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Several Probability Inequalities for Sublinear Expectations[1]

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Abstract

We derive in this paper that there are Boole’s inequality, Hölder's inequality, Khintchine inequality for sublinear expectations. These inequalities are well known under linear situation in probability. We prove also that Khintchine inequality for sublinear without of the identity distribution.

Keywords: Probability, Inequality, Sublinear expectation, Hölder, Khintchine

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1 Introduction

We all know that one of the most important properties of linear expectation is additivity. However, in many areas of applications, such additivity assumption is not reasonable, because if we use additive probabilities or additive expectations, we cannot model many uncertain phenomena well. Moreover, motivated by some problems in statistics, mathematical economics, quantum mechanics and finance, a number of scholars used non-additive probabilities or nonlinear expectations (for example Choquet expectation, g-expectation, sublinear expectation) to describe and interpret the phenomena (see for example [1, 2, 3, 7]).

For sublinear expectation, the additivity of linear expectation is promoted to subadditivity. Many scholars have already studied many kinds of properties of linear expectation under sublinear expectation (see for example [5, 6]).

In almost all branches of mathematics sciences, the inequality often plays a vital role. In many cases, it is even more important than the equation. The situation is the same as in probability and statistics. We also want to know whether the well known inequalities established or not under sublinear situation. This paper proved several probability inequalities which have not been proved under sublinear expectation.

The organization of this paper is as follows. Section 2 states the basic notions and the definition of sublinear expectation. Section 3 gives the proof of several probability inequalities under sublinear expectation.

2 Basic notions and sublinear expectation

We first give some basic notions which will be used later. ϕ is denoted as the empty set, $\mathbf{N}$ as natural numbers and $\mathbf{R}$ as real numbers. For a measurable space $(\Omega, \mathscr{F})$, $\mathcal{X}$ is used to denote the set of all bounded $\mathscr{F}-$ measurable functions. If we endow $\mathcal{X}$ with the supremum norm, then it is a Banach space. For a random variable $\xi \in \mathcal{X}$, we use $\sigma(\xi)$ to denote the $\sigma-$ field generated by ξ.

Definition 2.1. A functional $\mathscr{E} : \mathcal{X} \to \mathbf{R}$ is said to be a sublinear expectation if it satisfies the following properties:

(1) Monotonicity: If $\xi, \eta \in \mathcal{X}, \xi \geq \eta, \mathscr{E}(\xi) \geq \mathscr{E}(\eta)$.

(2) Constant preserving: For any $\lambda \in \mathbf{R}, \mathscr{E}(\lambda) = \lambda$.

(3) Subadditivity: For any $\xi, \eta \in \mathcal{X}, \mathscr{E}(\xi + \eta) \le \mathscr{E}(\xi) + \mathscr{E}(\eta)$.

(4) Positive homogeneity: For any $\lambda \ge 0$ and $\xi \in \mathcal{X}, \mathscr{E}(\lambda\xi) = \lambda \mathscr{E}(\xi)$.

Lemma 2.2. For a sublinear expectation $\mathscr{E}$, there exists a set $\mathscr{M}$ such that

$$\mathscr{E}(\xi) = \max_{\mu \in \mathscr{M}} E_\mu[\xi], \forall \xi \in \mathcal{X},$$

where μ is a finitely additive set function.

The proof of Lemma 2.2 could be found in [4, 6].

Let us denote

$$C(A) := \mathscr{E}(I_A) = \max_{\mu \in \mathscr{M}} \mu(A) \text{ and } c(A) := -\mathscr{E}(-I_A) = \min_{\mu \in \mathscr{M}} \mu(A)$$

Definition 2.3. If $C(A) = 0$, then the set A is named a $C-$polar set. We say $\xi_n \to \xi, C - q.s.$ if $\xi_n \to \xi$ pointwisely outside a $C-$polar set.

Definition 2.4. We say $\xi_n \to \xi$ in capacity if for any $\varepsilon, \delta > 0$, there exist an $N \in \mathbf{N}$ such that $C(|\xi_n - \xi| > \varepsilon) < \delta$ for any $n \ge N$.

Definition 2.5. A sublinear expectation $\mathscr{E}$ is said to be regular if for any sequence $\{\xi_n\}_{n \ge 1} \subset \mathcal{X}$ such that $\xi_n \downarrow 0$, we have $\mathscr{E}(\xi_n) \downarrow 0$.

Lemma.2.6. If the sublinear expectation $\mathscr{E}$ is regular, for any linear expectation E_μ dominated by $\mathscr{E}$, μ is a probability measure.

Proof. For any $A_n \downarrow \phi$, we have $\mathscr{E}(I_{A_n}) \downarrow 0$. If a linear expectation E_μ dominated by $\mathscr{E}$, then $\mu(A_n) \downarrow 0$, i.e. $\mu(\phi) = 0$. On the other hand, if $A_n \downarrow \Omega$, for a linear expectation E_μ dominated by $\mathscr{E}$, we can obtain that $\max_{\mu \in \mathscr{M}} \mu(A_n) = \mathscr{E}(I_{A_n}) = 1$, in the meantime $\min_{\mu \in \mathscr{M}} \mu(A_n) = -\mathscr{E}(-I_{A_n}) = 1$, i.e. $\mu(\Omega) = 0$.

3 Probability inequalities for sublinear expectations

There are many probability inequalities under linear expectations. This paper aims to introduce and prove several probability inequalities under sublinear expectations.

Theorem 3.1. (Boole's inequality) $\forall A, B \in \mathscr{F}$, we have
$C(AB) \geq 1 - C(A^c) - C(B^c)$

where A^c means the complementary set of A.

Proof. For any E_μ dominated by $\mathscr{E}$, we have

$$\begin{aligned} C(AB) + C(A^c) + C(B^c) &= \mathscr{E}(I_{AB}) + \mathscr{E}(I_{A^c}) + \mathscr{E}(I_{B^c}) \\ &\geq E_\mu[I_{AB}] + E_\mu[I_{A^c}] + E_\mu[I_{B^c}] \\ &= E_\mu[I_{AB} + I_{A^c} + I_{B^c}] \\ &= E_\mu[1] \\ &= 1 \end{aligned}$$

□

Theorem 3.2. (Hölder's inequality) For any $\xi, \eta \in \mathcal{X}, p > 1, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\mathscr{E}|\xi\eta| \leq \left(\mathscr{E}|\xi|^p\right)^{\frac{1}{p}} \left(\mathscr{E}|\eta|^q\right)^{\frac{1}{q}}$$

Proof. Without loss of generality, we suppose $0 < \mathscr{E}|\xi|^p, \mathscr{E}|\eta|^q < \infty$, or the inequality can be proved obviously. Because the function $-\log(x)$ is convex on $(0, \infty)$, then for any $a, b > 0$, we have

$$-\log\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \leq -\frac{1}{p}\log a^p - \frac{1}{q}\log b^q = -\log ab$$

i.e. $ab \leq \frac{a^p}{p} + \frac{b^q}{q}, 0 \leq a, b \leq \infty$.

$$\mathscr{E}\left(\frac{|\xi|}{\left(\mathscr{E}|\xi|^p\right)^{\frac{1}{p}}}\bullet\frac{|\eta|}{\left(\mathscr{E}|\eta|^q\right)^{\frac{1}{q}}}\right)\leq\mathscr{E}\left(\frac{1}{p}\left[\frac{|\xi|}{\left(\mathscr{E}|\xi|^p\right)^{\frac{1}{p}}}\right]^p+\frac{1}{q}\left[\frac{|\eta|}{\left(\mathscr{E}|\eta|^q\right)^{\frac{1}{q}}}\right]^q\right)$$

$$\leq\frac{1}{p}\mathscr{E}\left(\left[\frac{|\xi|}{\left(\mathscr{E}|\xi|^p\right)^{\frac{1}{p}}}\right]^p\right)+\frac{1}{q}\mathscr{E}\left(\left[\frac{|\eta|}{\left(\mathscr{E}|\eta|^q\right)^{\frac{1}{q}}}\right]^q\right)$$

$$=\frac{1}{p}+\frac{1}{q}$$

$$=1$$ □

Theorem 3.3. (Cauchy-Schwarz inequality) For any $\xi,\eta\in\boldsymbol{\mathcal{X}}$,

$$\mathscr{E}|\xi\eta|\leq\left(\mathscr{E}|\xi|^2\right)^{\frac{1}{2}}\left(\mathscr{E}|\eta|^2\right)^{\frac{1}{2}}$$

Proof. Let the $p=q=\frac{1}{2}$ in the Hölder's inequality and we can obtain the result. □

Theorem 3.4. (The promotion of Hölder's inequality) For any $\xi,\eta\in\boldsymbol{\mathcal{X}}$, $0<p<1$ and $q=\frac{-p}{1-p}$,

$$\mathscr{E}|\xi\eta|\geq\left(\mathscr{E}|\xi|^p\right)^{\frac{1}{p}}\left(\mathscr{E}|\eta|^q\right)^{\frac{1}{q}}$$

Proof. Let $\xi'=|\xi\eta|^p,\eta'=|\eta|^{-p}$

$$\mathscr{E}|\xi|^p=\mathscr{E}(\xi'\eta')\leq\left(\mathscr{E}(\xi')^{\frac{1}{p}}\right)^p\left(\mathscr{E}(\eta')^{\frac{1}{1-p}}\right)^{1-p}$$

$$=\left(\mathscr{E}|\xi\eta|\right)^p\left(\mathscr{E}(|\eta|)^{\frac{-p}{1-p}}\right)^{1-p}$$

Then we can obtain :

$$\mathscr{E}|\xi\eta| \geq \left(\mathscr{E}|\xi|^p\right)^{\frac{1}{p}}\left(\mathscr{E}|\eta|^q\right)^{\frac{1}{q}}$$ □

Theorem 3.5. (The Minkowski accompanied inequality) For any $r \geq 1$, $\xi_j \in \mathcal{X}$

$$\mathscr{E}\left(\sum_{j=1}^{n}|\xi_j|\right)^r \geq \sum_{j=1}^{n}\mathscr{E}|\xi_j|^r;$$

Proof. Any $r \geq 1$, for each $j = 1,\cdots,n$

$$\frac{|\xi_j|}{\left(\sum_{k=1}^{n}|\xi_k|^r\right)^{\frac{1}{r}}} \leq 1$$

Then we can get:

$$\frac{\sum_{j=1}^{n}|\xi_j|}{\left(\sum_{k=1}^{n}|\xi_k|^r\right)^{\frac{1}{r}}} \geq \frac{\sum_{j=1}^{n}|\xi_j|^r}{\sum_{k=1}^{n}|\xi_k|^r} = 1$$

which means:

$$\left(\sum_{j=1}^{n}|\xi_j|\right)^r \geq \sum_{k=1}^{n}|\xi_k|^r$$

Take sublinear expectation for both side of the inequality:

$$\mathscr{E}\left(\sum_{j=1}^{n}|\xi_j|\right)^r \geq \mathscr{E}\left(\sum_{j=1}^{n}|\xi_j|^r\right)$$

$$\geq \sum_{j=1}^{n}\mathscr{E}|\xi_j|^r$$ □

Theorem 3.6. (**Khintchine inequality**) Suppose $\xi_1,\cdots,\xi_n$ are independent and $\mathscr{E}$ is a regular sublinear expectation, $C(\xi_1=1)=C(\xi_1=-1)=\frac{1}{2}$, $b_1,\cdots,b_n$ are arbitrary real numbers, then for any $r>0$, there exist two constants $0<c_r\le c_r'<\infty$ such that

$$c_r\left(\sum_{j=1}^{n}b_j^2\right)^{\frac{r}{2}}\le \mathscr{E}\left|\sum_{j=1}^{n}b_j\xi_j\right|^r\le c_r'\left(\sum_{j=1}^{n}b_j^2\right)^{\frac{r}{2}}$$

Proof. Because $\mathscr{E}$ is a regular sublinear expectation, then from Lemma 2.6, we can know that for any linear expectation E_μ dominated by $\mathscr{E}$, μ is a probability measure. If we first suppose that $r=2k$, where k is integer. Then if we note $T_n=\sum_{j=1}^{n}b_j\xi_j$, from $\mathscr{E}(\xi)=\max_{\mu\in\mathscr{M}}E_\mu[\xi]$, we have:

$$\begin{aligned}\mathscr{E}(T_n^{2k})&=\max_{\mu\in\mathscr{M}}E_\mu(T_n^{2k})\\&=E\left(\sum A_{l_1,\cdots,l_j}b_{i_1}^{l_1}\cdots b_{i_j}^{l_j}\xi_{i_1}^{l_1}\cdots\xi_{i_j}^{l_j}\right)\\&=\mathscr{E}\left(\sum A_{l_1,\cdots,l_j}b_{i_1}^{l_1}\cdots b_{i_j}^{l_j}\xi_{i_1}^{l_1}\cdots\xi_{i_j}^{l_j}\right)\\&\le\sum A_{l_1,\cdots,l_j}b_{i_1}^{l_1}\cdots b_{i_j}^{l_j}\mathscr{E}\xi_{i_1}^{l_1}\cdots\xi_{i_j}^{l_j}\end{aligned},$$

where $l_1,\cdots,l_n$ are positive integers and satisfied $\sum_{u=1}^{j}l_u=2k$; $A_{l_1,\cdots,l_j}=\frac{(l_1+\cdots+l_j)!}{l_1!\cdots l_j!}$, $i_1,\cdots,i_j$ are integers in [1,n].

Because μ is a probability measure, $E_\mu(\xi_1)=0$, $\mathscr{E}(\xi_1)=\max_{\mu\in\mathscr{M}}E_\mu[\xi_1]=0$. When $l_1,\cdots,l_n$ are all evens, $\mathscr{E}\xi_{i_1}^{l_1}\cdots\xi_{i_j}^{l_j}=1$, or $\mathscr{E}\xi_{i_1}^{l_1}\cdots\xi_{i_j}^{l_j}=0$, so we have

$$\mathscr{E}T_n^{2k}\le\sum A_{2p_1,\cdots,2p_j}b_{i_1}^{2p_1}\cdots b_{i_j}^{2p_j},$$

where $p_1,\cdots,p_n$ are positive integers and satisfied $\sum_{u=1}^{j} p_u = k$. So we have

$$\mathscr{E} T_n^{2k} \le \sum \frac{A_{2p_1,\cdots,2p_j}}{A_{p_1,\cdots,p_j}} A_{p_1,\cdots,p_j} b_{i_1}^{2p_1} \cdots b_{i_j}^{2p_j} \le c_{2k} s_n^{2k},$$

where $s_n^2 = \sum_{j=1}^{n} b_j^2$,

$$c_{2k} = \sup \frac{A_{2p_1,\cdots,2p_j}}{A_{p_1,\cdots,p_j}} = \sup \frac{(2k)!}{(2p_1)!\cdots(2p_j)!} \frac{p_1!\cdots p_j!}{k!} \le k^k .$$

Then if $r = 2k$, for one $c'_{2k} \le k^k$, we can obtain the upper bound. For $r \le 2k$, we have

$$\left(\mathscr{E}|T_n|^r\right)^{\frac{1}{r}} \le \left(\mathscr{E}\, T_n^{2k}\right)^{\frac{1}{2k}} \le c_{2k}^{\frac{1}{2k}} s_n .$$

So for one $c'_{2k} \le k^{\frac{r}{2}}$, where k is the smallest integer which less than $\frac{r}{2}$. The upper bound is obtained.

For the lower bound, when $r \ge 2$, because the function $f(r) = \log E_\mu |\xi|^r$ is convex on $[0,\infty]$, while $\mathscr{E}(\xi) = \max_{\mu\in\mathscr{M}} E_\mu[\xi]$, then from $\left(\mathscr{E}|T_n|^r\right)^{\frac{1}{r}} \ge \left(\mathscr{E}\, T_n^2\right)^{\frac{1}{2}}$ we can obtain the result.

When $0 < r < 2$, let $r_1, r_2 > 0$, such that $r_1 + r_2 = 1, rr_1 + 4r_2 = 2$, then from Hölder's inequality, we can see that

$$s_n^2 \le \mathscr{E} T_n^2 \le \left(\mathscr{E}|T_n|^r\right)^{r_1} \left(\mathscr{E} T_n^4\right)^{r_2} \le \left(\mathscr{E}|T_n|^r\right)^{r_1} \left(2^{\frac{1}{2}} s_n\right)^{4r_2}$$

Then we can obtain that

$$\left(\mathscr{E}|T_n|^r\right)^{r_1} \ge 4^{-r_2} s_n^{2-4r_2} = 4^{-r_2} s_n^{rr_1}$$

$$\mathscr{E}|T_n|^r \ge 4^{\frac{-r_2}{r_1}} s_n^r .$$

Which means for $0 < r < 2$ and $c_r \geq 4^{\frac{-r_2}{r_1}} = 2^{-(2-r)}$ or $r \geq 2$ and $c_r \geq 1$, the lower bound is satisfied. □

Remark 3.7. The Khintchine inequality to linear expectations [8] depends on i.i.d. condition, but Theorem 3.6 shows that for sublinear expectation this i.i.d. condition can be weaken to independent only. The proof to Theorem 3.6 is similar to that in [8], which means exactly that linear expectation and sublinear expectation are interlinked sometimes.

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