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Locating Chromatic Number of Banana Tree

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Abstract

Locating chromatic number of a graph G, denoted by $\chi_L(G)$ is a combination of two concepts, namely coloring and partition dimension of graph. On the locating chromatic number of G, every vertex is partitioned into color classes. The distance of every vertex to the color classes is considered to produce a different color code. In this paper will be discussed about the locating chromatic number of banana tree.

Mathematics Subject Classification: 05C12, 05C15

Keywords: locating-chromatic number, color classes, banana tree

1 Introduction

Let G be a simple connected graph. Let c be a proper coloring of a graph G using k colors (k-coloring) for some positive integer k, where $c(v) \neq c(w)$ for adjacent vertices v and w in G. Let $\Pi = \{C_1, C_2, \ldots, C_k\}$ be a partition of $V(G)$ induced by c on $V(G)$, where the vertices of C_i are colored i for $1 \leq i \leq k$. The *color code of* u, $c_\Pi(u) = (d(u, C_1), d(u, C_2), \ldots, d(u, C_k))$, where $d(u, C_i) = \min\{d(u, x) | x \in C_i\}$ for $1 \leq i \leq k$. If distinct vertices of $V(G)$ have distinct color codes, then c is called a *k-locating coloring* of G. A *minimum k-locating coloring* is the *locating chromatic number*, denoted by $\chi_L(G)$.

The locating-chromatic number was introduced by Chartrand *et al.* [6] in 2002. The locating-chromatic number of a graph is an interesting topic to study because there has been no general theorem to determine the locating chromatic number of an arbitrary graph. Specially for tree, in 2011, Asmiati *et al.* [4] studied the locating-chromatic number for amalgamation of stars. Next, Asmiati *et al.* [3] determined the locating-chromatic number of firecracker graph and in 2014, Asmiati [2] discussed the locating chromatic number for non homogeneous amalgamation of stars, and last in 2016, Asmiati [1] determined locating chromatic numbers for non homogeneous caterpillars and firecracker graphs. Next, for characterization E.T. Baskoro and Asmiati [5] characterized trees whose locating-chromatic number three. Motivated by this, in this paper we determine the locating-chromatic number of banana tree.

The following definition of banana tree is taken from [7]. A *banana tree*, $B_{n,k}$ is a graph obtained by connecting one leaf of each n copies of an k-star graph (S_k) to a new vertex. We denote the new vertex as *root vertex*, denoted x. The vertices of distance 1 from the root vertex as the *intermediate vertices*(denoted by $m_i, i = 1, 2, \ldots, n$). The *center* of every S_k is denoted by $l_i, i = 1, 2, \ldots, n$. We denote the j-th leaf of the center l_i by l_{ij} $(j = 1, 2, \ldots, m-2)$.

Theorem 1.1 *[6] Let G be a simple connected graph and c be a locating coloring of G. If $v, w \in V(G)$ and $v \neq w$ such that $d(v, x) = d(w, x)$ for all $x \in V(G) - \{v, w\}$, then $c(v) \neq c(w)$. In particular, if v and w are non adjacent vertices of G such that neighborhood of v is equal to neighborhood of w, then $c(v) \neq c(w)$.*

Corollary 1.2 *[6] If G is a simple connected graph containing a vertex that is adjacent to k leaves of G, then $\chi_L(G) \geq k + 1$.*

2 Main Results

We know that for $n \geq 1$, $B_{n,1}$ is a star graph on $n + 1$ vertices. Then from Corollary 1, $\chi_L(B_{n,1}) = n + 1$. It is clearly that $\chi_L(B_{1,2}) = \chi_L(B_{1,3}) = 3$. The following result is about locating-chromatic number of $B_{n,k}$, namely $k \geq 2$.

Lemma 2.1 *If c is a $(a + 2)$-locating coloring of $B_{n,k}$, where $a \geq 1$ and $k = 2, 3$, then $n \leq (a + 1)^2$.*

Proof. Let c be a $(a+2)$-locating coloring of $B_{n,k}$, where $a \geq 1$ and $k = 2, 3$. For some t, the number of intermediate vertices m_i receiving the same color t, $t \neq 1$ does not exceed $(a + 1)$. Because one color is used for coloring the root vertex x, then the maximum number of n is $(a + 1)^2$. So, $n \leq (a + 1)^2$. □

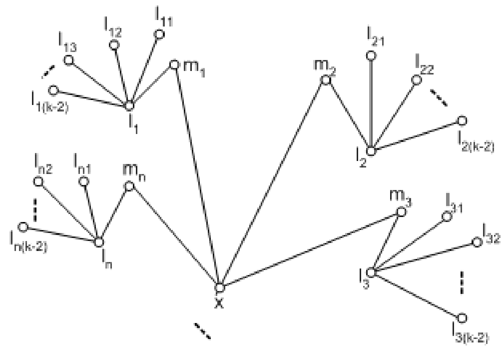

Figure 1: Construction of $B_{n,k}$

Theorem 2.2 *Given Banana tree $B_{n,k}$, where $k \geq 2$*

a. *If $a \geq 1$, $a^2 < n \leq (a+1)^2$, and $k = 2, 3$, then $\chi_L(B_{n,k}) = a + 2$.*

b. *If $k \geq 4$ and $1 \leq n \leq k-2$, then $\chi_L(B_{n,k}) = k - 1$, except $B_{2,4}$. $\chi_L(B_{2,4}) = 4$.*

PROOF. **Case a**. Since $n > a^2$, then by Lemma 2.1, $\chi_L(B_{n,k}) \geq a + 2$. On the other hand, if $n > (a+1)^2$, then by Lemma 2.1, $\chi_L(B_{n,k}) \geq a+2+1$. So, $\chi_L(B_{n,k}) \geq a + 2$, if $a^2 \leq n \leq (a+1)^2$.

The upper bound of $B_{n,k}$ for $k = 2, 3$ and $a^2 < n \leq (a+1)^2$. Let c be a $(a+2)$-coloring of $B_{n,k}$. Without loss of generality, we assign $c(x) = 1$, intermediate vertices m_i is colored by $2, 3, ..., a+2$, respectively, so that the number of intermediate vertices receiving the same color t, $t \neq 1$ does not exceed $(a+1)$. We can do like that, because $a^2 < n \leq (a+1)^2$. Next, if $c(m_i) = c(m_j), i \neq j$, then $c(l_i) \neq c(l_j)$. Specially for $k = 3$, $c(l_{i1}) = c(m_i)$, for every i.

We will show that color codes for every $v \in V(B_{n,k})$ is unique.

- If $c(x) = c(l_i)$, then $c_\Pi(x)$ contains at least two components of value 1, whereas for $c_\Pi(l_i)$ contains exactly one component of value 1. So, $c_\Pi(x) \neq c_\Pi(l_i)$.
- If $c(m_i) = c(m_j)$, $i \neq j$, then $c(l_i) \neq c(l_j)$. So $c_\Pi(m_i) \neq c_\Pi(m_j)$.
- If $c(m_i) = c(l_j)$, then $c_\Pi(m_i) \neq c_\Pi(l_j)$ because their color codes are different at least in the first ordinate.
- If $c(m_i) = c(l_{i1})$, then $c_\Pi(m_i) \neq c_\Pi(l_{i1})$ because $d(x, m_i) \neq d(x, l_{i1})$.
- If $c(m_i) = c(l_{j1})$, $i \neq j$, then $c_\Pi(m_i) \neq c_\Pi(l_{i1})$ because $c(l_i) \neq c(l_j)$.

- If $c(l_i) = c(l_j)$, $i \neq j$, then $c_\Pi(l_i) \neq c_\Pi(l_j)$ because $c(m_i) \neq c(m_j)$.
- If $c(l_i) = c(l_{j1})$, $i \neq j$, then $c_\Pi(l_i) \neq c_\Pi(l_{jl})$ because their color codes are different at least in the first ordinate.
- If $c(l_{i1}) = c(l_{j1})$, $i \neq j$, then $c_\Pi(l_{i1}) \neq c_\Pi(l_{j1})$ because $c(l_i) \neq c(l_j)$.

From all the above cases, color codes for all vertices of $B_{n,k}$, $a^2 \leq n \leq (a+1)^2$ is unique, then c is a locating-coloring. So, $\chi_L(B_{n,k}) \leq a+2$, $a^2 \leq n \leq (a+1)^2$.

Case b. The trivial lower bound trivial for $k \geq 4$ and $1 \leq n \leq k-2$. Since l_i, for every $i \in [1, n]$ adjacent to $(k-2)$ leaves, then by Corollary 1, $\chi_L(B_{n,k}) \geq k-1$.

The upper bound of $B_{n,k}$ for $1 \leq n \leq k-2$. Let c be a $(k-1)$-coloring of $B_{n,k}$. We assign color: $c(x) = 1$, center l_i, $c(l_i) = i+1$ for every $i \in [1, n]$. Intermediate vertices $m_i, i = 1, 2, \ldots, n$, colored one color of $\{2, 3, 4, ..., k-1\} \backslash \{c(l_i)\}$, and for leaves, $\{l_{ij} | j = 1, 2, \ldots, k-2\}$, we give color $\{1, 2, ..., k-1\} \backslash \{c(l_i)\}$ for any i. Set $c(l_i) \neq 1$, since if $c(l_i) = 1$, then color code of m_i will be the same with one of leaves l_i. We will show that color codes for every $v \in V(B_{n,k})$ are different.

- If $c(x) = c(l_{ik})$, we divide two cases.
  1. For $n = 1$ and $c(l_i) = p$. Then $c_\Pi(x) \neq c_\Pi(l_{ik})$, the color codes are different at least in the p th-ordinate, since $d(x, l_i) = 2$, whereas $d(l_i, l_{ik}) = 1$.
  2. For $n \geq 2$, there exists $c(m_i) \neq c(m_j), i \neq j$, then $c_\Pi(x)$ contains at least two components have value 1, whereas $c_\Pi(l_{ij})$ contains exactly one component of value 1. So, $c_\Pi(x) \neq c_\Pi(l_{ij})$. If $c(m_i) = q$ for every i, then $c_\Pi(x) \neq c_\Pi(l_{ik})$ since they are different at least in the q th-ordinate.
- If $c(m_i) = c(l_{ij})$, then $c_\Pi(m_i)$ contains exactly two components of value 1, whereas $c_\Pi(l_{ij})$ contains exactly one component of value 1. Thus, $c_\Pi(m_i) \neq c_\Pi(l_{ij})$.
- If $c(m_i) = c(m_j)$, $i \neq j$, then $c_\Pi(m_i) \neq c_\Pi(m_j)$, since $c(l_i) \neq c(l_j)$.
- If $c(m_i) = c(l_j)$, $i \neq j$, then $c_\Pi(m_i)$ contains exactly two components of value 1, whereas $c_\Pi(l_j)$ contains at least three components of value 1. So, $c_\Pi(m_i) \neq c_\Pi(l_j)$.
- If $c(l_i) = c(l_{st})$, $i \neq s$, then $c_\Pi(l_i)$ contains at least three components of value 1, whereas $c_\Pi(l_{sj})$ contains exactly one component of value 1. Thus, $c_\Pi(l_i) \neq c_\Pi(l_{sj})$

- If $c(l_{ik}) = c(l_{st})$ and $c(l_i) = x, c(l_s) = y$, then $c_\Pi(l_{ij})$ and $c_\Pi(l_{st})$ are different in xth and yth-ordinate.

Since the color codes for each vertex in $B_{n,k}$ is unique, then c is a locating-coloring. So, $\chi_L(B_{n,k}) \leq k-1$, for $1 \leq n \leq k-2$.

Next, we will show that $\chi_L(B_{2,4}) = 4$. Since for every l_i have two leaves, then by Corollary 1, $\chi_L(B_{2,4}) \geq 3$. For a contrary, suppose there exists a 3-locating coloring c on $B_{2,4}$. Since $c(x) = 1$, then $c(m_1) = 2$ dan $c(m_2) = 3$. Thus, $c(l_1) = 3$ and $c(l_2) = 2$. As a result, $c_\Pi(m_1) = c_\Pi(l_2)$ and $c_\Pi(m_2) = c_\Pi(l_1)$, a contradiction. So, $\chi_L(B_{2,4}) \geq 4$. If we assign a coloring like that and we change $c(m_1) = 4$, then color codes for every vertex of $B_{2,4}$ are different. Thus, $\chi_L(B_{2,4}) \leq 4$. So, $\chi_L(B_{2,4}) = 4$. □

Let S_k^i be a star graph ith-S_k in $B_{n,k}$ and $A_i = \{c(v) \mid \text{for every } v \in V(S_k^i)\}$.

Lemma 2.3 *Let c be a $(k+a)$-coloring of $B_{n,k}$ for $k \geq 4$ and $n \geq k-1$. Assume that all colors used in vertices m_i.*

a. $c(m_i) = c(m_j), i \neq j \Rightarrow c(l_i) \neq c(l_j)$.

b. $c(l_i) = c(l_j), i \neq j \Rightarrow A_i \neq A_j$.

If (1) and (2) are required then c is a $(k+a)$-locating coloring of $B_{n,k}$.

Lemma 2.4 *If c is a $(k+a)$-locating coloring of $B_{n,k}$ for $a \geq 1$ and $k \geq 4$, then $n \leq (k+a-1)^2$.*

Proof. Let c be a $(k+a)$-locating coloring of $B_{n,k}$, $k \geq 4$. Since one color is used by x, then for $c(m_i)$, we need $(k+a-1)$ colors. Next, there exists $(k+a-1)$ possibilities for coloring l_i. As a result, $n \leq (k+a-1)^2$. So, the maximum number of n is $(k+a-1)^2$. □

Theorem 2.5 *Locating-chromatic number of $B_{n,k}$ for $a \geq 1$ and $k \geq 4$ is*

$$\chi_L(B_{n,k}) = \begin{cases} k & ;\ k-1 \leq n \leq (k-1)^2, \\ k+a & ;\ (k+a-2)^2 < n \leq (k+a-1)^2. \end{cases}$$

Proof. We will determine the lower bound of $B_{n,k}$, where $k-1 \leq n \leq (k-1)^2$. Suppose that there exists $(k-1)$-locating coloring of $B_{n,k}$ for $n \geq k-1$. Since $n \geq k-1$, then $c(x) = c(l_i)$, for some i. Thus, the color codes of m_i will be the same with one of $\{l_{ij} | j = 1, 2, \ldots, k-2\}$, a contrary. So, $\chi_L(B_{n,k}) \geq k$.

The upper bound of $\chi_L(B_{n,k}) \leq k$ for $k-1 \leq n \leq (k-1)^2$. Let c be a k-coloring. Assign coloring as follows: $c(x) = 1$; intermediate vertices m_i, is given color $2, 3, \ldots, k$ such that the maximum number of m_i are given the same color is $(k-1)$. Center l_i given a color among $\{1, 2, 3, \ldots, k\}\backslash c(m_i)$ colors. Leaves, l_{ij} colored with $(k-2)$ colors among $\{1, 2, \ldots, k\}\backslash c(l_i)$ colors. To make sure that two centers receiving the same color will have color code different, then can be set $A_i \neq A_j$. By Lemma 2.3, c is a locating-coloring.

The lower bound of $B_{n,k}$ for $(k+a-2)^2 < n \leq (k+a-1)^2$. Since $n > (k+a-2)^2$, by Lemma 2.4, $\chi_L(B_{n,k}) \geq k+a$. On the other hand if $n > (k+a-1)^2$, then by Lemma 2.4, $\chi_L(B_{n,k}) \geq k+a+1$. So, $\chi_L(B_{n,k}) \geq k+a$, if $(k+a-2)^2 < n \leq (k+a-1)^2$.

The upper bound of $\chi_L(B_{n,k}) \leq k+a$ for $(k+a-2)^2 < n \leq (k+a-1)^2$. Without loss of generality, Let $c(x) = 1$ and intermediate vertices m_i are colored by $2, 3, ..., k+a$, such that the intermediate vertices number are colored by t does not exceed of $(k+a-1)$, for some t. We can do like that, because $(k+a-2)^2 < n \leq (k+a-1)^2$. Leaves l_i, colored among $\{1, 2, 3, \ldots, k+a\}\backslash c(m_i)$ colors. As a result, if $c(l_i) = c(l_n), i \neq n$, then we can be set $A_i \neq A_n$. By Lemma 2.3, c is a locating-coloring. So, $\chi_L(B_{n,k}) \leq k+a$ for $(k+a-2)^2 < n \leq (k+a-1)^2$.□

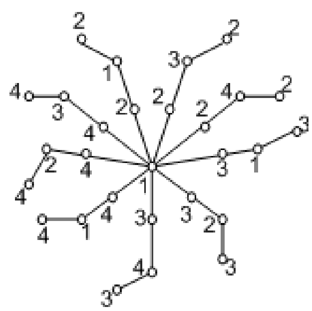

Figure 2: A minimum of locating coloring of $B_{9,2}$

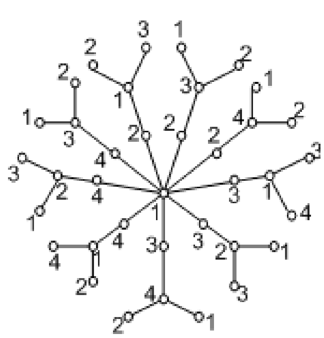

Figure 3: A minimum of locating coloring of $B_{9,3}$

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