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Necessary Conditions for a Generalized Absolutely $\boldsymbol{\delta}$ −Continuous of Real Valued Function

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Abstract

Let δ be a positive function on $[a, b] \subset \mathbb{R}$. By referring to system of fundamental δ −interval, the basic δ −calculus properties, and the concept of absolutely continuous, generalized absolutely continuous of a real valued function, this paper will explain about the properties of weakly (and strongly) absolutely δ −continuous and weakly (and strongly) generalized absolutely δ −continuous of a real valued function on a cell $[a, b] \subset \mathbb{R}$ with respect to the Lebesgue measure. Several studies on sufficient conditions for function $F: [a, b] \to \mathbb{R}$ are generalized strongly absolute δ − continuous among others are: (i) a δ −derivative of F exists at every $x \in [a, b]$; (ii) a δ −derivative of F exists nearly everywhere on $[a, b]$ and F is δ −continuous on $[a, b]$.

Keywords: absolute δ −continuous, strongly absolute δ −continuous, (strongly) generalized absolute δ −continuous, system of fundamental δ −interval

1 Introduction

Sufficient conditions for an absolutely continuous function with respect to Lebesgue measure of real valued function had been discussed (in [4], [5], [7,[8], [11], [12], [13], and [14]). The interval model that had been used in the concept of absolutely continuous was an elementary interval in $\mathbb{R}$ which defined in [2], p.44-45.

In mathematical analysis: the concept of continuity, absolute continuity, and generalized absolute continuity of a function are widely used in developing the theory of descriptive integral, like Newton integral, Lebesgue integral, Dendjoy integral ([2], [6]), and special Dendjoy integral ([3], [9], [15]). The properties of continuity, absolute continuity, and generalized absolute continuity of a real valued function which were used in the discussion of the above mentioned integral had been studied by mathematicians ([6], [15]).

Given a positive δ on a cell $[a, b]$, a system of fundamental $\delta-$interval at a point in $\mathbb{R}$ which is a generalization of the elementary system interval in $\mathbb{R}$ is constructed ([3]). Some basic properties of calculus on $\mathbb{R}$ associated with the system of fundamental $\delta-$interval are successfully studied ([6]). Referring to continuity, weakly and strongly absolute continuity, the weakly and strongly generalized absolute continuity used in defining descriptive integral mentioned above, the types of continuity relative to the fundamental $\delta-$interval are constructed and successfully proved that constructed function is a linear space ([10]).

Considering the importance of the concept of continuity, srongly absolute continuity, strongly generalized absolute continuity of a function, and based on the results of the study conducted by Indrati (in [6]) and Manuharawati (in [10]), this paper will explain some properties of $\delta-$continuous, strongly absolutely $\delta-$continuous, strongly generalized absolutely $\delta-$continuous of a real valued function on the set $X \subset \mathbb{R}$. Further, the necessary and sufficient conditions for a real valued function is having a strongly generalized absolutely $\delta-$continuous is explained.

2 Basic Concept

2.1 The System of Fundamental $\boldsymbol{\delta}-$interval

Let δ be a positive function defined on an interval $[a, b]$ and $x \in [a, b]$. An interval $x - \delta(x) \leq u < x < v \leq x + \delta(x)$ is called a fundamental $\delta-$interval at x. A collection of all fundamental $\delta-$interval at x is called a system of fundamental $\delta-$interval at x and denoted by $\mathcal{D}_x$. It is easy to understand that $\mathcal{D}_x \neq \emptyset$ and has properties ([3], p.: 4):

A1. For every $D_x \in \mathcal{D}_x$ and $s < x < t$,

$$D'_x = D_x \cap (s, t) \in \mathcal{D}_{x.}$$

A2. If $D'_x,\ {D_x}'' \in \mathcal{D}_x$, then

$$D'_x \cap D''_x \in \mathcal{D}_x.$$

A3. If A is index set and ${D_x}^\alpha \in \mathcal{D}_x$ for every $\alpha \in A$, then

$$D_x = \bigcup_{\alpha \in A} {D_x}^\alpha \in \mathcal{D}_{x.}$$

A4. For every $D_x \in \mathcal{D}_x$, D_x contains x and there exist $s, t \in D_x$ such that

$$s < x < t.$$

A5. If $D_x \in \mathcal{D}_x, u, v \in D_x$ and $u < x < v$, then there are $u_1, v_1 \in D_x$ with

$$u < u_1 < x < v_1 < v.$$

Since for every $x \in [a, b]$, $\mathcal{D}_x \neq \emptyset$, then for every $x \in [a, b]$, we can take exactly one fundamental $\delta-$interval $D_x \in \mathcal{D}_x$ and the collection of all D_x denoted by $\mathcal{G} = \{D_x\}$.

2.2 The δ −Continuity of a Function

Let $[a,b]$ be an interval on $\mathbb{R}$ with $a<b$ and $\delta:[a,b]\to\mathbb{R}^+$. Based on the fundamental δ −interval, and the concept of absolute continuity of function ([6]), it was constructed some types of absolute continuity as follow ([10]).

Definition 2.2.1 ([10]): *Given a positive function $\delta:[a,b]\to\mathbb{R}^+$, a set $X\subset\mathbb{R}$, and a function $F:[a,b]\to\mathbb{R}$.*

(i) *A function F is said to be weakly absolutely δ −continuous on X if for any real number $\varepsilon>0$, there exist a real number $\gamma>0$ such that for every sequence (x_i) on X there is a sequence of nonoverlapping fundamental δ − interval $\left(D_{x_i}\right)$, $D_{x_i}\in\mathcal{D}_{x_i}$ such that if $u_i\in D_{x_i}$ with $\sum_i|x_i-u_i|<\gamma$ then*

$$\sum_i|F(x_i)-F(u_i)|<\varepsilon.$$

The set of all weakly absolutely δ −continuous on X denoted by $AC_\delta(X)$.

(ii) *A function F is said to be generalized weakly absolutely δ −continuous on X if there is a sequence of sets (X_i) such that $X=\bigcup_i X_i$ and $F\in AC_\delta(X_i)$ for every i. The set of all generalized weakly absolutely δ −continuous on X denoted by $ACG_\delta(X)$.*

(iii) *A function F is said to be strongly absolutely δ −continuous on X if for any real number $\varepsilon>0$, there exist a real number $\gamma>0$ such that for every sequence (x_i) on X there is a sequence of nonoverlapping fundamental δ −interval $\left(D_{x_i}\right)$, $D_{x_i}\in\mathcal{D}_{x_i}$ such that if $u_i\in D_{x_i}$ with $\sum_i|x_i-u_i|<\gamma$ then*

$$\sum_i\omega(F;[a_i,b_i])<\varepsilon$$

with $a_i=min\{x_i,u_i\}$ and $b_i=max\{x_i,u_i\}$. The set of all strongly absolutely δ −continuous on a set $X\subset[a,b]$ denoted by $AC_\delta^*(X)$.

(iv) *A function F is said to be generalized strongly absolutely δ −continuous on a set X if there exist a sequence of sets (X_i) such that $X=\bigcup_i X_i$ and $F\in AC_\delta^*(X_i)$ for every i. The set of all generalized strongly absolutely δ −continuous on X denoted by $ACG_\delta^*(X)$.*

The following theorem will be used in discussion section.

Theorem 2.2.1 ([6]**):** *Let δ be a positive real function on $[a,b]$, $F:[a,b]\to\mathbb{R}$ be a function and $c\in[a,b]$. If $D_\delta F(c)$ exist, then F is δ −continuous at c.*

3. Result and Discussion

In this section $[a,b]\subset\mathbb{R}$ with $a<b$. Let $F:[a,b]\to\mathbb{R}$, $\delta:[a,b]\to\mathbb{R}^+$, and $A\subset[a,b]$, $B\subset[a,b]$. By the above concept and fundamental properties described in Section 2, we obtained some results that describe in theorems as follow.

Theorem 3.1 *If $F\in AC_\delta^*(A)$ and $F\in AC_\delta^*(B)$ then $F\in AC_\delta^*(A\cup B)$.*

Proof: W.l.o.g., it is sufficient to prove that $A \not\subset B$. Let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. Since $F \in AC_\delta^*(A)$, then there exists a real number $\gamma_1 > 0$ such that for any sequence (x_i) on A there is a a sequence of nonoverlapping fundamental δ −interval $(D_{x_i}{}')$, $D_{x_i}{}' \in \mathcal{D}_{x_i}$ such that if $u_i \in D_{x_i}{}' \cap A$ and $\sum_i |x_i - u_i| < \gamma_1$ we have

$$\sum_i \omega(F;[a_i,b_i]) < \frac{\varepsilon}{3}$$

with $a_i = min\{x_i, u_i\}$ and $b_i = max\{x_i, u_i\}$.
Since $F \in AC_\delta^*(B)$, then there exists a real number $\gamma_2 > 0$ such that for any sequence (x_i) on B there is a a sequence of nonoverlapping fundamental δ −interval $(D_{x_i}{}'')$, $D_{x_i}{}'' \in \mathcal{D}_{x_i}$ such that if $u_i \in D_{x_i}{}' \cap B$ and $\sum_i |x_i - u_i| < \gamma_2$ we have

$$\sum_i \omega(F;[a_i,b_i]) < \frac{\varepsilon}{3}$$

with $a_i = min\{x_i, u_i\}$ and $b_i = max\{x_i, u_i\}$.
If $\gamma = min\{\gamma_1, \gamma_2\}$, then $\gamma \in \mathbb{R}$ and $\gamma > 0$. Futher more, if (x_i) is a sequence on $A \cup B$, then there exists a sequence of nonoverlapping fundamental δ −interval (D_{x_i}), $D_{x_i} \in \mathcal{D}_{x_i}$, i.e.:

$$D_{x_i} = \begin{cases} D'_{x_i} \; if \;\; x_i \in A \\ D''_{x_i} \, if \;\; x_i \in B \\ D'_{x_i} \cap D''_{x_i} if x_i \in A \cap B \end{cases}.$$

If $u_i \in D_{x_i} \cap (A \cup B)$ with $\sum_i |x_i - u_i| < \gamma$, and

$$a_i = min\{x_i, u_i\}, \quad b_i = max\{x_i, u_i\},$$

then we have

$$\sum_i \omega(F;[a_i,b_i]) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon.$$

It means that $F \in AC_\delta^*(A \cup B)$. ■

Theorem 3.2 *If* $F \in AC_\delta$ *and* F *is* $\delta -$ *continuous at* $c \in [a,b]$ *then* $F \in AC_\delta(A \cup \{c\})$.

Proof: Let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. Since $F \in AC_\delta(A)$, then there exists a real number $\gamma > 0$ such that for any sequence (x_i) on A there exists a sequence of nonoverlapping fundamental δ −interval (D_{x_i}), $D_{x_i} \in \mathcal{D}_{x_i}$ such that for any $u_i \in D_{x_i} \cap A$ with $\sum_i |x_i - u_i| < \gamma$ we have

$$\textstyle\sum_i |F(x_i) - F(u_i)| < \frac{\varepsilon}{3}. \qquad (1)$$

Since F is δ −continuous at c, then there exists fundamental δ −interval $D_c \in \mathcal{D}_c$ such that for any $u \in D_c \cap [a,b]$ satisfy

$$|F(c) - F(u)| < \tfrac{\varepsilon}{3}. \qquad (2)$$

Set
$$D_c' = D_c \cap (c-\delta, c+\delta).$$

Clearly, $D_c{}' \in \mathcal{D}_c$. So, (2) is hold if $u \in D_c' \cap [a,b]$. Let (x_i) be a sequence on $A \cup \{c\}$. There are two cases, i.e.: $x_i \in A$ for every i or $x_k = c$ for some k.

(i) If $x_i \in A$ for every i, then (1) holds.

(ii) If $x_k = c$ for some k, take $D_{x_k}{}' = D_c{}'$.
If $u_i \in D_{x_i} \cap (A \cup \{c\})$ with $\sum_i |x_i - u_i| < \gamma$, then by (1) and (2) we have
$$\sum_i |F(x_i) - F(u_i)| = \sum_{i, i\neq k} |F(x_i) - F(u_i)| + |F(c) - F(u_k)| < \varepsilon.$$

From (i) and (ii), we have $F \in AC_\delta(A \cup \{c\})$. ■

Theorem 3.3 *If $F \in AC_\delta^*(A)$ and F is δ −continuous at $c \in [a,b]$ then $F \in AC_\delta^*(A \cup [c])$.*

Proof: Let ε be a positive real number. Since $F \in AC_\delta^*(A)$, then there exists a real number $\gamma > 0$, such that for any sequence (x_i) on A there exists a sequence of nonoverlapping fundamental δ −interval$\left(D_{x_i}\right)$, $D_{x_i} \in \mathcal{D}_{x_i}$ such that for any $u_i \in D_{x_i} \cap A$ with $\sum_i |x_i - u_i| < \gamma$ satisfy
$$\sum_i \omega(F; [a_i, b_i]) < \frac{\varepsilon}{3}. \tag{3}$$

with $\min\{x_i, u_i\}$ and $b_i = \max\{x_i, b_i\}$. Since F is δ −continuous at $c \in [a,b]$, then there exists a fundamental δ −interval $D_c \in \mathcal{D}_c$ such that for any $u \in D_c \cap [a,b]$, with $|u - c| < \gamma$ we have
$$|F(c) - F(u)| < \frac{\varepsilon}{3}. \tag{4}$$

Set $D_c' = D_c \cap (c-\delta, c+\delta)$. Then $D_c{}' \in \mathcal{D}_c$. So, if $u \in D_c' \cap [a,b]$, (2) holds. Consequently, we have
$$\omega(F; [s,t]) < \frac{\varepsilon}{3} \tag{5}$$

with $s = \min\{c, u\}$ and $t = \max\{c, u\}$.

Let (x_i) be any sequence on $A \cup \{c\}$. There exists two cases for such X_i, i. e.: for every i, $x_i \in A$ or there exists k such that $x_k = c$.

(i) If for every $i, x_i \in A$, then (3) holds.

(ii) If there exists k such that $x_k = c$, take $D_{x_k}{}' = D_c{}'$. If $u_i \in D_{x_i} \cap (A \cup \{c\})$ with $\sum_i |x_i - u_i| < \gamma$, then by (3) and (5) we have
$$\sum_i \omega(F; [a_i, b_i) \leq \sum_{i, i\neq k} \omega\,(F; [a_i, b_i]) + \omega(F; [s,t]) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$$
with $a_i = \min\{x_i, u_i\}$; $b_i = \max\{x_{i,}, u_i\}$; $s = \min\{c, u\}$; $t = \max\{t, u\}$.

From (i) and (ii), we have $F \in AC_\delta^*(A \cup \{c\})$. ■

Corollary 3.4: *If* $F \in ACG_\delta(A)$ *and* F *is continuous at* $c \in [a,b]$, *then* $F \in ACG_\delta(A \cup \{c\})$.

Proof: Since $F \in ACG_\delta(A)$ then there exists a sequence of sets (A_i) such that $A = \bigcup_i A_i$ and for avery i, $F \in AC_\delta(A_i)$. Since F is continuous at c, then by Theorem 3.2, $F \in AC_\delta(A_i \cup \{c\})$. Consequently, $F \in ACG_\delta(\bigcup_i(A_i \cup \{c\}))$ or $F \in ACG_\delta(A \cup \{c\})$. ■

Corollary 3.5: *If* $F \in ACG_\delta^*(A)$ *and* F *is continuous at* $c \in [a,b]$ *then* $F \in ACG_\delta^*(A \cup \{c\})$.

Proof: Since $F \in ACG_\delta^*(A)$, then there exists a sequence of sets (A_i) such that $A = \bigcup_i A_i$ and $F \in AC_\delta^*(A_i)$. By Theorem 3.3, $F \in AC_\delta^*(A_i \cup \{c\})$, for every i. So, $F \in ACG_\delta^*(A \cup \{c\})$. ■

Theorem 3.6: *If* $D_\delta F(x)$ *exists for every* $x \in [a,b]$, *then* $F \in ACG_\delta^*[a,b]$.

Proof: Let ε be a positive real number. Since $D_\delta F(x)$ exists for every $x \in [a,b]$, then for any $x \in [a,b]$ there exists a fundamental δ −interval $D_x{}' \in \mathcal{D}_x$ such that for any $u \in D_x' \cap [a,b]$, $u \neq x$ satisfies

$$\left|\frac{F(x)-F(u)}{x-u} - D_\delta F(x)\right| < \varepsilon$$

or

$$|F(x)-F(u)| < [|D_\delta F(x)| + \varepsilon]|x-u|.$$

For every $n, i \in \mathbb{N}$, set

$$S_n = \{x \in [a,b]: |D_\delta F(x) \le n|\} \text{ and } X_{n,i} = S_n \cap \left[a + \frac{i-1}{n}, a + \frac{i}{n}\right] .$$

It is easy to understand that $[a,b] = \bigcup_{n,i} X_{n,i}$. If $x \in [a,b]$ and

$$D_x = D_x{}' \cap \left[a + \frac{i-1}{n}, a + \frac{i}{n}\right],$$

then for any $u \in D_x, u \neq x$, we have

$$|F(x)-F(u)| < (n+\varepsilon)|x-u|.$$

Let n and i be certain the positive integer number. If (x_j) be a sequence on $X_{n,i}$, then there exists a sequence of fundamental δ −interval $\left(D_{x_j}\right)$ on $\mathcal{D}_{x_j}$. Take any $u_j \in D_{x_j} \cap X_{n,i}$ and

$$a_j = min\{x_j, u_i\}, \text{ and } b_j = max\{x_j, u_i\}.$$

Since $D_\delta F(x)$ exists for every $x \in [a,b]$, then by Theorem 2.2.1, F is δ −continuous on $[a,b]$. Since $[a_j, b_j] \subset [a,b]$, then for every j, $F \in C_\delta[a_j, b_j]$. So, there are $\alpha_j, \beta_j \in [a_j, b_j]$ such that

$$\omega(F;\ [a_j, b_j]) = |F(\alpha_j) - F(\beta_j)|.$$

Since $\alpha_j, \beta_j \in [a_j, b_j]$, then

$$|F(\alpha_j) - F(\beta_j)| < [n + \varepsilon]|x_j - u_j|.$$

Consequently,

$$\sum_j \omega(F;\ [a_j, b_j]) = \sum_j |F(\alpha_j) - F(\beta_j)| < [n + \varepsilon] \sum_j |x_j - u_j| < \varepsilon.$$

if $\sum_i |x_j - u_j| < \gamma = \frac{\varepsilon}{n+\varepsilon}$. It's meant that $F \in ACG_\delta^*[a, b]$. ■

Theorem 3.7: *If $D_\delta F(x)$ exist nearly everywhere on $[a, b]$, then $F \in ACG_\delta^*[a, b]$.*

Proof: Let ε be a positive real number. Since $D_\delta F(x)$ nearly everywhere on $[a, b]$, there exist a countable set $A \subset [a, b]$ such that $D_\delta F(x)$ exist for every $x \in [a, b] - A$. By Theorem 2.2.1, F is $\delta -$continuous on $[a, b] - A$. By Theorem 3.6, $F \in ACG_\delta^*([a, b] - A)$. Since A is a countable set, then by Theorem 3.5, $F \in ACG_\delta^*([a, b] - A) \cup A$. Since $([a, b] - A) \cup A = [a, b]$, then $F \in ACG_\delta^*[a, b]$. ■

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