The Generator of Second Homotopy Module of

\[ \langle x, y ; xyx = yxy \rangle \quad \text{and} \quad \langle a, b ; a^2, b^3 \rangle \]

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Abstract

This paper discuss about generator of second homotopy module of \( \langle x, y ; xyx = yxy \rangle \) and second homotopy module of \( \langle a, b ; a^2, b^3 \rangle \). It is shown that using Tietze transformation and operations on picture there are sequence of generators between their.

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1. Introduction

A picture over \( \mathcal{P} = \langle x ; r \rangle \) is called a set of generator second homotopy module \( \pi_2(\mathcal{P}) \) if \( \{ [P] ; P \in \mathcal{P} \} \) generate \( \mathbb{Z}G \) module \( \pi_2(\mathcal{P}) \) [1]. Therefore, set generator \( \mathcal{P} \) is generator iff each spherical picture over \( \mathcal{P} \) can be transformed to empty picture by using operation on picture [2].

Calculation generator of second homotopy module performed by [2] only to describe the generators of second homotopy module from one group presentation and [8] provides a simple application from [7].

This article discuss about generator of second homotopy module of \( \langle x, y ; xyx = yxy \rangle \) and second homotopy module of \( \langle a, b ; a^2, b^3 \rangle \) using [6]. In line with this, theory of Tietze transformation can be seen in [3] and [4]. In this transformation, the operations on picture used to get the generator of these second homotopy module. Operations on picture can be seen in [6].

We are going to prove the following lemma:
Lemma 1.1 Group presentation \( \langle x, y; xyx = yxy \rangle \) isomorphic to \( \langle a, b; a^2, b^3 \rangle \), and there are sequence of generator from \( \pi_2(\langle x, y; xyx = yxy \rangle) \) to \( \pi_2(\langle a, b; a^2 = b^3 \rangle) \).

2. Basic Theory

In this section we will introduce the basic concept which is needed in all articles. The reference to this basic theory such as [2, 3, 4].

Let \( x \) be a set (alphabet). A word \( W \) on \( x \) is the form \( x_1^{\varepsilon_1}x_2^{\varepsilon_2} \ldots x_n^{\varepsilon_n} \) where \( n \geq 0 \), \( x_i \in x \) and \( \varepsilon_i = \pm 1 \), \( i = 1, 2, \ldots, n \). Inverse of \( W \), denoted \( W^{-1} \) is word \( x_n^{-\varepsilon_n}x_{n-1}^{-\varepsilon_{n-1}} \ldots x_2^{-\varepsilon_2}x_1^{-\varepsilon_1} \). If \( x_i^{\varepsilon_i} \neq x_{i+1}^{-\varepsilon_{i+1}}, i = 1, \ldots, n - 1 \), then we say that it is reduced. Furthermore it is cyclically reduced if in addition \( x_1^{\varepsilon_1} \neq x_n^{-\varepsilon_n} \). Then we have a presentation \( \mathcal{P} = \langle x; r \rangle \), where \( r \) is a set of non-empty cyclically reduced words on \( x \). We say that \( \mathcal{P} \) is finite if \( x \) and \( r \) are both finite.

If \( F(x) \) is the free group on \( x \) and \( N = \langle \langle r \rangle \rangle \) is normal closure of \( r \) in \( F(x) \), then the quotient group \( G(\mathcal{P}) = F(x)/N \) is the group defined by \( \mathcal{P} \). Denote a typical of \( G(\mathcal{P}) \) by \( \tilde{W} = [W]N \) where \( W \) is a word on \( x \) and \([W]\) is the free equivalence class of \( W \). A group \( G \) is said to be finitely presented if \( G \) can be defined by a finite presentation (that is \( G = G(\mathcal{P}) \) for some finite presentation \( \mathcal{P} \)).

We may regard \( \mathcal{P} \) as a 2-complex. This complex has a single 0-cell the 1-cell are in bijective correspondence with \( x \), and the 2-cell are in bijective correspondence with \( r \) and are attached by the boundary path determined by the spelling of the corresponding member of \( r \). Thus there are homotopy group \( \pi_1(\mathcal{P}) \) and \( \pi_2(\mathcal{P}) \).

The element of the second homotopy module \( \pi_2(\mathcal{P}) \) can be represented by geometric configurations called spherical pictures.

A picture \( \mathcal{P} \) over \( \mathcal{P} \) is a geometric configuration consisting of the following:

a. A disc \( D^2 \) with basepoint \( 0 \) on \( \partial D^2 \).

b. Disjoint discs \( \Delta_1, \Delta_2, \ldots, \Delta_n \) in the interior of \( D^2 \). Each \( \Delta_i \) has a basepoint \( 0_i \) on \( \partial \Delta_i \).

c. A finite number of disjoint arcs \( \alpha_1, \alpha_2, \ldots, \alpha_m \) where each arc lies in the closure of \( D^2 - \bigcup_{i=1}^{n} \Delta_i \) and is either simple closed curve having trivial intersection with \( \partial D^2 \cup (\bigcup_{i=1}^{n} \partial \Delta_i) \), or is a simple non-closed curve which join two points of \( \partial D^2 \cup (\bigcup_{i=1}^{n} \partial \Delta_i) \), neither point being a basepoint. Each arc has a normal orientation, indicated by a short arrow meeting with the arc transversely and is labelled by an element of \( x \cup x^{-1} \).

d. If we travel around \( \partial \Delta_i \) once in clockwise direction starting from \( 0_i \) and read off the labels on arcs encountered (if we cross an arc, labelled \( x \) say, in the
direction of its normal orientation, then we read $x^{-1}$, then we obtain a word which belongs to $r \cup r^{-1}$. We call this word the label of $\Delta_i$.

We define $\partial P$ to be $\partial D^2$ and label on $P$ (denoted by $W(P)$) is word read off by traveling around $\partial P$ once in the clockwise direction starting from $O$. We say that $P$ is spherical picture if no arcs meet $\partial P$. If $P$ is spherical picture, we often omit $\partial P$.

Certain basic operation can be applied to a picture (spherical picture) $\mathcal{P}$ as follows: (D) deletion and (I) insertion floating circle, (D') deletion and insertion floating semicircle, (D'') deletion and insertion folding pair and (B) bridge move (see [6]).

Two pictures will be said to be equivalent if one can be transformed to the other by a finite number of operation D, I, D', D'' and B. We let $[P]$ denote the equivalence class containing $P$. Note that the set of equivalence of all spherical picture over $\mathcal{P}$ form an abelian group, denoted by $\pi_2(\mathcal{P})$ under the binary operation $[P_1] + [P_2] = [P_1 + P_2]$. The identity is the equivalence class containing the empty picture and the inverse of $[P]$ is $[-P]$. Then we can consider $\pi_2(\mathcal{P})$ as a left $\mathbb{Z}G(\mathcal{P})$-module where the $G(\mathcal{P})$-action is given by $\overline{W}. [P] = [P^W]$, $\overline{W} \in G$ and $P^W$ is the spherical picture obtained from spherical picture $P$ by surrounding it by a collection of concentric closed arcs with total label $W$. Then, we call $\pi_2(\mathcal{P})$ the second homotopy module of $\mathcal{P}$.

3. Proof of Lemma 1.1

By [2], generator of $\pi_2(\langle x, y; xyx = yxy \rangle)$ is

![Diagram]

and generator of $\pi_2(\langle a, b; a^2 = b^3 \rangle)$ is
Consider that $\pi_2((x,y;xyx = yxy))$ is generated by single picture, and $\pi_2((a, b; a^2 = b^3))$ as well. Thus, is obtained a different generator for each second homotopy module. However, there are sequence of generators of $\pi_2((x,y;xyx = yxy))$ to $\pi_2((a, b; a^2 = b^3))$.

By [3] and [5], we have sequence of Tietze transformation from $(a, b; a^2 = b^3)$ to $(x,y; xyx = yxy)$, i.e.,

\[
\langle x, y; xyx = yxy \rangle \cong \langle x, y, a; xyx = yxy, a = xyx \rangle \\
\cong \langle x, y, a, b; xyx = yxy, a = xyx, b = xy \rangle \\
\cong \langle x, y, a, b, xyx = yxy, a = xyx, b = xy, a^2 = b^3 \rangle \\
= \langle x, y, a, b, xyxy^{-1}x^{-1}y^{-1}, a^{-1}xyx, b^{-1}xy, a^2b^{-3} \rangle \\
\cong \langle a, b, x, y; a^{-1}xyx, b^{-1}xy, a^2b^{-3}, x = b^{-1}a, y = a^{-1}b^2 \rangle \\
\cong \langle a, b, x, y; a^2b^{-3}, x = b^{-1}a, y = a^{-1}b^2 \rangle \\
\cong \langle a, b; a^2b^{-3} \rangle \\
= \langle a, b; a^2 = b^3 \rangle
\]

Furthermore, we have sequence generator by [7], namely $P_1, P_2, P_3, P_4, P_5, P_6, P_7$ and $P_8$ respectively.
References


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