Necessary Conditions for a Generalized Absolutely

\( \delta \) –Continuous of Real Valued Function

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Abstract

Let \( \delta \) be a positive function on \([a, b] \subset \mathbb{R}\). By referring to system of fundamental \( \delta \)–interval, the basic \( \delta \)–calculus properties, and the concept of absolutely continuous, generalized absolutely continuous of a real valued function, this paper will explain about the properties of weakly (and strongly) absolutely \( \delta \)–continuous and weakly (and strongly) generalized absolutely \( \delta \)–continuous of a real valued function on a cell \([a, b] \subset \mathbb{R}\) with respect to the Lebesgue measure. Several studies on sufficient conditions for function \( F: [a, b] \rightarrow \mathbb{R} \) are generalized strongly absolute \( \delta \)–continuous among others are: (i) a \( \delta \)–derivative of \( F \) exists at every \( x \in [a, b] \); (ii) a \( \delta \)–derivative of \( F \) exists nearly everywhere on \([a, b]\) and \( F \) is \( \delta \)–continuous on \([a, b]\).

Keywords: absolute \( \delta \)–continuous, strongly absolute \( \delta \)–continuous, (strongly) generalized absolute \( \delta \)–continuous, system of fundamental \( \delta \)–interval

1 Introduction

Sufficient conditions for an absolutely continuous function with respect to Lebesgue measure of real valued function had been discussed (in [4], [5], [7],[8], [11], [12], [13], and [14]). The interval model that had been used in the concept of absolutely continuous was an elementary interval in \( \mathbb{R} \) which defined in [2], p.44-45.

In mathematical analysis: the concept of continuity, absolute continuity, and generalized absolute continuity of a function are widely used in developing the theory of descriptive integral, like Newton integral, Lebesgue integral, Dendjoy
integral ([2], [6]), and special Dendjoy integral ([3], [9], [15]). The properties of continuity, absolute continuity, and generalized absolute continuity of a real valued function which were used in the discussion of the above mentioned integral had been studied by mathematicians ([6], [15]).

Given a positive $\delta$ on a cell $[a, b]$, a system of fundamental $\delta$—interval at a point in $\mathbb{R}$ which is a generalization of the elementary system interval in $\mathbb{R}$ is constructed ([3]). Some basic properties of calculus on $\mathbb{R}$ associated with the system of fundamental $\delta$—interval are successfully studied ([6]). Referring to continuity, weakly and strongly absolute continuity, the weakly and strongly generalized absolute continuity used in defining descriptive integral mentioned above, the types of continuity relative to the fundamental $\delta$—interval are constructed and successfully proved that constructed function is a linear space ([10]).

Considering the importance of the concept of continuity, strongly absolute continuity, strongly generalized absolute continuity of a function, and based on the results of the study conducted by Indrati (in [6]) and Manuharawati (in [10]), this paper will explain some properties of $\delta$—continuous, strongly absolutely $\delta$—continuous, strongly generalized absolutely $\delta$—continuous of a real valued function on the set $X \subset \mathbb{R}$. Further, the necessary and sufficient conditions for a real valued function is having a strongly generalized absolutely $\delta$—continuous is explained.

2 Basic Concept

2.1 The System of Fundamental $\delta$—interval

Let $\delta$ be a positive function defined on an interval $[a, b]$ and $x \in [a, b]$. An interval $x - \delta(x) \leq u < x < v \leq x + \delta(x)$ is called a fundamental $\delta$—interval at $x$. A collection of all fundamental $\delta$—interval at $x$ is called a system of fundamental $\delta$—interval at $x$ and denoted by $\mathcal{D}_x$. It is easy to understand that $\mathcal{D}_x \neq \emptyset$ and has properties ([3], p. 4):

A1. For every $D_x \in \mathcal{D}_x$ and $s < x < t$, $D_x' = D_x \cap (s, t) \in \mathcal{D}_x$.

A2. If $D_x', D_x'' \in \mathcal{D}_x$, then $D_x' \cap D_x'' \in \mathcal{D}_x$.

A3. If $A$ is index set and $D_x^\alpha \in \mathcal{D}_x$ for every $\alpha \in A$, then $D_x = \bigcup_{\alpha \in A} D_x^\alpha \in \mathcal{D}_x$.

A4. For every $D_x \in \mathcal{D}_x$, $D_x$ contains $x$ and there exist $s, t \in D_x$ such that $s < x < t$.

A5. If $D_x \in \mathcal{D}_x$, $u, v \in D_x$ and $u < x < v$, then there are $u_1, v_1 \in D_x$ with $u < u_1 < x < v_1 < v$.

Since for every $x \in [a, b]$, $\mathcal{D}_x \neq \emptyset$, then for every $x \in [a, b]$, we can take exactly one fundamental $\delta$—interval $D_x \in \mathcal{D}_x$ and the collection of all $D_x$ denoted by $\mathcal{G} = \{D_x\}$.
2.2 The \( \delta \) – Continuity of a Function

Let \([a, b]\) be an interval on \(\mathbb{R}\) with \(a < b\) and \(\delta : [a, b] \rightarrow \mathbb{R}^+\). Based on the fundamental \(\delta\) – interval, and the concept of absolute continuity of function ([6]), it was constructed some types of absolute continuity as follow ([10])

**Definition 2.2.1** ([10]): Given a positive function \(\delta : [a, b] \rightarrow \mathbb{R}^+\), a set \(X \subset \mathbb{R}\), and a function \(F : [a, b] \rightarrow \mathbb{R}\).

(i) A function \(F\) is said to be weakly absolutely \(\delta\) – continuous on \(X\) if for any real number \(\varepsilon > 0\), there exist a real number \(\gamma > 0\) such that for every sequence \((x_i)\) on \(X\) there is a sequence of nonoverlapping fundamental \(\delta\) – interval \((D_{x_i})\), \(D_{x_i} \in \mathcal{D}_{x_i}\) such that if \(u_i \in D_{x_i}\) with \(\sum_i |x_i - u_i| < \gamma\) then

\[
\sum_i |F(x_i) - F(u_i)| < \varepsilon.
\]

The set of all weakly absolutely \(\delta\) – continuous on \(X\) denoted by \(AC_\delta(X)\).

(ii) A function \(F\) is said to be generalized weakly absolutely \(\delta\) – continuous on \(X\) if there is a sequence of sets \((X_i)\) such that \(X = \bigcup_i X_i\) and \(F \in AC_\delta(X_i)\) for every \(i\). The set of all generalized weakly absolutely \(\delta\) – continuous on \(X\) denoted by \(ACG_\delta(X)\).

(iii) A function \(F\) is said to be strongly absolutely \(\delta\) – continuous on \(X\) if for any real number \(\varepsilon > 0\), there exist a real number \(\gamma > 0\) such that for every sequence \((x_i)\) on \(X\) there is a sequence of nonoverlapping fundamental \(\delta\) – interval \((D_{x_i})\), \(D_{x_i} \in \mathcal{D}_{x_i}\) such that if \(u_i \in D_{x_i}\) with \(\sum_i |x_i - u_i| < \gamma\) then

\[
\sum_i \omega(F; [a_i, b_i]) < \varepsilon
\]

with \(a_i = \min\{x_i, u_i\}\) and \(b_i = \max\{x_i, u_i\}\). The set of all strongly absolutely \(\delta\) – continuous on a set \(X \subset [a, b]\) denoted by \(AC^*_\delta(X)\).

(iv) A function \(F\) is said to be generalized strongly absolutely \(\delta\) – continuous on a set \(X\) if there exist a sequence of sets \((X_i)\) such that \(X = \bigcup_i X_i\) and \(F \in AC^*_\delta(X_i)\) for every \(i\). The set of all generalized strongly absolutely \(\delta\) – continuous on \(X\) denoted by \(ACG^*_\delta(X)\).

The following theorem will be used in discussion section.

**Theorem 2.2.1** ([6]): Let \(\delta\) be a positive real function on \([a, b]\), \(F : [a, b] \rightarrow \mathbb{R}\) be a function and \(c \in [a, b]\). If \(D_\delta F(c)\) exist, then \(F\) is \(\delta\) – continuous at \(c\).

3. Result and Discussion

In this section \([a, b] \subset \mathbb{R}\) with \(a < b\). Let \(F : [a, b] \rightarrow \mathbb{R}\), \(\delta : [a, b] \rightarrow \mathbb{R}^+\), and \(A \subset [a, b]\), \(B \subset [a, b]\). By the above concept and fundamental properties described in Section 2, we obtained some results that describe in theorems as follow.

**Theorem 3.1** If \(F \in AC_\delta(A)\) and \(F \in AC^*_\delta(B)\) then \(F \in AC_\delta(A \cup B)\).
**Proof:** W.l.o.g., it is sufficient to prove that \( A \not\subset B \). Let \( \varepsilon \in \mathbb{R} \), \( \varepsilon > 0 \). Since \( F \in AC_\delta^*(A) \), then there exists a real number \( \gamma_1 > 0 \) such that for any sequence \((x_i)\) on \( A \) there is a a sequence of nonoverlapping fundamental \( \delta \)-interval \((D_{x_i}')\), \( D_{x_i}' \in \mathcal{D}_{x_i} \) such that if \( u_i \in D_{x_i}' \cap A \) and \( \sum_i |x_i - u_i| < \gamma_1 \) we have
\[
\sum_i \omega(F; [a_i, b_i]) < \frac{\varepsilon}{3}
\]
with \( a_i = \min \{x_i, u_i\} \) and \( b_i = \max \{x_i, u_i\} \).
Since \( F \in AC_\delta^*(B) \), then there exists a real number \( \gamma_2 > 0 \) such that for any sequence \((x_i)\) on \( B \) there is a a sequence of nonoverlapping fundamental \( \delta \)-interval \((D_{x_i}'')\), \( D_{x_i}'' \in \mathcal{D}_{x_i} \) such that if \( u_i \in D_{x_i}' \cap B \) and \( \sum_i |x_i - u_i| < \gamma_2 \) we have
\[
\sum_i \omega(F; [a_i, b_i]) < \frac{\varepsilon}{3}
\]
with \( a_i = \min \{x_i, u_i\} \) and \( b_i = \max \{x_i, u_i\} \).
If \( \gamma = \min \{\gamma_1, \gamma_2\} \), then \( \gamma \in \mathbb{R} \) and \( \gamma > 0 \). Further more, if \((x_i)\) is a sequence on \( A \cup B \), then there exists a sequence of nonoverlapping fundamental \( \delta \)-interval \((D_{x_i})\), \( D_{x_i} \in \mathcal{D}_{x_i} \), i.e.:
\[
D_{x_i} = \begin{cases} 
D_{x_i}' & \text{if } x_i \in A \\
D_{x_i}' & \text{if } x_i \in B \\
D_{x_i}' \cap D_{x_i}'' & \text{if } x_i \in A \cap B
\end{cases}
\]
If \( u_i \in D_{x_i} \cap (A \cup B) \) with \( \sum_i |x_i - u_i| < \gamma \), and
\[
a_i = \min \{x_i, u_i\}, \quad b_i = \max \{x_i, u_i\},
\]
then we have
\[
\sum_i \omega(F; [a_i, b_i]) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon.
\]
It means that \( F \in AC_\delta^*(A \cup B) \).

**Theorem 3.2** If \( F \in AC_\delta \) and \( F \) is \( \delta \)-continuous at \( c \in [a, b] \) then \( F \in AC_\delta(A \cup \{c\}) \).

**Proof:** Let \( \varepsilon \in \mathbb{R}, \varepsilon > 0 \). Since \( F \in AC_\delta(A) \), then there exists a real number \( \gamma > 0 \) such that for any sequence \((x_i)\) on \( A \) there exists a sequence of nonoverlapping fundamental \( \delta \)-interval \((D_{x_i})\), \( D_{x_i} \in \mathcal{D}_{x_i} \) such that for any \( u_i \in D_{x_i} \cap A \) with \( \sum_i |x_i - u_i| < \gamma \) we have
\[
\sum_i |F(x_i) - F(u_i)| < \frac{\varepsilon}{3}, \tag{1}
\]
Since \( F \) is \( \delta \)-continuous at \( c \), then there exists fundamental \( \delta \)-interval \( D_c \in \mathcal{D}_c \) such that for any \( u \in D_c \cap [a, b] \) satisfy
\[
|F(c) - F(u)| < \frac{\varepsilon}{3}, \tag{2}
\]
Set
\[ D'_c = D_c \cap (c - \delta, c + \delta). \]

Clearly, \( D'_c \in \mathcal{D}_c \). So, (2) is hold if \( u \in D'_c \cap [a, b] \). Let \( (x_i) \) be a sequence on \( A \cup \{c\} \). There are two cases, i.e.: \( x_i \in A \) for every \( i \) or \( x_k = c \) for some \( k \).

(i) If \( x_i \in A \) for every \( i \), then (1) holds.

(ii) If \( x_k = c \) for some \( k \), take \( D_{x_k}' = D_c' \).

If \( u_i \in D_{x_i} \cap (A \cup \{c\}) \) with \( \sum_i |x_i - u_i| < \gamma \), then by (1) and (2) we have
\[ \sum_i |F(x_i) - F(u_i)| = \sum_{i,i \neq k} |F(x_i) - F(u_i)| + |F(c) - F(u_k)| < \varepsilon. \]

From (i) and (ii), we have \( F \in AC_\delta(A \cup \{c\}) \). \( \square \)

**Theorem 3.3** If \( F \in AC_\delta^*(A) \) and \( F \) is \( \delta \) – continuous at \( c \in [a, b] \) then \( F \in AC_\delta^*(A \cup \{c\}) \).

**Proof:** Let \( \varepsilon \) be a positive real number. Since \( F \in AC_\delta^*(A) \), then there exists a real number \( \gamma > 0 \), such that for any sequence \( (x_i) \) on \( A \) there exists a sequence of nonoverlapping fundamental \( \delta \) – interval \( (D_{x_i}) \), \( D_{x_i} \in \mathcal{D}_c \) such that for any \( u \in D_{x_i} \cap A \) with \( \sum_i |x_i - u_i| < \gamma \) satisfy
\[ \sum_i \omega(F; [a_i, b_i]) < \frac{\varepsilon}{3}. \] (3)

with \( \min\{x_i, u_i\} \) and \( b_i = \max\{x_i, u_i\} \). Since \( F \) is \( \delta \) – continuous at \( c \in [a, b] \), then there exists a fundamental \( \delta \) – interval \( D_c \in \mathcal{D}_c \) such that for any \( u \in D_c \cap [a, b] \), with \( |u - c| < \gamma \) we have
\[ |F(c) - F(u)| < \frac{\varepsilon}{3}. \] (4)

Set \( D'_c = D_c \cap (c - \delta, c + \delta) \). Then \( D'_c \in \mathcal{D}_c \). So, if \( u \in D'_c \cap [a, b] \), (2) holds. Consequently, we have
\[ \omega(F; [s, t]) < \frac{\varepsilon}{3}. \] (5)

with \( s = \min\{c, u\} \) and \( t = \max\{c, u\} \).

Let \( (x_i) \) be any sequence on \( A \cup \{c\} \). There exists two cases for such \( X_i \), i.e.: for every \( i \), \( x_i \in A \) or there exists \( k \) such that \( x_k = c \).

(i) If for every \( i, x_i \in A \), then (3) holds.

(ii) If there exists \( k \) such that \( x_k = c \), take \( D_{x_k}' = D_c' \). If \( u_i \in D_{x_i} \cap (A \cup \{c\}) \) with \( \sum_i |x_i - u_i| < \gamma \), then by (3) and (5) we have
\[ \sum_i \omega(F; [a_i, b_i]) \leq \sum_{i,i \neq k} \omega(F; [a_i, b_i]) + \omega(F; [s, t]) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon \]

with \( a_i = \min\{x_i, u_i\}; \ b_i = \max\{x_i, u_i\}; \ s = \min\{c, u\}; \ t = \max\{t, u\} \).

From (i) and (ii), we have \( F \in AC_\delta^*(A \cup \{c\}) \). \( \square \)
Corollary 3.4: If $F \in ACG_\delta(A)$ and $F$ is continuous at $c \in [a, b]$, then $F \in ACG_\delta(A \cup \{c\})$.

Proof: Since $F \in ACG_\delta(A)$ then there exists a sequence of sets $(A_i)$ such that $A = \bigcup_i A_i$ and for every $i$, $F \in ACG_\delta(A_i)$. Since $F$ is continuous at $c$, then by Theorem 3.2, $F \in ACG_\delta(A_i \cup \{c\})$. Consequently, $F \in ACG_\delta(\bigcup_i (A_i \cup \{c\}))$ or $F \in ACG_\delta(A \cup \{c\})$. ■

Corollary 3.5: If $F \in ACG_\delta^*(A)$ and $F$ is continuous at $c \in [a, b]$ then $F \in ACG_\delta^*(A \cup \{c\})$.

Proof: Since $F \in ACG_\delta^*(A)$, then there exists a sequence of sets $(A_i)$ such that $A = \bigcup_i A_i$, and $F \in ACG_\delta^*(A_i)$. By Theorem 3.3, $F \in ACG_\delta^*(A_i \cup \{c\})$, for every $i$. So, $F \in ACG_\delta^*(A \cup \{c\})$. ■

Theorem 3.6: If $D_\delta F(x)$ exists for every $x \in [a, b]$, then $F \in ACG_\delta^*[a, b]$.

Proof: Let $\varepsilon$ be a positive real number. Since $D_\delta F(x)$ exists for every $x \in [a, b]$, then for any $x \in [a, b]$ there exists a fundamental $\delta$–interval $D_x' \in D_x$ such that for any $u \in D_x' \cap [a, b]$, $u \neq x$ satisfies

$$\left|\frac{F(x) - F(u)}{x - u} - D_\delta F(x)\right| < \varepsilon$$

or

$$|F(x) - F(u)| < |D_\delta F(x)| + \varepsilon|x - u|.$$

For every $n, i \in \mathbb{N}$, set

$$S_n = \{x \in [a, b] : |D_\delta F(x) \leq n\}$$

and

$$X_{n,i} = S_n \cap [a + \frac{i-1}{n}, a + \frac{i}{n}].$$

It is easy to understand that $[a, b] = \bigcup_{n,i} X_{n,i}$. If $x \in [a, b]$ and $D_x = D_x' \cap [a + \frac{i-1}{n}, a + \frac{i}{n}]$, then for any $u \in D_x$, $u \neq x$, we have

$$|F(x) - F(u)| < (n + \varepsilon)|x - u|.$$

Let $n$ and $i$ be certain the positive integer number. If $(x_j)$ be a sequence on $X_{n,i}$, then there exists a sequence of fundamental $\delta$–interval $(D_{x_j})$ on $D_{x_j}$. Take any $u_j \in D_{x_j} \cap X_{n,i}$ and

$$a_j = \min\{x_j, u_j\}, \text{ and } b_j = \max\{x_j, u_j\}.$$ 

Since $D_\delta F(x)$ exists for every $x \in [a, b]$, then by Theorem 2.2.1, $F$ is $\delta$–continuous on $[a, b]$. Since $[a_j, b_j] \subset [a, b]$, then for every $j$, $F \in C_\delta(a_j, b_j)$. So, there are $\alpha_j, \beta_j \in [a_j, b_j]$ such that

$$\omega(F; [a_j, b_j]) = |F(\alpha_j) - F(\beta_j)|.$$
Since \( \alpha_j, \beta_j \in [a_j, b_j] \), then
\[
|F(\alpha_j) - F(\beta_j)| < (n + \varepsilon)|x_j - u_j|.
\]
Consequently,
\[
\sum_j \omega(F; [a_j, b_j]) = \sum_j |F(\alpha_j) - F(\beta_j)| < (n + \varepsilon) \sum_j |x_j - u_j| < \varepsilon.
\]
if \( \sum_j |x_j - u_j| < \gamma = \frac{\varepsilon}{n+\varepsilon} \). It’s meant that \( F \in ACG^*_\delta[a, b] \).

**Theorem 3.7:** If \( D_\delta F(x) \) exist nearly everywhere on \([a, b]\), then \( F \in ACG^*_\delta[a, b] \).

**Proof:** Let \( \varepsilon \) be a positive real number. Since \( D_\delta F(x) \) nearly everywhere on \([a, b]\), there exist a countable set \( A \subset [a, b] \) such that \( D_\delta F(x) \) exist for every \( x \in [a, b] - A \). By Theorem 2.2.1, \( F \) is \( \delta \)-continuous on \([a, b] - A \). By Theorem 3.6, \( F \in ACG^*_\delta([a, b] - A) \). Since \( A \) is a countable set, then by Theorem 3.5, \( F \in ACG^*_\delta([a, b] - A) \cup A \). Since \(([a, b] - A) \cup A = [a, b]\), then \( F \in ACG^*_\delta[a, b] \).

**References**


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