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On Nearly m-Embedded Subgroups [1]

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Abstract

A subgroup H of a finite group G is said to be nearly m-embedded subgroup in G if G has a subgroup T and a $\{1 \leq G\}$-embedded subgroup C in G such that $G = HT$ and $T \cap H \leq C \leq H$. In this paper we investigate the structure of G under the assumption that some subgroups of P are nearly m-embedded in the normalizer of P in G, and some new criteria are obtained.

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1 Introduction

It is well known that the normalizer of Sylow subgroups of a group play an important role in the structure of groups. Let P be a Sylow subgroup of a group G. A question who is always interested in is the relation between the property of the normalizer of P and property of G, and many results have been obtained.

Theorem 1.1. *(Burnside) Let P be a Sylow p-subgroup of G. If $N_G(P) = C_G(P)$, then G is p-nilpotent.*

Hall in [1] got the following generalization of Burnside′s theorem:

Theorem 1.2. *Let P be a Sylow p-subgroup of G. If p'-elements of $N_G(P)$ are commute to the elements of P and the class size of P is less than p, then G is p-nilpotent.*

Wielandt, Ballester-Bolinches and Esteban-Romero proved the following respectively.

Theorem 1.3. *([2]) A group G is p-nilpotent if it has a regular Sylow p-subgroup whose G-normalizer is p-nilpotent.*

Theorem 1.4. *([3]) A group G is p-nilpotent if it has a modular Sylow p-subgroup whose G-normalizer is p-nilpotent.*

Let G be a group and H a subgroup of G. H is said to be s-permutable (or s-quasinormal, π-quasinormal) in G if H permutes with every Sylow subgroup of G; H is called c-normal in G if G has a normal subgroup T such that $G = HT$ and $H \cap T \leq H_G$, where H_G is the normal core of H in G; H is called c-supplemented in G permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the maximal s-permutable subgroups of G contained in H; H is said to be s-permutably embedded in G if for each prime p dividing $|H|$, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-permutable subgroup of G. More recently, authors in [4] introduced the following concept, which covers both weak s-permutably embedded property and weakly s-permutability.

Definition 1.5. *A subgroup H of a finite group G is said to be nearly m-embedded subgroup in G if G has a subgroup T and a $\{1 \leq G\}$-embedded subgroup C in G such that $G = HT$ and $T \cap H \leq C \leq H$.*

Let G be a group, p a prime and P a Sylow p-subgroup of G, we introduce a family of subgroups: $\mathcal{H}(P) = \{H \leq P \mid P' \leq H \leq \Phi(P)\}$. Now, we consider the structure of G under the assumption that some subgroups of P are nearly

m-embedded in the normalizer of the Sylow subgroup, and some new criteria are obtained. For example, we prove the following results:

Theorem A *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If there exists $H \in \mathcal{H}(P)$ such that H is nearly m-embedded in G and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is nearly m-embedded in $N_G(P)$, then G is p-nilpotent.*

Theorem B *Let G be a group and p a prime dividing the order of G with $(|G|, p-1) = 1$. Suppose that E is a normal subgroup of G such that G/E is p-nilpotent. Let P be a Sylow p-subgroup of E. If there exists $H \in \mathcal{H}(P)$ such that H is nearly m-embedded in G and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is nearly m-embedded in $N_G(P)$, then G is p-nilpotent.*

Theorem C *Let $\mathcal{F}$ be a saturated formation containing the class of all supersolvable groups $\mathcal{U}$, and assume that G is a group with a normal subgroup E satisfies $G/E \in \mathcal{F}$. Suppose that for any prime p dividing $|E|$ and $P \in Syl_p(E)$, there exists $H \in \mathcal{H}(P)$ such that H is nearly m-embedded in G and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is nearly m-embedded in $N_G(P)$, then $G \in \mathcal{F}$.*

All groups in this paper are finite. The notations and terminology are standard, as in [2].

2 Preliminaries

Lemma 2.1. *Let K and H be subgroups of G. Suppose that K is nearly m-embedded in G and H is normal in G. Then*

(1) If $H \leq K$, then K/H is nearly m-embedded in G/H.

(2) If $K \leq E \leq G$, then K is nearly m-embedded in E.

(3) If $(|H|, |K|) = 1$, then HK/H is nearly m-embedded in G/H.

Lemma 2.2. *Let N be a normal subgroup of a group G and M a subgroup of G. If M is $\{1 \leq G\}$-embedded in G, then NM is $\{1 \leq G\}$-embedded in G and NM/N is $\{1 \leq G\}$-embedded in G/N.*

Lemma 2.3. *Every $\{1 \leq G\}$-embedded subgroup of G is subnormal in G.*

Lemma 2.4. *Let $P/\Phi(P)$ be a minimal normal subgroup of a group $G/\Phi(P)$, where p is a prime divisor of $|G|$ and P a Sylow p-subgroup of G. If every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is nearly m-embedded in G, then P is cyclic.*

Proof Let P_1 be a proper subgroup of P. If P_1 is nearly m-embedded in G, we claim that $P_1 \leq \Phi(P)$. Let T be a supplement of P_1 in G such that $G = P_1T$

and $P_1 \cap T \leq C$, where C is a $\{1 \leq G\}$-embedded subgroup of G contained in P_1. Then $G = P_1T$ and $P = P \cap G = P \cap P_1T = P_1(P \cap T)$. Since $P/\Phi(P)$ is abelian, $(P \cap T)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$, and hence $(P \cap T)\Phi(P) \trianglelefteq G$. Since $P/\Phi(P)$ be a minimal normal Sylow p-subgroup of $G/\Phi(P)$, $P \cap T \leq \Phi(P)$ or $P \cap T = P$. If $P \cap T \leq \Phi(P)$, then $P = P_1(P \cap T) = P_1$, a contradiction. Now assume that $P \cap T = P$. Then $P_1 \leq P_1 \cap T \leq C \leq O_p(G) = P$.

Hence P_1 is $\{1 \leq G\}$-embedded in G. So $P_1\Phi(P)/\Phi(P)$ is $\{1 \leq G/\Phi(P)\}$-embedded in $G/\Phi(P)$ follows from Lemma 2.2. For any maximal subgroup $K/\Phi(P)$ of $G/\Phi(P)$, either $P_1\Phi(P)/\Phi(P) \leq K/\Phi(P)$ or $G/\Phi(P) = P_1\Phi(P)/\Phi(P) \cdot K/\Phi(P)$. Because P is the Sylow p-subgroup of G and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, there exists a maximal subgroup $M/\Phi(P)$ of $G/\Phi(P)$ such that $G/\Phi(P) = (P/\Phi(P)) \cdot (M/\Phi(P))$ and $(P/\Phi(P)) \cap (M/\Phi(P)) = 1$. If $G/\Phi(P) = P_1\Phi(P)/\Phi(P) \cdot M/\Phi(P)$, then we have $P = P_1\Phi(P) = P_1$, a contradiction. Hence $P_1\Phi(P)/\Phi(P) \leq M/\Phi(P)$, then $P_1\Phi(P)/\Phi(P) \leq (P/\Phi(P)) \cap (M/\Phi(P)) = 1$ and so $P_1 \leq \Phi(P)$.

If every maximal subgroup of P is nearly m-embedded in G, then by the above argument P has a unique maximal subgroup, which implies that P is cyclic.

If every cyclic subgroup of P with prime order and order 4 is nearly m-embedded in G, then we also have $|P/\Phi(P)| = p$ and then P is cyclic. Otherwise, let $K/\Phi(P)$ be any non-trivial cyclic subgroup of $P/\Phi(P)$. Let $x \in K \setminus \Phi(P)$ such that $T = \langle x \rangle \Phi(P)$. Then by the above argument, $\langle x \rangle \leq \Phi(P)$ and so $T = \Phi(P)$, a contradiction. This contradiction completes the proof the lemma.

Lemma 2.5. *Let P be a normal Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ and $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if every cyclic subgroup of P with prime order and order 4 is nearly m-embedded in G.*

Proof If G is p-nilpotent, then G has a normal p-complement T. Let P_1 be a every cyclic subgroup of P with prime order and order 4 . It follows that P_1 is a Sylow p-subgroup of P_1T. Let Q be a Sylow q-subgroup of G, where $q \neq p$ is a prime divisor of $|G|$. Then $Q \leq T$ and so $P_1TQ = QP_1T = P_1T$. By hypothesis, P is normal in G and so $P_1TP = PT$ is a subgroup of G, which implies that P_1T is an s-permutable subgroup of G and P_1 is s-permutably embedded in G. Hence P_1 is nearly m-embedded in G.

Conversely, assume that every cyclic subgroup of P with prime order and order 4 is nearly m-embedded in G and G is non-p-nilpotent. Let G be a counterexample with minimal order. Let M be a proper subgroup of G. Then $P \cap M$ is a normal Sylow p-subgroup of M. It follows from Lemma 2.1 that every cyclic subgroup of P with prime order and order 4 is nearly m-embedded in M and so M is p-nilpotent. By [10, VI, Theorem 24.2], $P/\Phi(P)$ is a G-chief factor of P. Now by Lemma 2.4, P is cyclic and G is p-nilpotent follows from Burside's Theorem, a contradiction.

Lemma 2.6. *Let Q be a normal Sylow q-subgroup of a group G such that G/Q is supersolvable, where q is a prime divisor of $|G|$. If every maximal subgroup of Q or every minimal subgroup of Q with prime order and order 4 is nearly m-embedded in G, then G is supersolvable.*

Proof Assume the result is false and let G be a counterexample with minimal order. By Lemma 2.1, it is easy to see that if every minimal subgroup of Q with prime order and order 4 is nearly m-embedded in G and G is non-supersolvable, then G is a minimal non-supersolvable, i.e., each proper subgroup of G is supersolvable and G is non-supersolvable. Now by [16], G has a normal Sylow p-subgroup P such that $G = PM$, where M is a supersolvable maximal subgroup of G and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$. If $P \neq Q$, then $G \lesssim G/P \times G/Q$ is supersolvable, a contradiction. Hence $P = Q$. Now Q is cyclic by Lemma 2.4 and then G is supersolvale, a contradiction.

So we may assume that every maximal subgroup of Q is nearly m-embedded in G. Let N be a minimal normal subgroup of G contained in Q. Let Q_1/N be a maximal subgroup of Q/N, then Q_1 is maximal in Q and by hypothesis and Lemma 2.1, Q_1/N is nearly m-embedded in G/N. So G/N satisfies the hypothesis and by induction G/N is supersolvable. It follows that N is the unique minimal normal subgroup of G contained in Q and $N \nleq \Phi(G)$. Hence $N = Q$. Now by Lemma 2.4 again, we have that Q is cyclic and so G is supersolvable, a contradiction. This contradiction complete the proof.

3 Proofs of Theorems A-C

Proof of Theorem A suppose the theorem is false and let G be a counterexample with minimal order. We proceed the proof by the following steps.

Step 1. H is a non-identity $\{1 \leq G\}$-embedded subgroup of G. Furthermore, G is not a non-abelian simple group.

Let T and C be subgroups of G such that $G = HT$, C is $\{1 \leq G\}$-embedded in G and $T \cap H \leq C \leq H$. Then $P = P \cap HT = H(P \cap T) = P \cap T$. It follows that $H \leq P \leq T$ and so $H = T \cap H \leq C \leq H$, which implies that H is $\{1 \leq G\}$-embedded in G.

If $H = 1$, then $P' \leq H = 1$ implies that P is abelian and so $P \leq C_G(P)$. Let Q be a Sylow q-subgroup of $N_G(P)$, where q is a prime dividing $|N_G(P)|$ different from p. It is easy to see that PQ satisfies the hypothesis by Lemma 2.1 and so PQ is p-nilpotent. Hence $Q \leq C_G(P)$ and so all p'-elements of $N_G(P)$ are contained in $C_G(P)$. It follows that $N_G(P) = C_G(P)$, which implies that

G is p-nilpotent by Burnside's Theorem, a contradiction. So we may assume that $H \neq 1$. Now by Lemma 2.3, G is not a non-abelian simple group.

Step 2. G has a unique minimal normal subgroup N, G/N is p-nilpotent. Furthermore, $O_{p'}(G) = 1$ and $N \nleq \Phi(G)$.

Let N be a minimal normal subgroup of G, consider the quotient group $\overline{G} = G/N$. Then $\overline{P} = PN/N$ is a Sylow p-subgroup of $\overline{G}$ and $N_{\overline{G}}(\overline{P}) = \overline{N_G(P)}$.

If every maximal subgroup of P is nearly m-embedded in $N_G(P)$, we claim that every maximal subgroup of $\overline{P}$ is nearly m-embedded in $N_{\overline{G}}(\overline{P})$. Let M/N be a maximal subgroup of PN/N. Then there exists a maximal subgroup P_1 of P such that $M = P_1N$ and $P \cap N = P_1 \cap N$ is a Sylow p-subgroup of N. By hypothesis, P_1 is nearly m-embedded in $N_G(P)$, so there exists a subgroup T of $N_G(P)$ such that $N_G(P) = P_1T$ and $P_1 \cap T \leq C$, where C is a $\{1 \leq G\}$-embedded subgroup of $N_G(P)$ contained in P_1. It follows that $N_{\overline{G}}(\overline{P}) = \overline{P_1} \cdot \overline{T} = M/N \cdot TN/N$. For any prime divisor $r \neq p$ of $|N \cap N_G(P)|$, T contains some Sylow r-subgroup of $N_G(P)$, which implies $(|(N \cap N_G(P)) : (P_1 \cap N)|, |(N \cap N_G(P)) : (T \cap N)|) = 1$. So $(P_1 \cap N)(T \cap N) = N \cap N_G(P) = N \cap P_1T$. By [15, A, Lemma 1.2], $P_1N \cap TN = (P_1 \cap T)N$. Then $\overline{P_1} \cap \overline{T} \leq \overline{C} \leq M/N$, where $\overline{C}$ is $\{1 \leq \overline{G}\}$-embedded in $\overline{G}$. Hence M/N is nearly m-embedded in $N_{\overline{G}}(\overline{P})$.

If every cyclic subgroup of P with prime order and order 4 is nearly m-embedded in $N_G(P)$, then by Lemma 2.5, we have that $N_G(P)$ is p-nilpotent and so $N_{\overline{G}}(\overline{P})$ is p-nilpotent. Now by Lemma 2.5 again, every cyclic subgroup of $\overline{P}$ with prime order and order 4 is nearly m-embedded in $N_{\overline{G}}(\overline{P})$.

Note that $(\overline{P})' \leq \overline{P'} \leq \overline{H} \leq \overline{\Phi(P)} \leq \Phi(\overline{P})$. It follows that $(\overline{P})' \leq \overline{H} \leq \Phi(\overline{P})$. Hence $\overline{H} \in \mathcal{H}(\overline{P})$. By Step 1 and Lemma 2.2, it is easy to see that G/N satisfies the hypotheses, so by induction G/N is p-nilpotent. Obviously N is the unique minimal normal subgroup of N. Furthermore, $O_{p'}(G) = 1$ and $N \nleq \Phi(G)$.

Step 3. $N \leq \Phi(P)$ and G is p-nilpotent.

By Step 1 and Lemma 2.3, $O_p(G) \neq 1$. By Step 2 $N \leq O_p(G)$ and $N \cap \Phi(G) = 1$, it follows that $O_p(G) \cap \Phi(G) = 1$. Now by [12, Lemma 2.6] we have $N = O_p(G) = C_G(O_p(G))$. By Lemma 2.3 again, we have $H \leq N$.

Let M be a maximal subgroup of G such that $G = NM$ and $N \cap M = 1$. Since H is $\{1 \leq G\}$-embedded in G, either $G = HM$ or $H \leq M$. If $H \leq M$, then $H \leq N \cap M = 1$, which contradicts to step 1. So $G = HM$ and $N = H \leq \Phi(P)$. It follows that $N \leq \Phi(G)$ and so G is p-nilpotent by Step 2, the finial contradiction. The theorem is proved.

Proof of Theorem B Assume that the result is false and let G with subgroup E be a minimal counterexample to the theorem in respect to $|G|+|E|$. By Lemma 2.2 and Theorem A, E is p-nilpotent. Let T be the normal p-complement of E, then $T \trianglelefteq G$. If $T \neq 1$, we consider G/T with subgroup E/T. It is easy to see that $E = PT$ and $(|P|, |T|) = 1$. With a similar argument as in Step 2 of Theorem A, we know that the hypothesis is still true for G/T with subgroup E/T, hence the minimal choice of G implies that G/M is p-nilpotent. Thus G is p-nilpotent, a contradiction. So we may assume that $T = 1$, i.e., $E = P$ is a p-group. Let K/P be the normal p-complement of G/P, this makes sense as $G/P = G/E$ is p-nilpotent. It is clear that there exists $H \in \mathcal{H}(P)$ such that H is nearly m-embedded in K and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is nearly m-embedded in $N_K(P)$, whence K is p-nilpotent by Theorem A, so that $K_{p'} charT \trianglelefteq G$ yielding that $K_{p'}$ is also a normal Hall p'-subgroup of G, i.e., G is p-nilpotent, a contradiction. This contradiction completes the proof of Theorem B. □

Proof of Theorem C Assume that the result is false and let G be a counterexample of minimal order. By Theorem A, G is a Sylow-tower group. Let $q = max\pi(G)$ and $Q \in Syl_q(G)$. Then $Q \trianglelefteq G$. By Lemma 2.1 and the proof of Theorem A, it is easy to see that G/Q satisfies the hypothesis and by induction G/Q is supersolvable. Now by Lemma 2.6, G is supersolvable.

4 Some Applications

Let G be a group and P a Sylow p-subgroup of G, where $p \in \pi(G)$. If $N_G(P)$ is p-nilpotent, then it is easy to see that $P' \in Syl_p((N_G(P))')$ and $\Phi(P) \in Syl_p(\Phi(N_G(P)))$. So Theorem A has the following corollaries:

Corollary 4.1. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If P' is nearly m-embedded in G and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is nearly m-embedded in $N_G(P)$, then G is p-nilpotent.*

Corollary 4.2. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If $\Phi(P)$ is nearly m-embedded in G and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is nearly m-embedded in $N_G(P)$, then G is p-nilpotent.*

Corollary 4.3. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If $(N_G(P))'$ is nearly m-embedded in G and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is nearly m-embedded in $N_G(P)$, then G is p-nilpotent.*

Corollary 4.4. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If $\Phi(N_G(P))$ is nearly m-embedded in G and every maximal subgroup of P or every cyclic subgroup of P with prime order and order 4 is nearly m-embedded in $N_G(P)$, then G is p-nilpotent.*

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