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The Generator of Second Homotopy Module of $\langle x, y; xyx = yxy\rangle$ and $\langle a, b\ ; a^2, b^3\rangle$

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Abstract

This paper discuss about generator of second homotopy module of $\langle x, y; xyx = yxy\rangle$ and second homotopy module of $\langle a, b\ ; a^2, b^3\rangle$. It is shown that using Tietze transformation and operations on picture there are sequence of generators between their.

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Keywords: second homotopy module, Tietze Transformation, generator

1. Introduction

A picture over $\mathcal{P} = \langle \boldsymbol{x}; \boldsymbol{r}\rangle$ is called a set of generator second homotopy module ($\pi_2(\mathcal{P})$) if $\{[\mathrm{P}]\text{:}\mathrm{P} \in \boldsymbol{P}\}$ generate $\mathbb{Z}G$ −module $\pi_2(\mathcal{P})$ [1]. Therefore, set generator $\boldsymbol{P}$ is generator iff each spherical picture over P can be transformated to empty picture by using operation on picture [2].

Calculation generator of second homotopy module performed by [2] only to describe the generators of second homotopy module from one group presentation and [8] provides a simple application from [7].

This article discuss about generator of second homotopy module of $\langle x, y; xyx = yxy\rangle$ and second homotopy module of $\langle a, b\ ; a^2, b^3\rangle$ using [6]. In line with this, theory of Tietze transformation can be seen in [3] and [4]. In this transformation, the operations on picture used to get the generator of these second homotopy module. Operations on picture can be seen in [6].

We are going to prove the following lemma:

Lemma 1.1 *Group presentation* $\langle x, y; xyx = yxy\rangle$ *isomorphic to* $\langle a, b\,; a^2, b^3\rangle$, *and there are sequence of generator from* $\pi_2(\langle x, y; xyx = yxy\rangle)$ *to* $\pi_2(\langle a, b\,; a^2 = b^3\rangle)$.

2. Basic Theory

In this section we will introduce the basic concept which is needed in all articles. The reference to this basic theory such as [2, 3, 4].

Let $\boldsymbol{x}$ be a set (alphabet). A *word* W on x is the form $x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_{n-1}^{\varepsilon_{n-1}} x_n^{\varepsilon_n}$ where $n \geq 0$, $x_i \in \boldsymbol{x}$ and $\varepsilon_i = \pm 1$, $i = 1, 2, \dots, n$. Inverse of W, denoted W^{-1} is word $x_n^{-\varepsilon_n} x_{n-1}^{-\varepsilon_{n-1}} \dots x_2^{-\varepsilon_2} x_1^{-\varepsilon_1}$. If $x_i^{\varepsilon_i} \neq x_{i+1}^{-\varepsilon_{i+1}}$, $i = 1, \dots, n-1$, then we say that it is *reduced*. Furthermore it is *cyclically reduced* if in addition $x_1^{\varepsilon_1} \neq x_n^{-\varepsilon_n}$. Then we have a presentation $\mathcal{P} = \langle \boldsymbol{x}; \boldsymbol{r}\rangle$, where r is a set of non-empty cyclically reduced words on $\boldsymbol{x}$. We say that $\mathcal{P}$ is finite if $\boldsymbol{x}$ and $\boldsymbol{r}$ are both finite.

If $F(\boldsymbol{x})$ is the free group on $\boldsymbol{x}$ and $N = \langle\langle \boldsymbol{r}\rangle\rangle$ is normal closure of $\boldsymbol{r}$ in $F(\boldsymbol{x})$, then the quotient group $G(\mathcal{P}) = F(\boldsymbol{x})/N$ is the group defined by $\mathcal{P}$. Denote a typical of $G(\mathcal{P})$ by $\bar{W} = [W]N$ where W is a word on $\boldsymbol{x}$ and $[W]$ is the free equivalence class of W. A group G is said to be *finitely presented* if G can be defined by a finite presentation (that is $G = G(\mathcal{P})$) for some finite presentation $\mathcal{P}$).

We may regard $\mathcal{P}$ as a 2-complex. This complex has a single 0-cell the 1-cell are in bijective correspondence with $\boldsymbol{x}$, and the 2-cell are in bijective correspondence with $\boldsymbol{r}$ and are attached by the boundary path determined by the spelling of the corresponding member of r. Thus there are homotopy group $\pi_1(\mathcal{P})$ and $\pi_2(\mathcal{P})$.

The element of the second homotopy module $\pi_2(\mathcal{P})$ can be represented by geometric configurations called spherical pictures.

A picture $\mathbb{P}$ over $\mathcal{P}$ is a geometric configuration consisting of of the following:

a. A disc D^2 with basepoint O on ∂D^2.

b. Disjoint discs $\Delta_1, \Delta_2, \dots, \Delta_n$ in the interior of D^2. Each Δ_i has a basepoint O_i on $\partial\Delta_i$.

c. A finite number of disjoint arcs $\alpha_1, \alpha_2, \dots, \alpha_m$ where each arc lies in the closure of $D^2 - \bigcup_{i=1}^n \Delta_i$ and is either simple closed curve having trivial intersection with $\partial D^2 \cup (\bigcup_{i=1}^n \partial\Delta_i)$, or is a simple non-closed curve which join two points of $\partial D^2 \cup (\bigcup_{i=1}^n \partial\Delta_i)$, neither point being a basepoint. Each arc has a normal orientation, indicated by a short arrow meeting with the arc transversely and is labelled by an element of $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$.

d. If we travel around $\partial\Delta_i$ once in clockwise direction starting from O_i and read off the labels on arcs encountered (if we cross an arc, labelled x say, in the direction of its normal orientation, then we read x^{-1}), then we obtain a word which belongs to $\boldsymbol{r} \cup \boldsymbol{r}^{-1}$. We call this word the label of Δ_i.

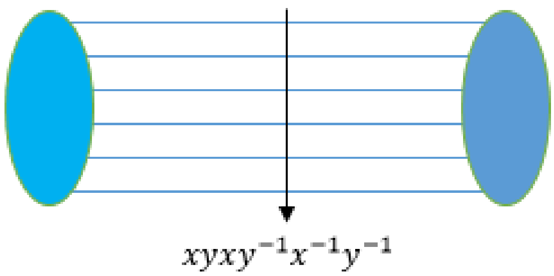
We define $\partial\mathcal{P}$ to be ∂D^2 and label on $\mathcal{P}$ (denoted by $W(\mathcal{P})$) is word read off by traveling around $\partial\mathcal{P}$ once in the clockwise direction starting from O. We say that $\mathcal{P}$ is spherical picture if no arcs meet $\partial\mathcal{P}$. If $\mathcal{P}$ is spherical picture, we often omit $\partial\mathcal{P}$.

Certain basic operation can be applied to a picture (spherical picture) $\mathbb{P}$ as follows: (D) deletion and (I) insertion *floating circle*, (D') deletion and insertion *floating semicircle*, (D'') deletion and insertion *folding pair* and (B) bridge move (see [6]).

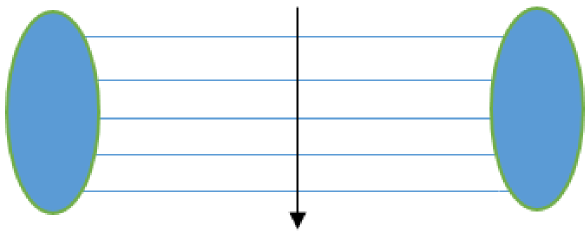
Two pictures will be said to be *equivalent* if one can be transformed to the other by a finite number of operation D, I, D', D'' and B. We let $[\mathcal{P}]$ denote the equivalence class containing $\mathcal{P}$. Note that the set of equivalence of all spherical picture over $\mathbb{P}$ form an abelian group, denoted by $\pi_2(\mathbb{P})$ under the binary operation $[\mathcal{P}_1] + [\mathcal{P}_2] = [\mathcal{P}_1 + \mathcal{P}_2]$. The identity is the equivalence class containing the empty picture and the inverse of $[\mathcal{P}]$ is $[-\mathcal{P}]$. Then we can consider $\pi_2(\mathbb{P})$ as a left $\mathbb{Z}G(\mathbb{P})$-module where the $G(\mathbb{P})$-action is given by $\overline{W}.[\mathcal{P}] = \left[\mathcal{P}^W\right]$, $\overline{W} \in G$ and $\mathcal{P}^W$ is the spherical picture obtained from spherical picture $\mathcal{P}$ by surrounding it by a collection of concentric closed arcs with total label W. Then, we call $\pi_2(\mathbb{P})$ the *second homotopy module* of $\mathbb{P}$.

3. Proof of Lemma 1.1

By [2], generator of $\pi_2(\langle x, y; xyx = yxy\rangle)$ is



and generator of $\pi_2(\langle a, b\,; a^2 = b^3\rangle)$ is

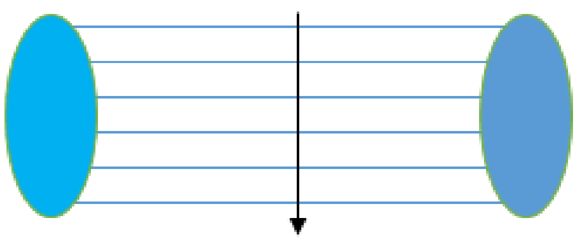


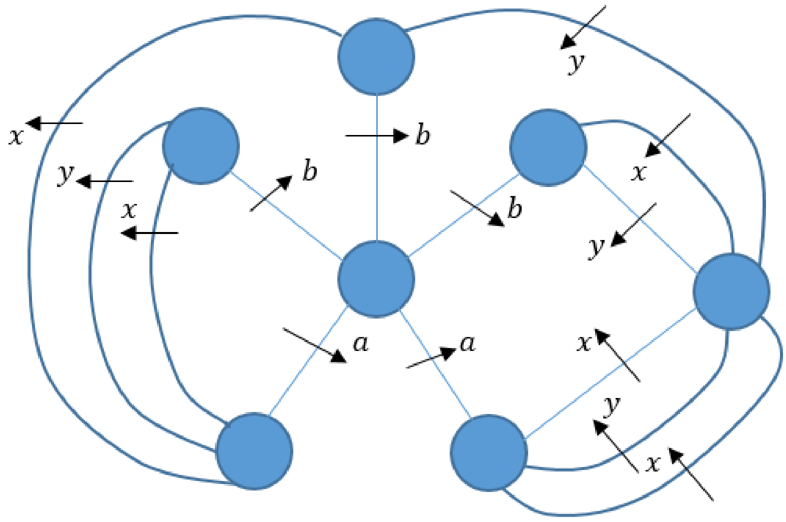
Consider that $\pi_2(\langle x, y; xyx = yxy\rangle)$ is generated by single picture, and $\pi_2(\langle a, b\,; a^2 = b^3\rangle)$ as well. Thus, is obtained a different generator for each second homotopy module. However, there are sequence of generators of $\pi_2(\langle x, y; xyx = yxy\rangle)$ to $\pi_2(\langle a, b\,; a^2 = b^3\rangle)$.

By [3] and [5], we have sequence of Tietze transformation from $\langle a, b\,; a^2 = b^3\rangle$ to $\langle x, y; xyx = yxy\rangle$, i.e.,

$$\begin{aligned}
\langle x, y; xyx = yxy\rangle &\cong \langle x, y, a; xyx = yxy, a = xyx\rangle \\
&\cong \langle x, y, a, b; xyx = yxy, a = xyx, b = xy\rangle \\
&\cong \langle x, y, a, b; xyx = yxy, a = xyx, b = xy, a^2 = b^3\rangle \\
&= \langle x, y, a, b; xyxy^{-1}x^{-1}y^{-1}, a^{-1}xyx, b^{-1}xy, a^2b^{-3}\rangle \\
&\cong \langle a, b, x, y; a^{-1}xyx, b^{-1}xy, a^2b^{-3}, x = b^{-1}a, y = a^{-1}b^2\rangle \\
&\cong \langle a, b, x, y; a^2b^{-3}, x = b^{-1}a, y = a^{-1}b^2\rangle \\
&\cong \langle a, b; a^2b^{-3}\rangle \\
&= \langle a, b; a^2 = b^3\rangle
\end{aligned}$$

Furthermore, we have sequence generator by [7], namely $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5, \mathcal{P}_6, \mathcal{P}_7$ and $\mathcal{P}_8$ respectively.

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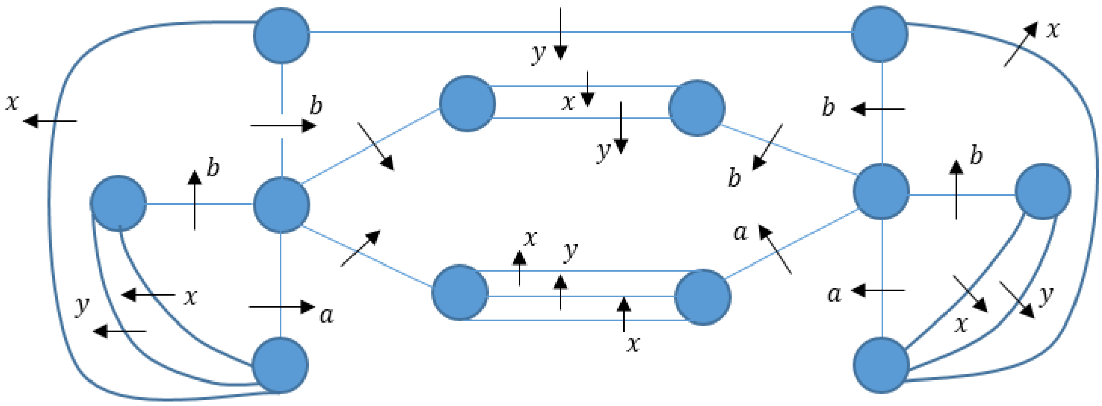
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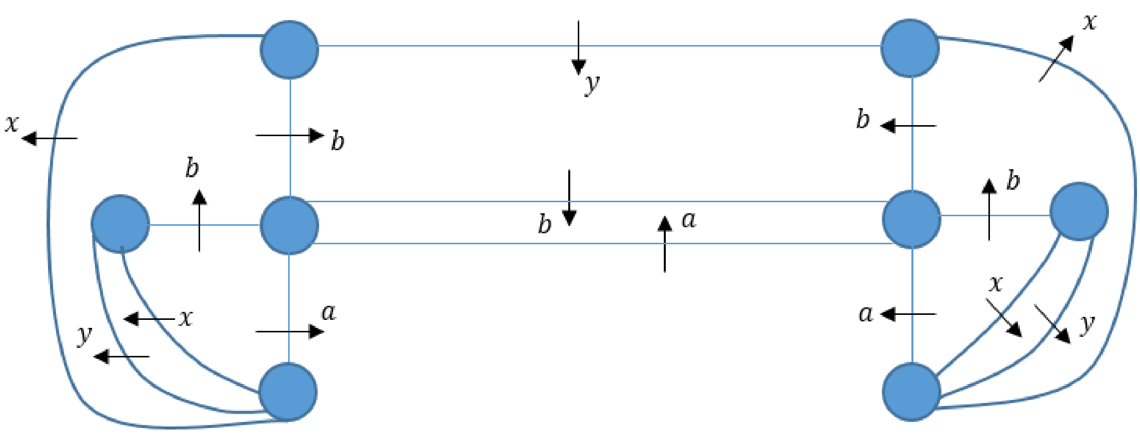
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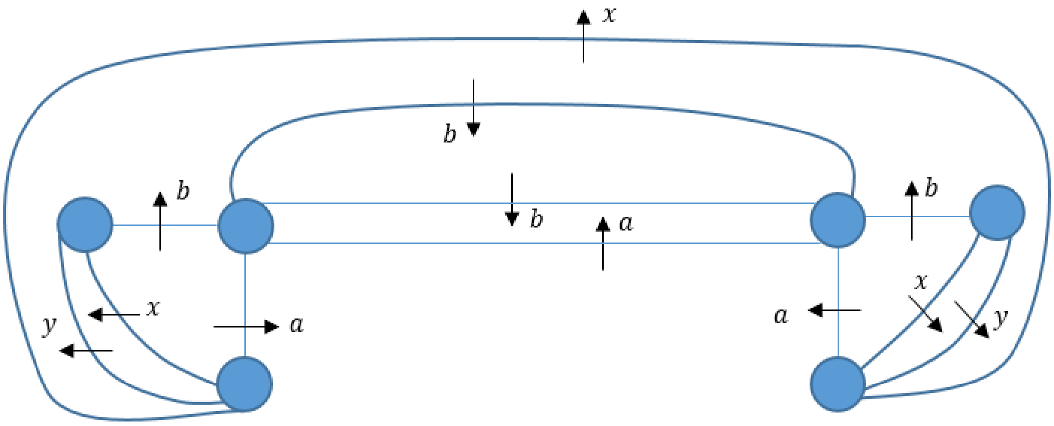
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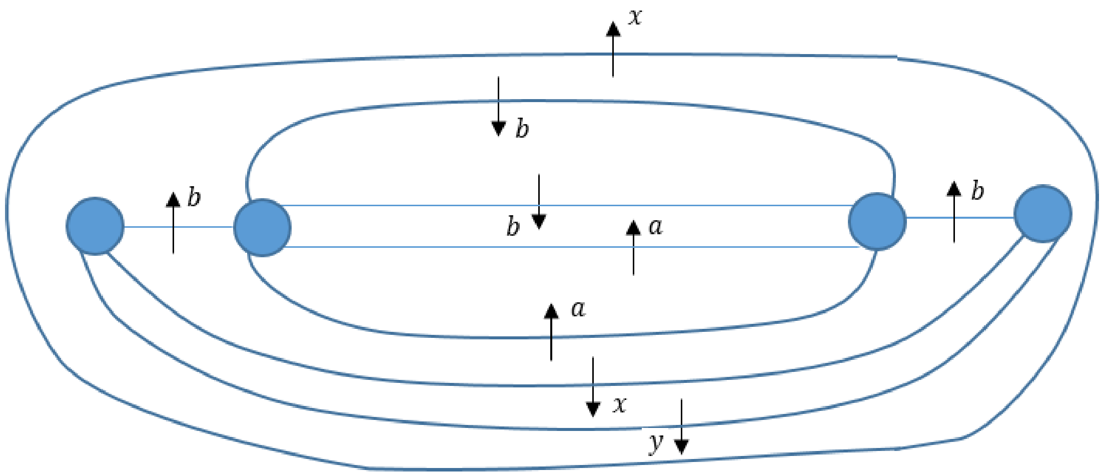
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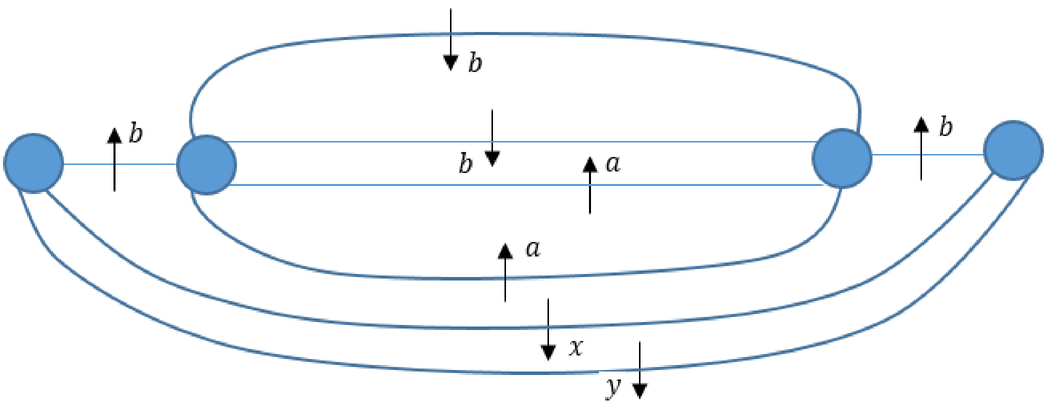
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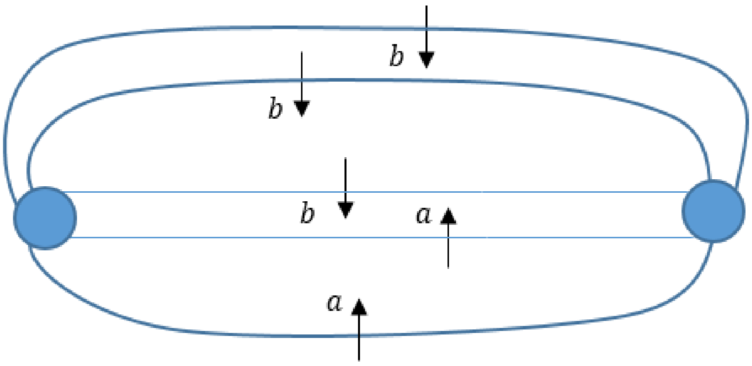

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