2[z + \log(1-z)]}{z}$$

$$\varphi_3(z) = \frac{1}{1-x} \log[\frac{1-xz}{1-z}] \;\text{ and }\; \varphi_4(z) = \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n, \operatorname{Re}\; c > 0 \qquad (z \in \mathbb{U}).$$

In the theorem below, we study inclusion with respect to the parameter α.

Theorem 3.4. *For $\alpha \geq 0$, we have $\mathcal{M}^{\alpha}_{k,2}[b, A, B; C, D] \subset \mathcal{T}_{k,2}[b, A, B; C, D]$.*

Proof. For $\alpha = 0$, the result is obvious. We assume that $\alpha > 0$ and using the integral representation for $f \in \mathcal{M}^{\alpha}_{k,2}[b, A, B; C, D]$, we have

$$f(z) = \frac{1}{\alpha} z^{\frac{1}{\alpha}-1} \int_0^z t^{\frac{1}{\alpha}-2} F(t)dt \qquad (z \in \mathbb{U}),$$

where $F \in \mathcal{I}_{k,2}[b, A, B; C, D]$ and $f(z) = \varphi_\alpha(z) * F(z)$. The function φ_α such that

$$\varphi_\alpha(z) = \sum_{n=1}^{\infty} \frac{1}{\alpha(n-1)+1} z^n \qquad (z \in \mathbb{U})$$

is convex. Using Theorem 3.2, we conclude that $f \in \mathcal{T}_{k,2}[b, A, B; C, D]$ and hence, we obtain the required result.

In the next theorems, we determine the growth and distortion bounds and various other related results.

Theorem 3.5. *Let $f \in \mathcal{T}_{k,m}[b, A, B; C, D]$. Then for $|z| = r < 1$ and $D \neq 0$, we have*

$$\frac{c}{1-Br}\left[\frac{(1-Dr)^{(\frac{m}{4}+\frac{1}{2})}}{(1+Dr)^{(\frac{m}{4}-\frac{1}{2})}}\right]^{\frac{C-D}{D}} \leq |f'(z)| \leq \frac{c_1}{1+Br}\left[\frac{(1+Dr)^{(\frac{m}{4}+\frac{1}{2})}}{(1-Dr)^{(\frac{m}{4}-\frac{1}{2})}}\right]^{\frac{C-D}{D}}$$

and

$$\exp\left(\frac{-m}{2}Cr\right)\frac{\Re}{1-Br} \leq |f'(z)| \leq \exp\left(\frac{m}{2}Cr\right)\frac{\Re_1}{1+Br}, \; D = 0.$$

where $c = \frac{k}{2}|b|(1-Ar)+|1-b|(1-Br)$ and $c_1 = \frac{k}{2}|b|(1+Ar)+|1-b|(1+Br)$.

Proof. By Definition 1.2, there exists $g \in \mathcal{V}_m[C, D]$ and $p \in \mathcal{P}_k[b, A, B]$, such that

$$\begin{aligned} f'(z) &= g'(z)p(z) = g'(z)\left[bh(z) + (1-b)\right], \qquad h \in \mathcal{P}_k[A, B] \\ &= g'(z)\left[b\left\{\left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z)\right\} + (1-b)\right], \end{aligned} \tag{5}$$

where $p_1, p_2 \in \mathcal{P}[A, B]$.

$$|f'(z)| \leq |g'(z)|\left[|b|\left\{\left(\frac{k}{4}+\frac{1}{2}\right)|p_1(z)| + \left(\frac{k}{4}-\frac{1}{2}\right)|p_2(z)|\right\} + |(1-b)|\right].$$

Using distortion bounds for $p_1, p_2 \in \mathcal{P}[A, B]$ proved in [6] and Lemma 2.1, we have the required result.

For $k=2$, $m=2$ and $b=1$, we obtain the bounds for function f such that $f \in \mathcal{K}[A,B;C,D]$, where the class $\mathcal{K}[A,B;C,D]$ is introduced by Silvia [26].

Theorem 3.6. *Let $f \in \mathcal{T}_{2,m}[b,A,B;C,D]$ and F be defined by*

$$F(z)=\frac{n+1}{z^n}\int_0^z t^{n-1}f(t)dt, n \in \mathbb{Z}^+ \qquad (z \in \mathbb{U}). \tag{6}$$

Then $F \in \mathcal{T}_{2,2}[b,A,B;1-2\beta,-1]$, for $|z|<r_1$, r_1 is given by (3).

Proof. Suppose that

$$1+\frac{1}{b}\left(\frac{f'(z)}{g'(z)}-1\right)=p(z) \qquad (z \in \mathbb{U}),$$

where $p \in \mathcal{P}[A,B]$ and $g \in \mathcal{V}_m[C,D]$. Let us consider

$$G(z)=\frac{n+1}{z^n}\int_0^z t^{n-1}g(t)dt, n \in \mathbb{Z}^+ \qquad (z \in \mathbb{U}).$$

Then by using lemma 2.6 for $\alpha=1$, we have $G \in \mathcal{C}(\frac{1-C}{1-D})$, for $|z|<r_1$, where r_1 is given by (3). Consider

$$1+\frac{1}{b}\left(\frac{F'(z)}{G'(z)}-1\right)=\frac{(\frac{1}{b}-1)\int_0^z t^n g'(t)dt+\frac{1}{b}\int_0^z t^n f'(t)dt}{\int_0^z t^n g'(t)dt}=\frac{N(z)}{D(z)}.$$

Also

$$\frac{N'(z)}{D'(z)}=\frac{(\frac{1}{b}-1)g'(z)+\frac{1}{b}f'(z)}{g'(z)}=\left(1+\frac{1}{b}\frac{f'(z)}{f'(z)}-1\right) \in \mathcal{P}[A,B].$$

Now, since D is m-valently starlike, therefore by using Lemma 2.2, we have

$$\frac{N(z)}{D(z)}=1+\frac{1}{b}\left(\frac{F'(z)}{G'(z)}-1\right) \in \mathcal{P}[A,B], \quad \text{for } |z|<r_1.$$

Hence, we have the desired proof.

For $b=1$ and $f \in \mathcal{T}_{2,m}[1,A,B;C,D]=\mathcal{T}_m[A,B;C,D]$, the function F defined by (6) is a close-to-convex function for $|z|<r_1$, where r_1 is given by (3), for details, see [12].

Theorem 3.7. *Let $f \in \mathcal{T}_{2,2}[b,A,B;C,D]$. Then the function F defined by* (2) *belongs to $\mathcal{S}^*(b)$.*

Proof. Let

$$J(z)=\int_0^z t^{n-1}(f(t))^{\beta}dt,\ \beta,\ n \text{ are positive integers.}$$

So from (2) on simplification, we have

$$\frac{zF'(z)}{F(z)}=\frac{1}{\beta}\left(\frac{zJ'(z)-nJ(z)}{J(z)}\right) \qquad (z\in\mathbb{U})\,.$$

Let

$$1+\frac{1}{b}\left(\frac{zF'(z)}{F(z)}-1\right)=\frac{(\frac{1}{b}-1)J(z)+\frac{1}{b}\left(zJ'(z)-nJ(z)\right)}{\beta J(z)}=\frac{N(z)}{D(z)}:N(0)=0=D(0).$$

By a result of Bernardi [2] and the relation $\mathcal{K}[A,B;C,D]\subset\mathcal{C}$ proved by Silvia [26], we have D is $(\beta+n-1)$-valent starlike. Also

$$\frac{N'(z)}{D'(z)}=\frac{1}{\beta}\left[\frac{(\frac{1}{b}-1)J'(z)+\frac{1}{b}\left((zJ'(z))'-nJ'(z)\right)}{J'(z)}\right]=1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\in\mathcal{P}.$$

Using Lemma 2.2 for $A=1,\ B=-1$, we have

$$\frac{N(z)}{D(z)}=1+\frac{1}{b}\left(\frac{zF'(z)}{F(z)}-1\right)\in\mathcal{P} \qquad (z\in\mathbb{U})\,.$$

This proves the required result.

Theorem 3.8. *Let $f\in\mathcal{T}_{2,m}[b,1,-1;1-2\beta,-1]=\mathcal{T}_m(b;\beta)$. Then*

$$n^2||a_{n+1}|-|a_n||\le e^2 n^{(\frac{k}{2}+1)(1-\beta)}(\frac{1}{4})^{(1-\beta)}|b|(k(1-\beta)+1).$$

Proof. We have

$$F(z)=(zf'(z))'=g'(z)\left[H(z)p(z)+zp'(z)\right], \qquad (H\in\mathcal{P}_k(\beta))\,. \tag{7}$$

As in [16], for $g\in\mathcal{V}_k(\beta)$, there exists $g_1\in\mathcal{V}_k$ such that

$$g'(z)=(g_1'(z))^{1-\beta}\,\frac{\left(\frac{s_1(z)}{z}\right)^{(\frac{k}{4}+\frac{1}{2})(1-\beta)}}{\left(\frac{s_2(z)}{z}\right)^{(\frac{k}{4}-\frac{1}{2})(1-\beta)}}, \qquad (s_1,\ s_2\in\mathcal{S}^*)\,.$$

Thus equation (7) becomes

$$F(z)=(zf'(z))'=\frac{(s_1(z))^{(\frac{k}{4}+\frac{1}{2})(1-\beta)}}{(z)^{(1-\beta)}\left(s_2(z)\right)^{(\frac{k}{4}-\frac{1}{2})(1-\beta)}}\left[H(z)p(z)+zp'(z)\right].$$

Since s_1 is univalent, we choose $z_1 = z_1(r)$ with $|z_1| = r$, $Q(r) = \max_{|z|=r} |(z - z_1)s_1(z)| \leq \frac{2r^2}{1-r^2}$, see [5] and write

$$\frac{1}{2\pi}\int_0^{2\pi} |(z-z_1)F(z)|d\theta \leq \frac{Q(r)}{r}\left[\frac{1}{2\pi}\int_0^{2\pi} \frac{(|s_1(z)|)^{[(\frac{k}{4}+\frac{1}{2})(1-\beta)]}\left(|H(z)p(z)+zp'(z)|\right)}{|z|^{(1-\beta)}|s_2(z)|^{(\frac{k}{4}-\frac{1}{2})(1-\beta)}}\right]d\theta,$$

where we have used the Schwarz inequality. Thus

$$|n^2a_n-(n+1)2z_1a_{n+1}| \leq \frac{r^2}{\pi(1-r^2)}\frac{1}{r^{(n+1-\beta)}}\int_0^{2\pi} \frac{(|s_1(z)|)^{[(\frac{k}{4}+\frac{1}{2})(1-\beta)]}|H(z)p(z)+zp'(z)|}{|z|^{(1-\beta)}|s_2(z)|^{(\frac{k}{4}-\frac{1}{2})(1-\beta)}}d\theta.$$

Using the well-known distortion bounds for $s_1,\ s_2 \in \mathcal{S}^*$, see [4], we have

$$|n^2a_n-(n+1)^2z_1a_{n+1}| \leq \frac{1}{r^{n-1}}\frac{2}{1-r^2}\left[\frac{1}{(1-r)^2}\right]^{(\frac{m}{4}+\frac{1}{2})[(1-\beta)-1]}(4)^{(\frac{m}{4}-\frac{1}{2})(1-\beta)}I, \tag{8}$$

where

$$I = \left(\frac{1}{2\pi}\int_0^{2\pi}|H(z)|^2d\theta * \frac{1}{2\pi}\int_0^{2\pi}|p(z)|^2d\theta\right)^{\frac{1}{2}} + \frac{r}{2\pi}\int_0^{2\pi}|p'(z)|d\theta. \tag{9}$$

Since $p \in \mathcal{P}(b)$, we can write $p(z) = bh(z) + (1-b)$, $h \in \mathcal{P}$ or $p'(z) = bh'(z)$ and

$$\frac{1}{2\pi}\int_0^{2\pi}|h'(z)|d\theta \leq \frac{2}{1-r^2},\ \text{see [4]}. \tag{10}$$

Using Lemma 2.3 and inequality (10) in (9), we have

$$\begin{aligned}&\left[\frac{1+(m^2(1-\beta^2)-1)r^2}{1-r^2}\right]^{\frac{1}{2}}\left[\frac{1+(4|b|^2-1)r^2}{1-r^2}\right]^{\frac{1}{2}} + \frac{2r|b|}{1-r^2}\\ &= \frac{1}{1-r^2}\left[\left(1+(m^2(1-\beta^2)-1)r^2\right)^{\frac{1}{2}}\left(1+(4|b|^2-1)r^2\right)^{\frac{1}{2}}+2r|b|\right].\end{aligned}$$

This result along with (8) yields

$$|n^2a_n-(n+1)^2z_1a_{n+1}| \leq \frac{1}{r^{n+1}}\frac{2r}{(1+r)^2}\left[\frac{1}{(1-r)^2}\right]^{(\frac{m}{4}+\frac{1}{2})(1-\beta)}\Psi(m,r,b,\beta),$$

where

$$\Psi(m,r,b,\beta) = 2^{(\frac{m}{2}-1)(1-\beta)}\left[\left(1+(m^2(1-\beta^2)-1)r^2\right)^{\frac{1}{2}}\left(1+(4|b|^2-1)r^2\right)^{\frac{1}{2}}+2r|b|\right].$$

We take $r = r_n = \left(\frac{n}{n+1}\right)^2$. Because $|z_1| = r_n$, it follows that

$$n^2||a_n|-|a_{n+1}| \leq e^2n^{(\frac{m}{4}+\frac{1}{2})[(1-\beta)}\left(\frac{1}{4}\right)^{(1-\beta)}|b|\left[m(1-\beta)+1\right].$$

4 Conclusion

In this paper, we introduced some subclasses of analytic functions with bounded radius rotation involving subordination and established integral and convolution preserving properties. We also determined estimates for the growth and distortion bounds, inclusions, conditions for starlikeness and bounds on coefficients differences. Most of our findings are related with the existing known results found in literature of the subject.

Acknowledgements. The authors would like to thank Honorable Rector CIIT, Islamabad Prof. Dr. SM Junaid Zaidi (HI, SI) and Worthy Vice Chancellor MUST, Mirpur Prof. Dr. Habib-ur-Rahman (FCSP, SI) for their untiring efforts for the promotion of research conducive environment in their relevant Institutions.

References

[1] H.S. Al-Amiri and T.S. Fernando, On close-to-convex functions of complex order, *Int. J. Math. Math. Sci.*, **13** (1990), 321-330. https://doi.org/10.1155/s0161171290000473

[2] S.D. Bernardi, Convex and starlike univalent functions, *Trans. Am. Math. Soc.*, **35** (1969), 429-446. https://doi.org/10.1090/s0002-9947-1969-0232920-2

[3] S.Z.H. Bukhari and K.I. Noor, Some subclasses of analytic functions with bounded Mocanu variation, *Acta. Math. Hungarica*, **126** (2010), 199-208. https://doi.org/10.1007/s10474-009-9043-8

[4] A.W. Goodman, *Univalent Functions*, Vol. I, II, Mariner Publ Comp. Tampa, Florida, 1983.

[5] G. Golusin, On distortion theorems and coefficients of univalent functions, *Mat Sb.*, **19** (1946), 183-203.

[6] W. Janowski, Some extremal problems for certain families of analytic functions I, *Ann. Polon. Math.*, **28** (1973), 297-326.

[7] W. Kaplan, Close-to-convex schlicht functions, *Michigan Math. J.*, **1** (1952), 169-185. https://doi.org/10.1307/mmj/1028988895

[8] K.I. Noor, Applications of certain operators to the classes related with generalized Janowski functions, *Integral Transforms Spec. Funct.*, **21** (2010), 557-567. https://doi.org/10.1080/10652460903424261

[9] K.I. Noor, Some properties of analytic functions with bounded radius rotation, *Complex Var. Elliptic Eq.*, **54** (2009), 865-877. https://doi.org/10.1080/17476930902998878

[10] K.I. Noor, On close-to-convex functions of complex order and related subclasses, *Analysis, Geometry and Groups: A Riemann Legacy Volume*, (eds.) H. M. Srivastava, Th. M. Rassias, Hardonic Press, Palm Harbor, Florida, USA, 1993, 313-335.

[11] K.I. Noor, On radii of convexity and starlikeness of some classes of analytic functions, *Int. J. Math. Math. Sci.*, **14** (1991), 741-746. https://doi.org/10.1155/s016117129100100x

[12] K.I. Noor, Some radius of convexity problems for analytic functions of bounded boundary rotations, *Punjab Univ. J. Math.*, **21** (1988), 71-81.

[13] K.I. Noor, On quasi-convex functions and related topics, *Int. J. Math. Math. Sci.*, **10** (1987), 241-258. https://doi.org/10.1155/s0161171287000310

[14] K.I. Noor, *On Close-to-Convex and Related Functions*, Ph.D. Thesis, Swansea Uni. Wales, UK, 1972.

[15] K.I. Noor and S.Z.H. Bukhari, On analytic functions related with generalized Robertson functions, *Appl. Math. Comput.*, **215** (2009), 2965–2970. https://doi.org/10.1016/j.amc.2009.09.043

[16] K.S. Padmanabhan and R. Parvatham, Properties of a class of functions with bounded boundary rotation, *Ann. Polon. Math.*, **31** (1975), 311-323.

[17] R. Parvatham, T.N. Shanmugam, On analytic functions with reference to an integral operator, *Bull. Australian Math. Soc.*, **28** (1983), 207-215. https://doi.org/10.1017/s0004972700020888

[18] B. Pinchuk, Functions with bounded boundary rotation, *Isr. J. Math.*, **10** (1971), 6-16. https://doi.org/10.1007/bf02771515

[19] Y. Polatoğlu and H.E. Özkan, New subclass of starlike functions of complex order, *J. Prime Res. Math.*, **2** (2006), 157-169.

[20] Y. Polatoğlu and M. Bolcal, A coefficient inequality for the class of analytic functions in the unit disk, *International Journal of Mathematics and Mathematical Sciences*, **59** (2003), 3753–3759. https://doi.org/10.1155/s0161171203208322

[21] Y. Polatoğlu and M. Bolcal, An investigation on the Janowski convex function of complex order, *J. İstanbul Kültür Univ.*, **3** (2003), 65-70.

[22] Y. Polatoğlu, M. Bolcal, A. Şen and E. Yavuz, A study on the generalization of Janowski functions in the unit disk, *Acta. Math. Acad. Paedag. Nyíregyháziensis*, **22** (2006), 27–31.

[23] Ch. Pommerenke, On close-to-convex analytic functions, *Trans. Am. Math. Soc.*, **14** (1965), 176-186.
https://doi.org/10.1090/s0002-9947-1965-0174720-4

[24] S. Ruscheweyh and T. Sheil-Small, Hadamard products of Schlicht functions and the Polya-Schoenberg conjecture, *Comment. Math. Helv.*, **48** (1973), 119-135. https://doi.org/10.1007/bf02566116

[25] H. Silverman and E.M. Silvia, Subclasses of starlike functions subordinate to convex functions, *Canad. J. Math.*, **37** (1985), 48-61.
https://doi.org/10.4153/cjm-1985-004-7

[26] E.M. Silvia, Subclasses of close-to-convex functions, *Int. J. Math. Math. Sci.*, **6** (1983), 449-458. https://doi.org/10.1155/s0161171283000393

Received: December 21, 2016; Published: January 24, 2017