On Minimal $X$-ss-Semipermutable Subgroups of Finite Groups

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Abstract

In this paper, we investigate the influence of minimal $X$-ss-semipermutable subgroups on the structure of finite groups and give some new criteria of $p$-nilpotency of finite groups.

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1 Introduction

Throughout the following, $G$ always denotes a finite group. Most of the notation is standard and can be found in [2, 7].
Let $A$ and $B$ be subgroups of a group $G$. $A$ is said to permute with $B$ if $AB = BA$. It is known that $AB$ is a subgroup of $G$ if and only if $A$ permutes with $B$. Some generalizations of permutable subgroups were introduced. For example, $A$ is said to be $s$-semipermutable in $G$ [9] if $AP = PA$ for any Sylow $p$-subgroup $P$ of $G$ with $(|A|, p) = 1$. Let $X$ be a nonempty subset of $G$. Then $A$ is said to be $X$-permutable with $B$ [3] if there exists some element $x$ in $X$ such that $AB^x = B^x A$. $A$ is said to be $X$-$s$-semipermutable in $G$ [4] if $A$ is $X$-permutable with every Sylow subgroup of some supplement $T$ of $A$ in $G$.

There were many papers related with the applications of partially permutable subgroups of various types (for example the work in [1, 3, 4, 6, 8, 9]). As a continuation, the concept of $X$-$ss$-semipermutability [10] was introduced:

Let $X$ be a nonempty subset of a group $G$. Let $H$ be a subgroup of a group $G$. Then we say that $X$-$ss$-semipermutable in $G$ if $H$ has a supplement $T$ in $G$ such that $H$ is $X$-permutable with every Sylow $p$-subgroups of $T$ with $(p, |H|) = 1$.

Obviously, the $X$-permutability and $X$-$s$-semipermutability imply the $X$-$ss$-semipermutability. However, the converse does not hold. For example, let $G = [C_5]C_4$, where $C_5$ is a group of order 5 and $C_4$ is the automorphism group of $C_5$ of order 4. Let $X = 1$ and $H$ be a subgroup of $C_4$ of order 2. Then $H$ is $X$-ss-semipermutable in $G$, but not $X$-$s$-semipermutable in $G$.

In this paper, we will analyze the structure of finite groups with minimal $X$-$ss$-semipermutable subgroups and give some new criteria of $p$-nilpotency of finite groups.

2 Preliminaries

Throughout this paper, we will use $X_{ss}(H)$ to denote the set of all such supplements $T$ of $H$ in $G$ that $H$ is $X$-permutable with every Sylow $p$-subgroups of $T$ with $(p, |H|) = 1$.

**Lemma 2.1** [10] Let $A$ be a subgroup of a group $G$, $X$ be a nonempty subset of $G$ and let $N$ be a normal subgroup of $G$.

1. If $A$ is $X$-$ss$-semipermutable in $G$, then $AN/N$ is $XN/N$-$s$-semipermutable in $G/N$.
2. If $A$ is $X$-$ss$-semipermutable in $G$, $A \leq D \leq G$ and $X \subseteq D$, then $A$ is $X$-$ss$-semipermutable in $D$.
3. If $A$ is $X$-$ss$-semipermutable in $G$ and $X \subseteq D$, then $A$ is $D$-$ss$-semipermutable in $G$.
4. If $T \in X_{ss}(A)$ and $A \leq N_G(X)$, then $T^x \in X_{ss}(A)$ for any $x \in X$.

**Lemma 2.2** Let $P$ be a $p$-subgroup of $G$, $Q$ a $q$-subgroup of $G$ and $PQ \leq G$. If $R$ is a subnormal subgroup of $G$, then $PQ \cap R = (P \cap R)(Q \cap R)$.
Proof. Since $(|PQ : P|, |PQ : Q|) = 1$, $(|PQ \cap R : P|, |PQ \cap R : Q|) = 1$. By [2, Lemma 3.8.2], $PQ \cap R = (PQ \cap R \cap P)(PQ \cap R \cap Q) = (P \cap R)(Q \cap R)$.

Lemma 2.3 [9] Let $A$ be a subgroup of a group $G$. If $A$ is $s$-semipermutable in $G$ and $A \leq H \leq G$, then $A$ is $s$-semipermutable in $H$.

3 Main results

Theorem 3.1 Let $G$ be a group, $p$ be the smallest prime dividing $|G|$ and $X$ be a soluble normal subgroup of $G$. Suppose that every subgroup of $G$ of order $p$ or 4 (if the Sylow $p$-subgroup of $G$ is a non-abelian 2-group) is $X$-ss-semipermutable in $G$. Then $G$ is $p$-nilpotent.

Proof. Suppose that the statement is false and let $G$ be a counterexample of minimal order. We prove the theorem by the following steps:

1. $O_p'(G) = 1$

   If $O_p'(G) \neq 1$. Since $X$ is a soluble normal subgroup of $G$, $XO_p'(G)/O_p'(G)$ is a soluble normal subgroup of $G/O_p'(G)$. Let $K/O_p'(G)$ be a subgroup of $G/O_p'(G)$ of order $p$ or 4 (if the Sylow $p$-subgroup of $G/O_p'(G)$ is a non-abelian 2-group), then there exists a subgroup $L$ of $G$ of order $p$ or 4 (if Sylow $p$-subgroup of $G$ is a non-abelian 2-group) such that $K = LO_p'(G)$. By Lemma 2.1, $G/O_p'(G)$ satisfies the hypothesis. The choice of $G$ yields that $G/O_p'(G)$ is $p$-nilpotent. Consequently $G$ is $p$-nilpotent, a contradiction. Hence $O_p'(G) = 1$.

2. $O_p(G) \neq 1$

   Suppose that $O_p(G) = 1$. Since $O_p'(G) = 1$, $X = 1$. Let $R$ be a minimal subnormal subgroup of $G$. If $|R| = q$, where $q$ is a prime divisor of $|G|$. Then $R \leq O_q(G)$, a contradiction. Therefore $R$ is a non-abelian simple subgroup.

   Let $H$ be a subgroup of $G$ of order $p$, then $H$ is $X$-ss-permutable in $G$. Set $T \in X_{ss}(H)$, then $G = HT$. Let $Q \in Syl_q(G)$ and $M \in Syl_q(T)$, where $q \neq p$. Then there exists an element $g$ of $G$ such that $Q = M^g$. Since $H \leq G = N_G(X)$, $T^g \in X_{ss}(H)$. Thus $HQ = QH$. Hence $H$ is $s$-semipermutable in $G$. For any $a \in R$, $HQ^a \leq G$. By Lemma 2.2, $HQ^a \cap R = (H \cap R)(Q^a \cap R) = (H \cap R)(Q \cap R)^a$. Since $HQ^a \cap R$ is a $pq$-group, $(H \cap R)(Q \cap R)^a$ is solvable. It follows that $(H \cap R)(Q \cap R)^a \neq R$. Hence $R$ is not a simple subgroup by [5, Theorem 3]. This contradiction shows that $O_p(G) \neq 1$.

3. $O_p(G) \leq Z_\infty(G)$

   Since $p$ is the smallest prime dividing $|G|$, it is equivalent to prove that every $G$-chief factor $L/K$ in $O_p(G)$ is of prime order. Assume that the assertion is not true and let $L/K$ be a counterexample with $|K|$ minimal, that is, $L/K$ is non-cyclic but for every chief factor $U/V$ of $G$ below $O_p(G)$ with $|V| < |K|$, $U/V$ is cyclic. Let $R/K$ be a chief factor of $P/K$, where $P$ is a Sylow $p$-subgroup of $G$ and $R \leq L$. Then $R = \langle a \rangle$ for any $a \in R \setminus K$. Let $H = \langle a \rangle$. 

If $|H| = p$ or 4 (if $P$ is non-abelian 2-group). Then by the hypothesis, $H$ is $X$-ss-semipermutable in $G$. Set $T \in X_{ss}(H)$, then $G = HT$. Let $Q \in Syl_{q}(T)$, where $q \neq p$. Then $HQ^x = Q^xH$ for some $x \in X$. Since $Q^x$ is a Sylow $q$-subgroup of $G$, $HQ^x \cap L = (H \cap L)(Q^x \cap L) = H$ by Lemma 2.2. Thus $H$ is normal in $HQ^x$. It follows that $R/K$ is normal in $HQ^xK/K$. Since $R/K$ is a chief factor of $P/K$, $R/K$ is normal in $G/K$. The choice of $L/K$ shows that $L/K = R/K$ is cyclic. This contradiction means that all elements of $R \setminus K$ of order $p$ and order 4 (if $P$ is a non-abelian 2-group) are contained in $K$. Since $L/K = (R/K)^{G/K} = R^{G}/K$, we have that all elements of $L$ of order $p$ and 4 (if $P$ is a non-abelian 2-group) are contained in $K$. Let $U/V$ be any chief factor of $G$ below $K$. Then, by the choice of $L/K$, $U/V$ is of order $p$ and so $G/C_{G}(U/V)$ is abelian of exponent dividing $p - 1$. Put $W = \cap_{U \leq K}C_{G}(U/V)$, where $U/V$ is a $G$-chief. Then $W$ is normal in $G$ and $G/W$ is abelian of exponent dividing $p - 1$. Let $Q$ be any Sylow $q$-subgroup of $W$, where $q \neq p$. Then by [2, A(12.3)], $Q$ acts trivially on $K$. Moreover, since all elements of $L$ of order $p$ and 4 (if $P$ is a non-abelian 2-group) are contained in $K$, $Q$ acts trivially on $L/K$ by the well-known Blackburn’s theorem, from which we conclude that $W/C_{W}(L/K)$ is a $p$-group. It follows that $W \leq C_{G}(L/K)$ by [2, Lemma 1.7.11]. Since $G/W = G/\cap_{U \leq K}C_{G}(U/V)$ is abelian of exponent dividing $p - 1$, also $G/C_{G}(L/K)$ is. Now, by [7, I, Lemma 1.3], we have that $L/K$ is of order $p$. This contradiction shows that (3) holds.

(4) $F^{*}(G) = F(G) = O_{p}(G)$.

Let $F = F^{*}(G)$. By (1), $F(G) = O_{p}(G)$ and $O_{p}(F) = 1$. Then by $F$ is a quasinilpotent normal subgroup of $G$, $O_{p}(F) = O_{p}(G)$ is the maximal normal subgroup of $F$. Thus the soluble normality of $X$ shows that $X \cap F \leq O_{p}(F)$. Set $\overline{X} = XO_{p}(F)/O_{p}(F), \overline{F} = F/O_{p}(F)$. Then $\overline{X} \cap \overline{F} = 1$ and hence $\overline{F} \leq C_{G}\overline{X}$. If $F \neq F(G)$. Let $R/O_{p}(F)$ be a minimal subnormal subgroup of $G/O_{p}(F)$ and $R \leq F$. We assume that $R$ is not $p$-nilpotent. If not, let $S$ be a normal Hall $p'$-subgroup. Since $p$ is the smallest prime diving $|G|$, $S$ is soluble. Then the minimal subnormal subgroup of $S$ is prime order and contained in $O_{p'}(G)$. By $O_{p'}(G) = 1$, $S = 1$. It follows that $R$ is $p$-group and therefore $R \leq O_{p}(F)$. This contraction shows that $R$ is not $p$-nilpotent. Let $\overline{G} = G/O_{p}(F), \overline{R} = R/O_{p}(F)$ Since $\overline{R}$ is the minimal subnormal subgroup of $\overline{G}$, $\overline{R}$ is a non-abelian simple subgroup. Let $M$ be a minimal non-$p$-nilpotent subgroup of $R$. Thus $M = [A]B$, where $A$ is a Sylow $p$-subgroup of $M$, $exp(A) = p$ or 4 and $B$ is a $p'$-subgroup of $M$. If $A \leq O_{p}(F)$, then by (3) $M$ is $p$-nilpotent. If $A \notin O_{p}(F)$, then there exists an element $a$ of $A$ such that $a \in A \setminus O_{p}(F)$. Let $H = \langle a \rangle$, then $|H| = p$ or $|H| = 4$. By the hypothesis, $H$ is $X$-ss-semipermutable in $G$. Let $\overline{H} = HO_{p}(F)/O_{p}(F)$. Hence $\overline{H}$ is $\overline{X}$-ss-permutable in $\overline{G}$. Set $T \in X_{ss}(\overline{H})$, then $G = \overline{H} T$. Let $\overline{Q} \in Syl_{q}(\overline{G})$ and $\overline{M} \in Syl_{q}(T)$, where $q \neq p$. Then there exists an element $\overline{g}$ of $\overline{G}$ such that $\overline{Q} = \overline{M}^{\overline{g}}$. Since $\overline{H} \leq \overline{F} \leq C_{\overline{G}}\overline{X}, \overline{T}^{\overline{g}} \in X_{ss}(\overline{H})$. Thus $\overline{H} \overline{Q} = \overline{Q} \overline{H}$. Hence $\overline{H}$ is
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Since $H \leq R$, $H$ is $s$-semipermutable in $R$ by Lemma 2.3. Let $K \in Syl_pR$. For any $\alpha \in R$, $H K^{\alpha}$ is $pq$-subgroup of $R$. Since $R$ is a non-abelian simple subgroup, $H K^{\alpha} \neq R$. Hence $R$ is not a simple subgroup by [5, Theorem 3]. This contradiction shows that $F \neq F(G)$.

(5) Final contradiction.

Put $W = \cap_{U \leq K} C_G(U/V)$, where $U/V$ is a $G$-chief in $O_p(G)$. Since $F(G) \leq C_G(U/V)$, $F(G) \leq W$. Suppose that $F(G) \neq W$ and let $R/F(G)$ be a minimal normal subgroup of $G/F(G)$ with $R \leq W$. Thus $R/F(G)$ is quasinilpotent and so is $R$. It follows that $R \leq F(G)$, a contradiction. Thus, $F(G) = W$. Since $G/C_G(U/K)$ is abelian of exponent dividing $p - 1$ by the preceding argument (3), $G/F(G) = G/W$ is abelian of exponent dividing $p - 1$. By (3), $G$ is $p$-nilpotent. Thus the proof is complete.

Corollary 3.2 Let $G$ be a group and $p$ be the smallest prime diving $|G|$. Suppose that every subgroup of $G$ of order $p$ or $4$ (if the Sylow $p$-subgroup of $G$ is a non-abelian $2$-group) is $s$-semipermutable in $G$. Then $G$ is $p$-nilpotent.

Corollary 3.3 Let $G$ be a soluble group and $p$ be the smallest prime diving $|G|$. Suppose that every subgroup of $G$ of order $p$ or $4$ (if the Sylow $p$-subgroup of $G$ is a non-abelian $2$-group) is $G$-permutable in $G$. Then $G$ is $p$-nilpotent.

Corollary 3.4 Let $G$ be a group and $p$ be the smallest prime diving $|G|$. Suppose that every subgroup of $G$ of order $p$ or $4$ (if the Sylow $p$-subgroup of $G$ is a non-abelian $2$-group) is $F(G)$-permutable in $G$. Then $G$ is $p$-nilpotent.

Corollary 3.5 Let $G$ be a group, $p$ be the smallest prime diving $|G|$ and $X$ be a soluble normal subgroup of $G$. Suppose that every subgroup of $G$ of order $p$ or $4$ (if the Sylow $p$-subgroup of $G$ is a non-abelian $2$-group) is $X$-$ss$-semipermutable in $G$. Then $G$ is $p$-nilpotent.

Corollary 3.6 Let $G$ be a group and $X$ be a soluble normal subgroup of $G$. Suppose that every primary cyclic subgroup of $G$ (if the Sylow $2$-subgroup of $G$ is a non-abelian $2$-group) is $X$-$ss$-semipermutable in $G$. Then $G$ is a Sylow Tower group.

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