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On Dependent Elements and Free Actions in Inverse Semirings

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Abstract

The purpose of this paper is to introduce the notion of dependent elements and free actions in inverse semirings. We consider some mappings on semiprime inverse semirings and prove that they are free actions.

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Keywords: Inverse semiring, semiprime inverse semiring, left centralizer,derivation, generalized derivation, dependent element, free action

1 Introduction

By semiring we mean a nonempty set S with two binary operations '+' and '.' such that $(S,+)$ and $(S,.)$ are semigroups where $+$ is commutative with absorbing zero 0 (i.e; $a+0=0+a=a,\ a.0=0.a=0\ \forall a\in S$) and both left and right distributive laws holds in S. Introduced by Karvellas[7], a semiring S is called inverse semiring if for every $a\in S$ there exists a unique element $\acute{a}\in S$ such that $a+\acute{a}+a=a$ and $\acute{a}+a+\acute{a}=\acute{a}$, $\acute{a}$ is called pseudo inverse of a. Karvellas [7] proved that for all $a,b\in S$, $\acute{(a.b)}=\acute{a}.b=a.\acute{b}$ and $\acute{a}\acute{b}=ab$. In [2], Bandlet and Petrich also considered inverse semirings with some conditions namely (A-1)-(A-4). Throughout this paper, S we will denote inverse semiring which satisfies (A-2) condition of Bandlet and Petrich[2] i.e; for every $a\in S$, $a+\acute{a}$ is in center $Z(S)$ of S. This class of inverse semiring has been identified and studied as MA semirings in [3, 4]. It is significant to note that commutative inverse semiring and a distributive lattice are generalization of MA semirings. Also, if R is non-commutative ring and S is inverse semiring satisfying A_2 then $S_1 = \{(r,s) : r \in R, s \in S\}$ is a non-commutative inverse semiring where addition and multiplication are pointwise and $\acute{(r,s)} = (-r, \acute{s})$.An ideal J of S is inverse ideal if $j \in J$ implies $\acute{j} \in J$. For more examples, we refer readers to [3].

By [3], $xy + y\acute{x} = xy + \acute{y}x$ means commutator in inverse semiring and it will be denoted by $[x,y]$. We will make use of commutator identites $[xy,z] = x[y,z] + [x,z]y$ and $[x,yz] = [x,y]z + y[x,z]$(for proof see [3]). S is prime if $aSb = (0)$ implies $a = 0$ or $b = 0$ and semiprime if $aSa = (0)$ implies $a = 0$. A mapping $d : S \to S$ is called derivation if $d(x+y) = d(x) + d(y)$ and $d(xy) = d(x)y + xd(y), \forall x,y \in S$. If α and β are automorphisms of S then an additive mapping d is (α,β) derivation if $d(xy) = d(x)\alpha(y) + \beta(x)d(y), x,y \in S$ .Following [13], an additive mapping $T : S \to$ is called left(right) centralizer if $T(xy) = T(x)y \ (xT(y)), \forall x,y \in S$ and T is centralizer if it is both left and right centralizer.

The concept of free action was introduced by Murray and von Neumann [9] and von Neumann for commutative von Neumann algebras[10]. Recently, Vukman and Kosii-Ulble[12] further explored dependent elements of certain mappings on prime and semiprime rings.
Here in this paper, we define dependent element in inverse semirings as follows: An element $a \in S$ is dependent element of the mapping $F : S \to S$ if $F(x)a + \acute{a}x = 0, \forall x \in S$. Consider $S_2 = \{\begin{pmatrix} 0 & 0 \\ (r_1,s) & (r_2,s) \end{pmatrix}, r_1, r_2 \in R, s \in S\} \subseteq M_2(S_1)$ then S_2 is inverse semiring with respect to usual operation of addition and multiplication of matrices. Define $F : S_2 \to S_2$ by $F\begin{pmatrix} 0 & 0 \\ (r_1,a) & (r_2,a) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ (r_1,a) & 0 \end{pmatrix}, \forall (r,a) \in S_1$. If we fix $(t,b) \in S_1$ then $c = \begin{pmatrix} 0 & 0 \\ (t,b) & 0 \end{pmatrix}$is dependent element of F. The set of all dependent element of the mapping F will be represented by $D(F)$. If the only dependent element of a mapping $F : S \to S$ is zero then F is known as free action. Motivated by the work of Laraji and Thaheem[8] and then Chadhary and Samman[1], we prove that an element a is dependent element of left centralizer T on semiprime inverse semiring S iff it is in center of S and $T(a) + \acute{a} = 0$ holds. We also consider a few mappings related to centralizer and derivation and show them they are free actions.

We start with the following useful Lemma.

Lemma 1.1.{Lemma 1.1, [11]} Let S be inverse semiring then $a + b = 0$

implies that $a = \acute{b}$,$\forall a, b \in S$.

Theorem 2.1. Let S be a semiprime inverse semiring and T be a left centralizer on S then $a \in D(T)$ if and only if $a \in Z(S)$ and $T(a) + \acute{a} = 0$.

Proof. Let $a \in D(T)$ then

$$T(x)a + \acute{a}x = 0 \tag{1}$$

Replacing x by xy and post multiplying (1) by z and using Lemma 1.1, we get $T(x)yaz = axyz$. Replacing x by xyz in (1), we have $T(x)yza + \acute{a}xyz = 0$. Thus, we get $[a, z]T(x) = 0$. By (1) and Lemma 1.1., we obtain $[a, z]a = 0$. Replacing z with zwz, we have $[a, z]wa = 0$. Put $w = wz$, we get $[a, z]wza = 0$. Also, we have $[a, z]waz = 0$. Adding pseudo inverse of $[a, z]wza = 0$ in last equation and then using semiprimeness, we have $[a, z] = 0$.By Lemma 1.1 $a \in Z(S)$. Thus $T(a)x = T(x)a, \forall x \in S$. From this and (1) we obtain $T(a) + \acute{a} = 0$. Conversely, let $a \in Z(S)$ and $T(a) + \acute{a} = 0$ then $T(x)a + \acute{a}x = (T(a) + \acute{a})x = 0, \forall x \in S$. Hence $a \in D(T)$.

Corollary 2.2. If T is left centralizer on semiprime inverse semiring S then $D(T)$ is subsemiring of $Z(S)$.

Proof: Let $a, b \in D(T)$, then by theorem 2.1 $a, b \in Z(S)$ and $T(a) + \acute{a} = 0, T(b) + \acute{b} = 0$. Thus $a + b \in Z(S)$ and $T(a + b) + (a + b)' = 0$ which implies that $a + b \in D(T)$. Also if $a \in D(T)$ then $T(x)a + a\acute{x} = 0, \forall x \in S$ which shows $\acute{a} \in D(T)$. Hence $D(T)$ is subsemiring of $Z(S)$.

Corollary 2.3. If T is left centralizer on semiprime inverse semiring S then $J = Ann(D(T))$ is inverse ideal of S such that $T(J) \subseteq J$.

Proof. As $D(T) \in Z(S)$ so an easy calculation shows that J is ideal. Also, if $i \in J$ then $ai = 0, \forall a \in D(T)$ which implies that $\acute{a}i = 0, \forall a \in D(T)$. Thus $Ann(D(T))$ is inverse ideal of S. As $D(T) \subseteq Z(S)$ so $T(i)a = T(ia) = T(ai) = 0, \forall a \in D(T)$. This shows that $T(J) \subseteq J$.

Theorem 2.4. Every left centralizer T on prime inverse semiring is a free action provided T is not an identity map.

Proof. Let $a \in D(T)$ then $T(x)a + \acute{a}x = 0, \forall x \in S$. Replacing x with xy and using Theorem 2.1., we have $(T(x) + \acute{x})ya = 0,\ \forall x, y \in S$. This implies that either $a = 0$ or $T(x) + \acute{x} = 0$. If $T(x) + \acute{x} = 0$ then by Lemma 1.1 , T becomes identity map.

Theorem 2.5. Let T be a left centralizer of a semiprime inverse semiring S. Then $\psi : S \to S$, defined by $\psi(x) = [T(x), x], x \in S$ is a free action.

Proof. Let $a \in D(\psi)$ then by definition

$$[T(x), x]a + \acute{a}x = 0, x \in S \tag{2}$$

By lemma 1.1. , we have $[T(x), x]a = ax, x \in S$. Linearizing (2), we get

$$[T(x), y]a + [T(y), x]a = 0 \tag{3}$$

Replacing y with ay , we get

$$a[T(x), y]a + [T(x), a]ya + T(a)[y, x]a + [T(a), x]ya = 0 \tag{4}$$

Using lemma 1.1 in (3) and using it in (4), we have $a[T(y), x]\acute{a} + [T(x), a]ya + T(a)[y, x]a + [T(a), x]ya = 0$. Replacing x and y with a, we get

$$a[T(a), a]\acute{a} + [T(a), a]a^2 + T(a)[a, a]a + [T(a), a]a^2 = 0 \tag{5}$$

Now, $[T(a), a]a^2 + T(a)[a, a]a = (T(a)a + \acute{a}T(a))a^2 + T(a)(a + \acute{a})a^2 = (T(a)a + T(a)\acute{a} + \acute{a}T(a) + T(a)a)a^2 = (T(a)a + \acute{a}T(a))a^2 = [T(a), a]a^2$. Thus from (5) we obtain, $a[T(a), a]\acute{a} + [T(a), a]a^2 + [T(a), a]a^2 = 0$. But $[T(x), x] = ax$ so we have, $\acute{a^3} + a^3 + a^3 = 0$ or $a^3 = 0$. Post-multiplying (2) by a^2, we get $axa^2 = 0$. This gives $a^2 = 0$. Again, from (2), we obtain $a = 0$. This proves that ψ is a free action.

Theorem 2.6. Let S be a semiprime inverse semiring and $d : S \to S$ a derivation then the mapping $\varphi : S \to S$, defined by $\varphi(x) = [d(x), x], x \in S$, is a free action.

Proof. Let $a \in D(\varphi)$ then by definition

$$[d(x), x]a + \acute{a}x = 0, \forall x \in S \tag{6}$$

By lemma 1.1. we get, $[d(x), x]a = ax, x \in S$. Linearizing (6),we have

$$[d(x), y]a + [d(y), x]a = 0, x, y \in S \tag{7}$$

Replacing y with xy in (7), we have $x[d(x), y]a + x[d(y), x]a + 2[d(x), x]ya + d(x)[y, x]a + [x, x]d(y)a = 0$ or $x[d(x), y]a + x[d(y), x]a + 2[d(x), x]ya + d(x)[y, x]a + x(x + \acute{x})d(y)a = 0$ or $x[d(x), y]a + x(d(y)x, \acute{x}d(y))a + 2[d(x), x]ya + d(x)[y, x]a + xd(y)(x + \acute{x})a = 0$ or $x[d(x), y]a + x(d(y)x + \acute{x}d(y) + d(y)x + d(y)\acute{x})a + 2[d(x), x]ya + d(x)[y, x]a = 0$ or $x\{[d(x), y]a + [d(y), x]a\} + 2[d(x), x]ya +$

$d(x)[y,x]a = 0$. From (7), we have $2[d(x),x]ya + d(x)[y,x]a = 0$. Replacing y with ya in last equation and then using it again, we get

$$d(x)y[a,x]a = 0 \tag{8}$$

Replacing y by xy in above relation, we get

$$d(x)xy[a,x]a = 0 \tag{9}$$

Multiplying (8) by x on the left, we have $xd(x)y[a,x]a = 0$. From this and (9), we get $[d(x),x]y[a,x]a = 0$. Replacing y by ay we have,

$$axy[a,x]a = 0 \tag{10}$$

Replacing y by a^2y , we have

$$axa^2y[a,x]a = 0 \tag{11}$$

Multiplying (10) on the left by a and replacing y by ay , we have $a^2xay[a,x]a = 0$. Adding pseudo inverse of last equation in (11) and then replacing y by ya, we arrive at $a[a,x]a = 0, \forall x \in S$. In particular, $a[d(a),a]a = 0$. Thus $a^3 = 0$ which implies that $a = 0$ (see proof of last theorem). This shows that φ is a free action on S.

Following [1], we canonically define (α,β) generalized derivation of inverse semiring S. Let α and β be automorphisms of S then an additive mapping $G : S \to S$ is called generalized (α,β) derivation with the associated (α,β) derivation d, if there exists an (α,β) derivation d of S such that $G(xy) = \alpha(x)G(y) + d(x)\beta(y), \forall x, y \in S$. For $G = d$, G is (α,β) derivation and for $d = 0$ and $\alpha = I$ (identity map on S), G is right centralizer.

Theorem 2.7. Let $\tau : S \to S$ be a generalized (α,β)-derivation with the associated (α,β)-derivation d of S then $a \in D(\tau)$ implies $a \in D(\alpha + d)$.

Proof. Let $a \in D(\tau)$ then

$$\tau(x)a + \acute{a}x = 0, \forall x \in S \tag{12}$$

By lemma 1.1 we have $\tau(x)a = ax, x \in S$. Replacing x by xy in (12) and using the last relation, we get

$$(\tau(x)\acute{a} + \alpha(x)a)y + d(x)\beta(y)a = 0 \tag{13}$$

Multiplying (13) by z on the right and using Lemma 1.1., we have

$$(\tau(x)\acute{a} + \alpha(x)a)yz = d(x)\beta(y)\acute{a}z \tag{14}$$

Replacing y with yz in (13), we have $(\tau(x)\acute{a} + \alpha(x)a)yz + d(x)\beta(yz)a = 0$. This together with (14), we get $d(x)\beta(y)[\beta(z)a + \acute{a}z] = 0$. But β is onto, so we have $d(x)y[\beta(z)a + \acute{a}z] = 0$. Replacing y with $[\beta(z)a + \acute{a}z]yd(x)$ and using semiprimeness of S, we get $d(x)[\beta(z)a + \acute{a}z] = 0$. By lemma 1.1, we have $d(x)\beta(z)a = d(x)az, x, z \in S$. From (13) and semiprimeness of S , we have $\tau(x)\acute{a} + (\alpha + d)(x)a = 0$. Thus $(\alpha + d)(x)a + \acute{a}x = 0$ and hence, $a \in D(\alpha + d)$.

Corollary 2.8. Every (α, β)-derivation of a semiprime inverse semiring is a free action.

Proof. Put $\tau = d$ in above theorem then d is (α, β)-derivation so , we have $(\alpha + \tau)(x)a + \acute{a}x = 0$. This implies $\alpha(x)a + \tau(x)a + \acute{a}x = 0$. From (12), we get $\alpha(x)a = 0, x \in S$. As α is onto, we have $xa = 0, \forall x \in S$ or $a = 0$. Thus τ is free action.

Theorem 2.9. Let S be a semiprime inverse semiring and T be a centralizer on S and d a derivation of S then $\mu = d \circ T$ is a free action.
Proof. Let $a \in D(\mu)$ then by definition

$$(d \circ T)(x)a + \acute{a}x = 0, x \in S \tag{15}$$

Replacing x by xy in above equation, we get

$$d \circ T(x)ya + T(x)d(y)a + \acute{a}xy = 0 \tag{16}$$

Multiplying (15) by y from right side and using Lemma 1.1., we get $d \circ T(x)ay = axy, \forall x, y \in S$. From this and (16) we have, $d \circ T(x)[a, y] + T(x)d(y)\acute{a} = 0, x, y \in S$. Replacing y by ay we get, $d \circ T(x)\{[a, a]y + a[a, y]\} + T(x)d(a)y\acute{a} + T(x)ad(y)\acute{a} = 0$. But $(a + \acute{a})ay + a(ay + y\acute{a}) = (aay + \acute{a}ay + aay + ay\acute{a}) = a[a, y]$. Therefore, $d \circ T(x)a[a, y] + T(x)d(a)y\acute{a} + T(x)ad(y)\acute{a} = 0$. Using Lemma 1.1. in (15) and using it in last equation , we get

$$ax[a, y] + T(x)d(a)y\acute{a} + T(x)ad(y)\acute{a} = 0 \tag{17}$$

Multiplying left side of (17) by z, we get

$$zax[a, y] + zT(x)d(a)y\acute{a} + zT(x)ad(y)\acute{a} = 0 \tag{18}$$

Replacing x by zx in (17), we have $azx[a, y] + T(zx)d(a)y\acute{a} + T(zx)ad(y)\acute{a} = 0$. Applying lemma 1.1, in last equation and using it in (18), we get $[a, z]x[a, y] = 0$, which implies that $[a, z] = 0, z \in S$. This implies that $T(x)d(y)a = 0$. Replacing y by $T(y)$ in last equation, we get $T(x)d \circ T(y)a = 0$. Applying lemma 1.1 in (15) and using it in last equation, we get $T(x)\acute{a}y = 0$ or $T(x)a = 0, x \in S$. This shows that $d(T(x))a + T(x)d(a) = 0$. Multilying on left side of by a, we get $d(T(x))a^2 + T(x)d(a)a = 0$, or $d(T(x))a^2 = 0$. Applying lemma 1.1 in (15) and using it in last equation, we have $a = 0$. Hence $d \circ T$ is a free action.

Theorem 2.10. Let T be a left centralizer of a semiprime inverse semiring then $\eta = T(x)x + xT(x)$ is a free action on S.

Proof. Let $a \in D(\eta)$ then by definition

$$(T(x)x + xT(x))a + a'x = 0, x \in S \tag{19}$$

From Lemma 1.1. we get, $(T(x)x + xT(x))a = ax, x \in S$. Linearizing (19) , we get

$$(T(x)y + T(y)x + yT(x) + xT(y))a = 0 \tag{20}$$

Replacing both x and y by a in last equation, we have $a^2 + a^2 = 0$.By Lemma 1.1. $a^2 = a'^2$. Now replacing y by xa in (20), we get $(T(x)x + xT(x)a + xaT(x) + T(x)ax)a = 0$. Thus $axa + xaT(x)a + T(x)axa = 0$. Replacing x by a in last equation, we have $a^3 + a^2T(a)a + T(a)a^3 = 0$, But $a^2 = a^{2'}$ so, $a^3 + a'^2T(a)a + T(a)a^3 = 0$. Now, pre-multiplying (19) by a and replacing x by a, we have

$$aT(a)a^2 + a^2T(a)a + aa'^2 = 0 \tag{21}$$

Multiplying (19) on right side by a, we get $T(a)a^3 + aT(a)a^2 + aa'^2 = 0$. This together with (21) and Lemma 1.1. we get, $T(a)a^3 + a'^2T(a)a = 0$. Thus $a^3 = 0$ which implies that $a = 0$. Hence η is a free action.

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