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On Minimal X-ss-Semipermutable Subgroups of Finite Groups

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Abstract

In this paper, we investigate the influence of minimal X-ss-semipermutable subgroups on the structure of finite groups and give some new criteria of p-nilpotency of finite groups.

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1 Introduction

Throughout the following, G always denotes a finite group. Most of the notation is standard and can be found in [2, 7].

Let A and B be subgroups of a group G. A is said to permute with B if $AB = BA$. It is known that AB is a subgroup of G if and only if A permutes with B. Some generalizations of permutable subgroups were introduced. For example, A is said to be *s-semipermutable* in G [9] if $AP = PA$ for any Sylow p-subgroup P of G with $(|A|, p) = 1$. Let X be a nonempty subset of G. Then A is said to be *X-permutable* with B [3] if there exists some element x in X such that $AB^x = B^xA$. A is said to be *X-s-semipermutable* in G[4] if A is X- permutable with every Sylow subgroup of some supplement T of A in G. There were many papers related with the applications of partially permutable subgroups of various types (for example the work in [1, 3, 4, 6, 8, 9]).

As a continuation, the concept of X-ss-semipermutability[10] was introduced:

Let X be a nonempty subset of a group G. Let H be a subgroup of a group G. Then we say that *X-ss-semipermutable* in G if H has a supplement T in G such that H is X-permutable with every Sylow p-subgroups of T with $(p, |H|) = 1$.

Obviously, the X-permutability and X-s-semipermutability imply the X-ss-semipermutability. However, the converse does not hold. For example, let $G = [C_5]C_4$, where C_5 is a group of order 5 and C_4 is the automorphism group of C_5 of order 4. Let $X = 1$ and H be a subgroup of C_4 of order 2. Then H is X-ss-semipermutable in G, but not X-s-semipermutable in G.

In this paper, we will analyze the structure of finite groups with minimal X-ss-semipermutable subgroups and give some new criteria of p-nilpotency of finite groups.

2 Preliminaries

Throughout this paper, we will use $X_{ss}(H)$ to denote the set of all such supplements T of H in G that H is X-permutable with every Sylow p-subgroups of T with $(p, |H|) = 1$.

Lemma 2.1 *[10] Let A be a subgroup of a group G, X be a nonempty subset ofG and let N be a normal subgroup of G.*

(1) If A is X-ss-semipermutable in G, then AN/N is XN/N-s-semipermutable in G/N.

(2) If A is X-ss-semipermutable in G, $A \leq D \leq G$ and $X \subseteq D$, then A is X-ss-semipermutable in D.

(3) If A is X-ss-semipermutable in G and $X \subseteq D$, then A is D-ss-semipermutable in G.

(4) If $T \in X_{ss}(A)$ and $A \leq N_G(X)$, then $T^x \in X_{ss}(A)$ for any $x \in X$.

Lemma 2.2 *Let P be a p-subgroup of G, Q a q-subgroup of G and $PQ \leq G$. If R is a subnormal subgroup of G, then $PQ \cap R = (P \cap R)(Q \cap R)$.*

Proof. Since $(|PQ : P|, |PQ : Q|) = 1$, $(|PQ \cap R : P|, |PQ \cap R : Q|) = 1$. By [2, Lemma 3.8.2], $PQ \cap R = (PQ \cap R \cap P)(PQ \cap R \cap Q) = (P \cap R)(Q \cap R)$.

Lemma 2.3 *[9] Let A be a subgroup of a group G. If A is s-semipermutable in G and $A \leq H \leq G$, then A is s-semipermutable in H.*

3 Main results

Theorem 3.1 *Let G be a group, p be the smallest prime diving $|G|$ and X be a soluble normal subgroup of G. Suppose that every subgroup of G of order p or 4 (if the Sylow p-subgroup of G is a non-abelian 2-group) is X-ss-semipermutable in G. Then G is p-nilpotent.*

Proof. Suppose that the statement is false and let G be a counterexample of minimal order. We prove the theorem by the following steps:

(1) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$. Since X is a soluble normal subgroup of G, $XO_{p'}(G)/O_{p'}(G)$ is a soluble normal subgroup of $G/O_{p'}(G)$. Let $K/O_{p'}(G)$ is a subgroup of $G/O_{p'}(G)$ of order p or 4 (if the Sylow p-subgroup of $G/O_{p'}(G)$ is a non-abelian 2-group), then there exists a subgroup L of G of order p or 4 (if Sylow p-subgroup of G is a non-abelian 2-group) such that $K = LO_{p'}(G)$. By Lemma 2.1, $G/O_{p'}(G)$ satisfies the hypothesis. The choice of G yields that $G/O_{p'}(G)$ is p-nilpotent. Consequently G is p-nilpotent, a contradiction. Hence $O_{p'}(G) = 1$.

(2) $O_p(G) \neq 1$

Suppose that $O_p(G) = 1$. Since $O_{p'}(G) = 1$, $X = 1$. Let R be a minimal subnormal subgroup of G. If $|R| = q$, where q is a prime divisor of $|G|$. Then $R \leq O_q(G)$, a contradiction. Therefore R is a non-abelian simple subgroup. Let H be a subgroup of G of order p, then H is X-ss-permutable in G. Set $T \in X_{ss}(H)$, then $G = HT$. Let $Q \in Syl_q(G)$ and $M \in Syl_q(T)$, where $q \neq p$. Then there exists an element g of G such that $Q = M^g$. Since $H \leq G = N_G(X)$, $T^g \in X_{ss}(H)$. Thus $HQ = QH$. Hence H is s-semipermutable in G. For any $a \in R$, $HQ^a \leq G$. By Lemma 2.2, $HQ^a \cap R = (H \cap R)(Q^a \cap R) = (H \cap R)(Q \cap R)^a$. Since $HQ^a \cap R$ is a pq-group, $(H \cap R)(Q \cap R)^a$ is solvable. It follows that $(H \cap R)(Q \cap R)^a \neq R$. Hence R is not a simple subgroup by [5, Theorem 3]. This contradiction shows that $O_p(G) \neq 1$.

(3) $O_p(G) \leq Z_\infty(G)$.

Since p is the smallest prime diving $|G|$, it is equivalent to prove that every G-chief factor L/K in $O_p(G)$ is of prime order. Assume that the assertion is not true and let L/K be a counterexample with $|K|$ minimal, that is, L/K is non-cyclic but for every chief factor U/V of G below $O_p(G)$ with $|V| < |K|$, U/V is cyclic. Let R/K be a chief factor of P/K, where P is a Sylow p-subgroup of G and $R \leq L$. Then $R =< a > K$ for any $a \in R \setminus K$. Let $H =< a >$.

If $|H| = p$ or 4 (if P is non-abelian 2-group). Then by the hypothesis, H is X-ss-semipermutable in G. Set $T \in X_{ss}(H)$, then $G = HT$. Let $Q \in Syl_q(T)$, where $q \neq p$. Then $HQ^x = Q^xH$ for some $x \in X$. Since Q^x is a Sylow q-subgroup of G, $HQ^x \cap L = (H \cap L)(Q^x \cap L) = H$ by Lemma 2.2. Thus H is normal in HQ^x. It follows that R/K is normal in HQ^xK/K. Since R/K is a chief factor of P/K, R/K is normal in G/K. The choice of L/K shows that $L/K = R/K$ is cyclic. This contradiction means that all elements of $R \setminus K$ of order p and order 4 (if P is a non-abelian 2-group) are contained in K. Since $L/K = (R/K)^{G/K} = R^G/K$, we have that all elements of L of order p and 4 (if P is a non-abelian 2-group) are contained in K. Let U/V be any chief factor of G below K. Then, by the choice of L/K, U/V is of order p and so $G/C_G(U/V)$ is abelian of exponent dividing $p-1$. Put $W = \cap_{U \leq K} C_G(U/V)$, where U/V is a G-chief. Then W is normal in G and G/W is abelian of exponent dividing $p-1$. Let Q be any Sylow q-subgroup of W, where $q \neq p$. Then by [2, A(12.3)], Q acts trivially on K. Moreover, since all elements of L of order p and 4 (if P is a non-abelian 2-group) are contained in K, Q acts trivially on L/K by the well-known Blackburn's theorem, from which we conclude that $W/C_W(L/K)$ is a p-group. It follows that $W \leq C_G(L/K)$ by [2, Lemma 1.7.11]. Since $G/W = G/\cap_{U \leq K} C_G(U/V)$ is abelian of exponent dividing $p-1$, also $G/C_G(L/K)$ is. Now, by [7, I, Lemma 1.3], we have that L/K is of order p. This contradiction shows that (3) holds.

(4) $F^*(G) = F(G) = O_p(G)$.

Let $F = F^*(G)$. By (1), $F(G) = O_p(G)$ and $O_{p'}(F) = 1$. Then by F is a quasinilpotent normal subgroup of G, $O_p(F) = O_p(G)$ is the maximal normal subgroup of F. Thus the soluble normality of X shows that $X \cap F \leq O_p(F)$. Set $\overline{X} = XO_p(F)/O_p(F), \overline{F} = F/O_p(F)$. Then $\overline{X} \cap \overline{F} = 1$ and hence $\overline{F} \leq C_{\overline{G}}\overline{X}$. If $F \neq F(G)$. Let $R/O_p(F)$ be a minimal subnormal subgroup of $G/O_p(F)$ and $R \leq F$. We assume that R is not p-nilpotent. If not, let S be a normal Hall p'-subgroup. Since p is the smallest prime diving $|G|$, S is soluble. Then the minimal subnormal subgroup of S is prime order and contained in $O_{p'}(G)$. By $O_{p'}(G) = 1$, $S = 1$. It follows that R is p-group and therefore $R \leq O_p(F)$. This contraction shows that R is not p-nilpotent. Let $\overline{G} = G/O_p(F), \overline{R} = R/O_p(F)$ Since $\overline{R}$ is the minimal subnormal subgroup of $\overline{G}$, $\overline{R}$ is a non-abelian simple subgroup. Let M be a minimal non-p-nilpotent subgroup of R. Thus $M = [A]B$, where A is a Sylow p-subgroup of M, $exp(A) = p$ or 4 and B is a p'-subgroup of M. If $A \leq O_p(F)$, then by (3) M is p-nilpotent. If $A \nleq O_p(F)$, then there exists an element a of A such that $a \in A \setminus O_p(F)$. Let $H =< a >$, then $|H| = p$ or $|H| = 4$. By the hypothesis, H is X-ss-semipermutable in G. Let $\overline{H} = HO_p(F)/O_p(F)$. Hence $\overline{H}$ is $\overline{X}$-ss-permutable in $\overline{G}$. Set $\overline{T} \in \overline{X}_{ss}(\overline{H})$, then $\overline{G} = \overline{H}\,\overline{T}$. Let $\overline{Q} \in Syl_q(\overline{G})$ and $\overline{M} \in Syl_q(\overline{T})$, where $q \neq p$. Then there exists an element $\overline{g}$ of $\overline{G}$ such that $\overline{Q} = \overline{M}^{\overline{g}}$. Since $\overline{H} \leq \overline{F} \leq C_{\overline{G}}\overline{X}$, $\overline{T}^{\overline{g}} \in \overline{X}_{ss}(\overline{H})$. Thus $\overline{H}\,\overline{Q} = \overline{Q}\,\overline{H}$. Hence $\overline{H}$ is

s-semipermutable in $\overline{G}$. Since $\overline{H} \leq \overline{R}$, $\overline{H}$ is s-semipermutable in $\overline{R}$ by Lemma 2.3. Let $\overline{K} \in Syl_q\overline{R}$. For any $\alpha \in \overline{R}$, $\overline{H}\ \overline{K}^a$ is pq-subgroup of $\overline{R}$. Since$\overline{R}$ is a non-abelian simple subgroup, $\overline{H}\ \overline{K}^a \neq \overline{R}$. Hence $\overline{R}$ is not a simple subgroup by [5, Theorem 3]. This contradiction shows that $F \neq F(G)$.

(5) Final contradiction.

Put $W = \cap_{U \leq K} C_G(U/V)$, where U/V is a G-chief in $O_p(G)$. Since $F(G) \leq C_G(U/V)$, $F(G) \leq W$. Suppose that $F(G) \neq W$ and let $R/F(G)$ be a minimal normal subgroup of $G/F(G)$ with $R \leq W$. Thus $R/F(G)$ is quasinilpotent and so is R. It follows that $R \leq F(G)$, a contradiction. Thus, $F(G) = W$. Since $G/C_G(U/K)$ is is abelian of exponent dividing $p-1$ by the preceding argument (3), $G/F(G) = G/W$ is abelian of exponent dividing $p-1$. By (3), G is p-nilpotent. Thus the proof is complete.

Corollary 3.2 *Let G be a group and p be the smallest prime diving $|G|$. Suppose that every subgroup of G of order p or 4 (if the Sylow p-subgroup of G is a non-abelian 2-group) is s-semipermutable in G. Then G is p-nilpotent.*

Corollary 3.3 *Let G be a soluble group and p be the smallest prime diving $|G|$. Suppose that every subgroup of G of order p or 4 (if the Sylow p-subgroup of G is a non-abelian 2-group) is G-permutable in G. Then G is p-nilpotent.*

Corollary 3.4 *Let G be a group and p be the smallest prime diving $|G|$. Suppose that every subgroup of G of order p or 4 (if the Sylow p-subgroup of G is a non-abelian 2-group) is $F(G)$-permutable in G. Then G is p-nilpotent.*

Corollary 3.5 *Let G be a group, p be the smallest prime diving $|G|$ and X be a soluble normal subgroup of G. Suppose that every subgroup of G of order p or 4 (if the Sylow p-subgroup of G is a non-abelian 2-group) is X-s-semipermutable in G. Then G is p-nilpotent.*

Corollary 3.6 *Let G be a group and X be a soluble normal subgroup of G. Suppose that every primary cyclic subgroup of G (if the Sylow 2-subgroup of G is a non-abelian 2-group) is X-ss-semipermutable in G. Then G is a Sylow Tower group.*

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