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On the Split Common Fixed Point Problem for Nonexpansive Semigroups in Banach Spaces

Jian-Qiang Zhang and Gang Wang[1]

College of Statistics and Mathematics
Yunnan University of Finance and Economics
Kunming, Yunnan, 650221, P.R. China

Abstract

In this paper, an iteration method is proposed for finding a split common fixed point of two nonexpansive semigroups in Banach spaces. The iterative scheme proposed is proved to converge weakly and strongly to a split common fixed point of two nonexpansive semigroups in Banach spaces under some suitable conditions. The results presented in the paper are new and improve and extend some recent corresponding results.

Mathematics Subject Classifications: 47H09, 47J25

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1 Introduction

Let E be a real normed linear space and C be a nonempty closed convex subset of E. The mapping $T : C \to C$ is said to be nonexpansive if for all $x, y \in C$

$$\|Tx - Ty\| \leq \|x - y\|. \tag{1.1}$$

[1]Corresponding author
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Definition 1.1 ([1]) *A one-parameter family* $\mathcal{F} := \{T(t) : t \geq 0\}$ *of* E *into itself is called a strongly continuous semigroup of Lipschitzian mappings on* E *if it satisfies the following conditions:*

(i) $T(0)x = x$, *for all* $x \in E$;

(ii) $T(s+t) = T(s)T(t)$, *for all* $s, t \geq 0$;

(iii) *for each* $x \in E$, *the mapping* $t \mapsto T(t)x$ *is continuous;*

(iv) *for each* $t > 0$, *there exists a bounded measurable function* $L(t) : [0, \infty) \to [0, \infty)$ *such that*

$$\|T(t)x - T(t)y\| \leq L(t)\|x - y\|, \ for\ all\ x, y \in E; \tag{1.2}$$

A strongly continuous semigroup of Lipchitszian mappings $\mathcal{F}$ is called strongly continuous semigroup of nonexpansive mappings if $L(t) = 1$ for all $t > 0$. We denote by $F(\mathcal{F})$ the set of common fixed points of $\mathcal{F}$, that is,

$$F(\mathcal{F}) := \{x \in E : T(t)x = x, 0 \leq t < \infty\} = \bigcap_{t \geq 0} F(T(t)). \tag{1.3}$$

If $\mathcal{F}$ satisfies $(i) - (iii)$ and

$$\limsup_{t\to\infty} \sup_{x \in D} \|T(t)x - T(s)T(t)x\| = 0, \ for\ all\ s > 0\ and\ bounded\ D \subseteq C, \tag{1.4}$$

then $\mathcal{F}$ is called uniformly asymptotically regular on C.

As well known, the construction of fixed points of nonexpansive mappings, and of common fixed points of nonexpansive semigroups is an important problem in the theory of nonexpansive mappings and its applications, in particular, in image recovery, convex feasibility problem, and signal processing problem (see, for example [2-5]).

Iterative approximation of fixed point for nonexpansive mappings, asymptotically nonexpansive mappings, nonexpansive semi-groups, and asymptotically nonexpansive semigroupps in Hilbert or Banach spaces has been studied extensively by many authors (see, for example, [6-9] and the references therein).

Let H_1 and H_2 be two real Hilbert spaces, C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. The split feasibility problem is formulated as finding a point $q \in H_1$ with the property:

$$q \in C \quad \text{and} \quad Aq \in Q, \tag{1.5}$$

where $A : H_1 \to H_2$ is a bounded linear operator.

Assuming that $SFP(1.5)$ is consistent (i.e (1.5) has a solution), it is not hard to see that $x \in C$ solve (1.5) if and only if it solves the following fixed point equation

$$x = P_C(I - \gamma A^*(I - P_Q)A)x, \qquad x \in C, \tag{1.6}$$

where P_C and P_Q are the (orthogonal) projections onto C and Q, respectively, $\gamma > 0$ is any positive constant, and A^* denotes the adjoint of A.

If C and Q are the sets of fixed points of two nonlinear mappings, respectively, and C and Q are nonempty closed convex subsets, then q is said to be a split common fixed point for the two nonlinear mappings. That is, the split common fixed point problem $(SCFP)$ for mappings S and T is to find a point $q \in H_1$ with the property:

$$q \in C := F(S) \quad \textit{and} \quad Aq \in Q := F(T), \tag{1.7}$$

where $F(S)$ and $F(T)$ denote the sets of fixed points of S and T, respectively. We use Γ to denote the set of solution of $SCFP(1.7)$, that is $\Gamma = \{q \in F(S) : Aq \in F(T)\}$.

Since each nonempty closed convex subset of a Hilbert space is the set of fixed points of its projection, so the split common fixed point problem can be considered as a generalization of the split feasibility problem and the convex feasibility problem. The split common fixed point problems was introduced by Moudafi [10] in 2010. In [10], Moudafi proposed an iteration scheme and obtained a weak convergence theorem of the split common fixed point problem for demicontractive mappings in the setting of two Hilbert spaces. Since then, the split common fixed point problems of other nonlinear mappings in the setting of two Hilbert spaces have been studied by some authors, see, for instance, [2,11-14]. In [15], Cholamjiak et al proved a strong convergence theorem for split feasibility problem involving a uniformly asymptotically regular nonexpansive semigroup and a total asymptotically strict pseudocontractive mapping in Hilbert spaces.

In 2015, Takahashi [16] first attempted to introduce and consider the split feasibility problem and split common null point problem in the setting of Banach spaces. By using hybrid methods and Halpern's type methods and under suitable conditions some strong and weak convergence theorems for such problems are proved in the setting of one Hilbert space and one Banach space.

Recently, in [17], Tang et al proved a weak convergence theorem and a strong convergence theorem for split common fixed point problem involving a quasi-strict pseudo-contractive mapping and an asymptotical nonexpansive mapping in the setting of two Banach spaces, under the following assumptions:

(1) E_1 is a real uniformly convex and 2-uniformly smooth Banach space having the Opial property and the best smoothness constant k satisfying $0 < k < \frac{1}{\sqrt{2}}$.

(2) E_2 is a real Banach space.

(3) $A : E_1 \to E_2$ be a bounded linear operator and A^* is the adjoint of A.

(4) $S : E_1 \to E_1$ is an $\{l_n\}$-asymptotically nonexpansive mapping with

$\{l_n\} \subset (1, \infty)$ and $l_n \to 1$. $T : E_2 \to E_2$ is a τ-quasi-strict pseudocontractive mapping with $F(S) \neq \emptyset$ and $F(T) \neq \emptyset$, and T is demiclosed at zero.

In this paper, motivated and inspired by the recent research works on split common fixed point problems, we introduce a new iteration method to approximate a split common fixed point of two nonexpansive semigroups in the setting of two Banach spaces. Under some suitable conditions, the iterative scheme proposed is shown to converge strongly and weakly to a split common fixed point of two nonexpansive semigroups.

2 Preliminaries

Let E be a real Banach space with the dual E^*. The normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \forall x \in E, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between E and E^*.

A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} \leq 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. The modulus of convexity of E is defined by

$$\delta_E(\epsilon) = inf\{1 - \|\frac{1}{2}(x+y)\| : \|x\|,\ \|y\| \leq 1,\ \|x-y\| \geq \epsilon\},$$

for all $\epsilon \in [0, 2]$. E is said to be uniformly convex if $\delta_E(0) = 0$, and $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. A Hilbert space is 2-uniformly convex, while L^p is $max\{p, 2\}-$uniformly convex for every $p > 1$.

Let $\rho_E : [0, \infty) \to [0, \infty)$ be the modulus of smoothness of E defined by

$$\rho_E(t) = sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x \in U, \|y\| \leq t\}.$$

A Banach space E is said to be uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$. A typiacal example of uniformly smooth Banach space is L^p, where $p > 1$. More precisely, L^p is $min\{p, 2\}$-uniformly smooth for every $p > 1$. Let q be a fixed real number with $q > 1$, then a Banach space E is said to be $q-$uniformly smooth if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^q$ for all $t > 0$. It is well known that every $q-$uniformly smooth Banach space is uniformly smooth.

Lemma 2.1 ([18]) *Given a number $r > 0$. A real Banach space E is uniformly convex if and only if there exists a continuous strictly increasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x-y\|),$$

for all $x, y \in E$, $t \in [0,1]$, *with* $\|x\| \le r$ *and* $\|y\| \le r$.

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be demiclosed at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x^* \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tx^* = p$.

A mapping $T : C \to C$ is said to be semi$-$compact, if for any sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0, (n \to \infty)$, there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{x_j}\}$ converges strongly to $x^* \in C$.

A Banach space E is said to satisfy Opial property if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$, for any $y \in E$ with $y \neq x$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|. \tag{2.3}$$

Lemma 2.2 ([19]) *Let E be a real uniformly convex Banach space, let K be a nonempty closed convex subset of E, and let $T : K \to K$ be a nonexpansive mapping. Then $I - T$ is demiclosed at zero.*

Lemma 2.3 ([20]) *Let $\{a_n\}$ and $\{\alpha_n\}$ be two nonnegative real number sequences and satisfy*

$$a_{n+1} \le (1 + \alpha_n)a_n, \quad \forall n \ge 1,$$

where $a_n \ge 0$, $\alpha_n \ge 0$ and $\sum_{n=1}^{\infty} \alpha_n < \infty$. Then

(1) $\lim_{n\to\infty} a_n$ *exists;*

(2) *if* $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 ([18]) *Let E be a 2-uniformly smooth Banach space with the best smoothness constants $K > 0$. Then the following inequality holds:*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, Jx\rangle + 2\|Ky\|^2, \forall x, y \in E. \tag{2.4}$$

3 Main Results

Theorem 3.1 *Let E_1 be a real uniformly convex and 2-uniformly smooth Banach space satisfying Opial's condition and with the best smoothness constant k satisfying $0 < k < \frac{1}{\sqrt{2}}$, E_2 be a real Banach space, and $A : E_1 \to E_2$ be a bounded linear operator and A^* be the adjoint of A, Let $\{S(t) : t \ge 0\} : E_1 \to E_1$ be a uniformly asymptotically regular nonexpansive semigroup with*

$C := \bigcap_{t\geq 0} F(S(t)) \neq \emptyset$, $\{T(t) : t \geq 0\} : E_2 \to E_2$ *be a uniformly asymptotically regular nonexpansive semigroup with* $Q := \bigcap_{t\geq 0} F(T(t)) \neq \emptyset$*, respectively. Let* $\{x_n\}$ *be a sequence generated by:* $x_1 \in E_1$

$$\begin{cases} z_n = x_n + \gamma J_1^{-1} A^* J_2(T(t_n) - I)Ax_n, \quad \forall n \geq 1, \\ x_{n+1} = (1-\alpha_n)z_n + \alpha_n S(t_n)(z_n), \end{cases} \tag{3.1}$$

where $\{t_n\}$ *is sequence of real numbers,* $\{\alpha_n\}$ *is a sequence in (0, 1) and* γ *is a positive constant satisfying*

(1) $t_n > 0$ *and* $lim_{n\to\infty} t_n = \infty$*;*

(2) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ *and* $0 < \gamma < \frac{1-2k^2}{\|A\|^2}$.

(I) If $\Gamma = \{p \in C : Ap \in Q\} \neq \emptyset$*, then the sequence* $\{x_n\}$ *converges weakly to a split common fixed point* $x^* \in \Gamma$*.*

(II)In addition, if $\Gamma = \{p \in C : Ap \in Q\} \neq \emptyset$ *and there exists at least one* $S(t) \in \{S(t) : t \geq 0\}$ *is semi-compact, then* $\{x_n\}$ *converges strongly to a split common fixed point* $x^* \in \Gamma$*.*

Proof. Now we prove the conclusion (I).

We shall divide the proof into four steps.

Step 1. We first show that the limit $lim_{n\to\infty}\|x_n - p\|$ exists for any $p \in \Gamma$.

For any given $p \in \Gamma$, then $p \in C$ and $Ap \in Q$. It follows from (3.1) and Lemma 2.4 that

$$\begin{aligned}
\|z_n - p\|^2 &= \|(x_n - p) + \gamma J_1^{-1} A^* J_2(T(t_n) - I)Ax_n\|^2 \\
&\leq \|\gamma J_1^{-1} A^* J_2(T(t_n) - I)Ax_n\|^2 + 2\gamma\langle x_n - p, A^* J_2(T(t_n) - I)Ax_n\rangle \\
&\quad + 2k^2\|x_n - p\|^2 \\
&\leq \gamma^2\|A\|^2\|(T(t_n) - I)Ax_n\|^2 + 2\gamma\langle Ax_n - Ap, J_2(T(t_n) - I)Ax_n\rangle \\
&\quad + 2k^2\|x_n - p\|^2 \\
&= \gamma^2\|A\|^2\|(T(t_n) - I)Ax_n\|^2 + 2k^2\|x_n - p\|^2 \\
&\quad + 2\gamma\langle Ax_n - T(t_n)Ax_n + T(t_n)Ax_n - T(t_n)Ap, J_2(T(t_n) - I)Ax_n\rangle \\
&= \gamma^2\|A\|^2\|(T(t_n) - I)Ax_n\|^2 - 2\gamma\|Ax_n - T(t_n)Ax_n\|^2 \\
&\quad + 2\gamma\langle T(t_n)Ax_n - T(t_n)Ap, J_2(T(t_n) - I)Ax_n\rangle + 2k^2\|x_n - p\|^2 \\
&\leq \gamma^2\|A\|^2\|(T(t_n) - I)Ax_n\|^2 - 2\gamma\|(T(t_n) - I)Ax_n\|^2 \\
&\quad + \gamma[\|T(t_n)Ax_n - T(t_n)Ap\|^2 + \|(T(t_n) - I)Ax_n\|^2] + 2k^2\|x_n - p\|^2 \\
&\leq \gamma(\gamma\|A\|^2 - 1)\|(T(t_n) - I)Ax_n\|^2 + \gamma\|A\|^2(t_n)\|x_n - p\|^2 + 2k^2\|x_n - p\||^2 \\
&= (\gamma\|A\|^2 + 2k^2)\|x_n - p\|^2 - \gamma(1 - \gamma\|A\|^2)\|(T(t_n) - I)Ax_n\|^2. \quad (3.2)
\end{aligned}$$

It follows from (3.1), (3.2) and Lemma 2.1 that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|z_n - p + \alpha_n(S(t_n)z_n - z_n)\|^2 \\
&\leq (1-\alpha_n)\|z_n - p\|^2 + \alpha_n\|S(t_n)z_n - p\|^2 - \alpha_n(1-\alpha_n)g(\|z_n - S(t_n)z_n\|) \\
&\leq (1-\alpha_n)\|z_n - p\|^2 + \alpha_n\|z_n - p\|^2 - \alpha_n(1-\alpha_n)g(\|z_n - S(t_n)z_n\|) \\
&= \|z_n - p\|^2 - \alpha_n(1-\alpha_n)g(\|z_n - S(^n t_n)z_n\|) \\
&\leq (\gamma\|A\|^2 + 2k^2)\|x_n - p\|^2 - \gamma(1-\gamma\|A\|^2)\|(T(t_n) - I)Ax_n\|^2 \\
&\quad - \alpha_n(1-\alpha_n)g(\|z_n - S(t_n)z_n\|.
\end{aligned} \tag{3.3}
$$

Since $0 < k < \frac{1}{\sqrt{2}}$, $0 < \gamma < \frac{1-2k^2}{\|A\|^2}$, so, $0 < \gamma\|A\|^2 + 2k^2 < 1$. This implies that the sequence $\{||x_n - p||\}$ is decreasing, therefore the limit $lim_{n\to\infty}\|x_n - p\|$ exists. This also means that $\{x_n\}$ is bounded. Further, it follows from (3.2) that $\{z_n\}$ is bounded, too.

Step 2. We prove that $lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $lim_{n\to\infty}\|z_{n+1} - z_n\| = 0$.

It follows from (3.3) that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (\gamma\|A\|^2 M + 2k^2)\|x_n - p\|^2 \\
&\quad - \gamma(1-\gamma\|A\|^2)\|(T(t_n) - I)Ax_n\|^2 - \alpha_n(1-\alpha_n)g(\|z_n - S(t_n)z_n\|).
\end{aligned} \tag{3.4}
$$

From (3.4), we have

$$
\begin{aligned}
&\gamma(1-\gamma\|A\|^2)\|(T(t_n) - I)Ax_n\|^2 + \alpha_n(1-\alpha_n)g(\|z_n - S(t_n)z_n\|) \\
&\quad \leq (\gamma\|A\|^2 + 2k^2)\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.
\end{aligned} \tag{3.5}
$$

This implies that

$$lim_{n\to\infty}\|(T(t_n) - I)Ax_n\| = 0, \tag{3.6}$$

and

$$lim_{n\to\infty}g(\|z_n - S(t_n)z_n\|) = 0. \tag{3.7}$$

By virtue of Lemma 2.1 and the property of g, we may get

$$lim_{n\to\infty}\|z_n - S(t_n)z_n\| = 0. \tag{3.8}$$

It follows from (3.1) that

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|(1-\alpha_n)z_n+\alpha_n S(t_n)z_n-x_n\|\\ &= \|(1-\alpha_n)[x_n+\gamma J_1^{-1}A^*J_2(T(t_n)-I)Ax_n]+\alpha_n S(t_n)z_n-x_n\|\\ &= \|(1-\alpha_n)\gamma J_1^{-1}A^*J_2(T(t_n)-I)Ax_n+\alpha_n(S(t_n)z_n-x_n)\|\\ &= \|(1-\alpha_n)\gamma J_1^{-1}A^*J_2(T(t_n)-I)Ax_n+\alpha_n(S(t_n)z_n-z_n)+\alpha_n(z_n-x_n)\|\\ &= \|(1-\alpha_n)\gamma J_1^{-1}A^*J_2(T(t_n)-I)Ax_n+\alpha_n(S(t_n)z_n-z_n)\\ &\quad +\alpha_n\gamma J_1^{-1}A^*J_2(T(t_n)-I)Ax_n\|\\ &= \|\gamma J_1^{-1}A^*J_2(T(t_n)-I)Ax_n+\alpha_n(S(t_n)z_n-z_n)\|\\ &\le \|\gamma J_1^{-1}A^*J_2(T(t_n)-I)Ax_n\|+\alpha_n\|S(t_n)z_n-z_n\|.\end{aligned} \tag{3.9}$$

It follows from (3.6), (3.8) and (3.9) that

$$lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.10}$$

In addition, since

$$\begin{aligned}\|(T(t_n)-I)Ax_{n+1}\| &= \|(T(t_n)-I)Ax_{n+1}-(T(t_n)-I)Ax_n+(T(t_n)-I)Ax_n\|\\ &\le 2\|A\|\|x_{n+1}-x_n\|+\|(T(t_n)-I)Ax_n\|,\end{aligned}$$

it follows from (3.6) and (3.10) that

$$lim_{n\to\infty}\|(T(t_n)-I)Ax_{n+1}\|=0. \tag{3.11}$$

Similarly,

$$\begin{aligned}&||z_{n+1}-z_n||\\ &= ||x_{n+1}+\gamma J_1^{-1}A^*J_2(T(t_n)-I)Ax_{n+1}-x_n-\gamma J_1^{-1}A^*J_2(T(t_n)-I)Ax_n||\\ &\le \|(x_{n+1}-x_n)+\gamma J_1^{-1}A^*J_2(T(t_n)-I)Ax_{n+1}-\gamma J_1^{-1}A^*J_2(T(t_n)-I)Ax_n||\\ &\le \|x_{n+1}-x_n\|+\|\gamma J_1^{-1}A^*J_2(T(t_n)-I)Ax_{n+1}\|+\|\gamma J_1^{-1}A^*J_2(T(t_n)-I)Ax_n||\end{aligned} \tag{3.12}$$

In view of (3.6), (3.10) (3.11) and (3.12), we have

$$lim_{n\to\infty}\|z_{n+1}-z_n\|=0. \tag{3.13}$$

In addition,

$$\|x_n-z_n\|=\|J_1(x_n-z_n)\|=\|\gamma A^*J_2(T(t_n)-I)Ax_n\|\le\gamma\|A\|\|(T(t_n)-I)Ax_n\|, \tag{3.14}$$

from (3.6), we have

$$lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.15}$$

Step 3. We prove that $lim_{n\to\infty}\|z_n-S(t)z_n\|=0$ and $lim_{n\to\infty}\|(T(t)-I)Az_n\|=0$.

Since $\{S(t) : t \geq 0\}$ and $\{T(t) : t \geq 0\}$ are uniformly asymptotically regular, and $lim_{n\to\infty} t_n = \infty$, then for all $t \geq 0$,

$$lim_{n\to\infty} \|S(t)S(t_n)z_n - S(t_n)z_n\| \leq \limsup_{n\to\infty x\in C} \|S(t)S(t_n)x - S(t_n)x\| = 0, \tag{3.16}$$

and

$$\begin{aligned} &lim_{n\to\infty} \|T(t)(T(t_n) - I)Az_n - (T(t) - I)Az_n\| \\ &\quad \leq \limsup_{n\to\infty x\in C} \|T(t)(T(t_n) - I)Ax - (T(t) - I)Ax\| = 0. \end{aligned} \tag{3.17}$$

Since $\{S(t)x\}$ is continuous on t for all $x \in E_1$, and

$$\|z_n - S(t)z_n\| \leq \|z_n - S(t_n)z_n\| + \|S(t_n)z_n - S(t)S(t_n)z_n\| + \|S(t)S(t_n)z_n - S(t)z_n\|, \tag{3.18}$$

itt follows from (3.6) and (3.16) that

$$lim_{n\to\infty} \|z_n - S(t)z_n\| = 0. \tag{3.19}$$

Similarly,

$$lim_{n\to\infty} \|(T(t) - I)Ax_n\| = 0. \tag{3.20}$$

Step 4. We prove that $\{x_n\}$ converges weakly to $x^* \in \Gamma$.

By the reflexivity of Banach space E_1 and boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to x^*. By using (3.15) this implies that $\{z_{n_i}\}$ of $\{z_n\}$ converges weakly to x^*, too. Since $S(t)$ is asymptotically nonespansive for all $t \geq 0$, it is demiclosed at zero, we know from Lemma 2.2 that $x^* \in F(S(t))$.

On the other hand, since A is a bounded linear operator, we know that $\{Ax_{n_i}\}$ converges weakly to Ax^*. it follows from (3.20) that $\lim_{n_i\to\infty} \|(T(t) - I)Ax_{n_i}\| = 0$. Since $T(t)$ is demiclosed at zero for all $t \geq 0$, we have $Ax^* \in F(T(t))$. This means that $x^* \in \Gamma$.

We now prove that $\{x_n\}$ converges weakly to $x^* \in \Gamma$.

In fact, if there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $y^* \in E_1$, we also know that $y^* \in F(T(t))$. By the assumption that E_1 satisfies Opial's condition, we have

$$\begin{aligned} \liminf_{n_i\to\infty} \|x_{n_i} - x^*\| &< \liminf_{n_i\to\infty} \|x_{n_i} - y^*\| \\ &= \liminf_{n\to\infty} \|x_n - y^*\| \\ &= \liminf_{n_j\to\infty} \|x_{n_j} - y^*\| \\ &< \liminf_{n_i\to\infty} \|x_{n_i} - x^*\| \\ &= \liminf_{n\to\infty} \|x_n - x^*\| = \liminf_{n_i\to\infty} \|x_{n_i} - x^*\|. \end{aligned} \tag{3.21}$$

This is a contradiction. Therefore $\{x_n\}$ converges weakly to $x^* \in \Gamma$. The proof of conclusion(I) is completed.

Next, we prove the conclusion(II).

Since there exists at least one $S(t) \in \{S(t) : t \geq 0\}$ is semi-compact and $lim_{n\to\infty}||z_n - S(t)z_n|| = 0$ for all $t \geq 0$, there exist subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that $\{z_{n_j}\}$ converges strongly to $\mu^* \in E_1$. By using (3.15) again, we know that the subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges strongly to μ^*. Due to $\{x_n\}$ converges weakly to x^*, we obtain $\mu^* = x^*$. By $lim_{n\to\infty}\|x_n - x^*\|$ exists and $lim_{n\to\infty}\|x_{n_j} - x^*\| = 0$, we know that $\{x_n\}$ converges strongly to $x^* \in \Gamma$. This completes the proof of the conclusion(II).

From Theorem 3.1, we can directly obtain the folloeing corollary.

Corollary 3.2 *Let E_1 be a real uniformly convex and 2-uniformly smooth Banach space satisfying Opial's condition and with the best smoothness constant k satisfying $0 < k < \frac{1}{\sqrt{2}}$, E_2 be a real Banach space, and $A : E_1 \to E_2$ be a bounded linear operator and A^* is the adjoint of A, $S : E_1 \to E_1$ and $T : E_2 \to E_2$ be two nonexpansive mappings with $C := F(S) \neq \emptyset$ and $Q := F(T) \neq \emptyset$, respectively. Let $\{x_n\}$ be a sequence generated by: $x_1 \in E_1$*

$$\begin{cases} z_n = x_n + \gamma J_1^{-1} A^* J_2(T - I)Ax_n, & \forall n \geq 1, \\ x_{n+1} = (1 - \alpha_n)z_n + \alpha_n S z_n, \end{cases} \tag{3.22}$$

where $\{\alpha_n\}$ is a sequence in (0, 1) satisfying $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, γ is a positive constant satisfying $0 < \gamma < \frac{1-2k^2}{\|A\|^2}$.

(I) If $\Gamma = \{p \in C : Ap \in Q\} \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to a split common fixed point $x^* \in \Gamma$.

(II)In addition, if $\Gamma = \{p \in C : Ap \in Q\} \neq \emptyset$ and S is semi-compact, then $\{x_n\}$ converges strongly to a split common fixed point $x^* \in \Gamma$.

References

[1] Rabian Wangkeeree and Pakkapon Preechasilp, The General Iterative Methods for Asymptotically Nonexpansive Semigroups in Banach Spaces, *Abstract and Applied Analysis*, **2012** (2012), Article ID 695183, 1-20. http://dx.doi.org/10.1155/2012/695183

[2] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image construction, *Inverse Probl.*, **20** (2003), 103–120. http://dx.doi.org/10.1088/0266-5611/20/1/006

[3] P.L. Combettes, The convex feasibility problem in image recovery, Chapter in: *Advances in Imaging and Electron Physics*, vol. 95, Academic

Press, New York, 1996, 155–270.
http://dx.doi.org/10.1016/s1076-5670(08)70157-5

[4] S. Kitahara, W. Takahashi, Image recovery by convex combinations of sunny nonexpansive retractions, *Topol. Methods Nonlinear Anal.*, **2** (1993), 333–342.

[5] S. Plubtieng, K. Ungchittrakool, Hybrid iterative methods for convex feasibility problems and fixed point problems of relatively nonexpansive mappings in Banach spaces, *Fixed Point Theory and Applications*, **2008** (2008), Article ID 583082, 1-19. http://dx.doi.org/10.1155/2008/583082

[6] T. Suzuki, On strong convergence to common fixed points of nonexpansive semigroups in Hilbert spaces, *Proc. Am. Math. Soc.*, **131** (2003), 2133–2136. http://dx.doi.org/10.1090/s0002-9939-02-06844-2

[7] S.S. Chang, H.W. Joseph Lee, C.K. Chan, Convergence theorem of common fixed points for Lipschitzian pseudocontraction semigroups in Banach spaces, *Appl. Math. Comput.*, **212** (2009), 60–65. http://dx.doi.org/10.1016/j.amc.2009.01.086

[8] Y. Yao, N. Shahzad, New methods with perturbations for non-expansive mappings in Hilbert spaces, *Fixed Point Theory and Applications*, **2011** (2011), 79. http://dx.doi.org/10.1186/1687-1812-2011-79

[9] Y.J. Cho, L. Ciric, S.H. Wang, Convergence theorems for nonexpansive semigroups in CAT(0) spaces, *Nonlinear Anal.*, **74** (2011), 6050-6059. http://dx.doi.org/10.1016/j.na.2011.05.082

[10] A. Moudafi, The split common fixed-point problem for demicontractive mappings, *Inverse Problems*, **26** (2010), 055007.
http://dx.doi.org/10.1088/0266-5611/26/5/055007

[11] S.S. Chang, L. Wang, Y.K. Tang and L. Yang, The split common fixed foint problem for total asymptotically strictly pseudocontractive mappings, *Journal of Applied Mathematics*, **2012** (2012), Article ID 385638, 1-13. http://dx.doi.org/10.1155/2012/385638

[12] S.S. Chang, Y.J. Cho, J.K. Kim, W.B. Zhang and L. Yang, Multiple-set split feasibility problems for asymptotically strict pseudocontractions, *Abstract and Applied Analysis*, **2012** (2012), Article ID 491760, 1-12. http://dx.doi.org/10.1155/2012/491760

[13] J. Quan, S.S. Chang, X. Zhang, Multiple-set split feasibility problems for k-strictly pseudononspreading mapping in Hilbert spaces, *Abstract and Applied Analysis*, **2013** (2013), Article ID 342545, 1-5. http://dx.doi.org/10.1155/2013/342545

[14] A. Moudafi, A note on the split common fixed-point problem for quasi-nonexpansive operators, *Nonlinear Anal.*, **74** (2011), 4083-4087. http://dx.doi.org/10.1016/j.na.2011.03.041

[15] Prasit Cholamjiak and Yekini Shehu, Iterative approximation for split common fixed point problem involving an asymptotically nonexpansive semigroup and a total asymptotically strict pseudocontraction, *Fixed Point Theory and Applications*, **2014** (2014), 131. http://dx.doi.org/10.1186/1687-1812-2014-131

[16] W. Takahashi, Iterative methods for split feasibility problems and split common null point problems in Banach spaces, *The 9th International Conference on Non- linear Analysis and Convex Analysis*, Chiang Rai, Thailand, Jan. 21-25, 2015.

[17] Jinfang Tang, Shih-sen Chang, Lin Wang and Xiongrui Wang, On the split common fixed point problem for strict pseudocontractive and asymptotically nonexpansive mappings in Banach spaces, *Journal of Inequalities and Applications*, **2015** (2015), 305. http://dx.doi.org/10.1186/s13660-015-0832-z

[18] H.K. Xu, Inequalities in Banach spaces with applications, *Nonlinear Analysis: Theory, Methods and Applications*, **16** (1991), 1127-1138. http://dx.doi.org/10.1016/0362-546x(91)90200-k

[19] J. Gornicki, Weak convergence theorems for asymptotically nonexpansive mappings in uniformly convex Banach spaces, *Commentationes Mathematicae Universitatis Carolinae*, **30** (1989), no. 2, 249-252.

[20] K.K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, *J. Math. Anal. Appl.*, **178** (1993), 301–308. http://dx.doi.org/10.1006/jmaa.1993.1309

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