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The Jacobsthal and Jacobsthal-Lucas Numbers via Square Roots of Matrices

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Abstract

In this paper, the Jacobsthal and Jacobsthal-Lucas numbers with specialized rational subscripts are studied by using square roots of the matrices F^n and S^n. We also reveal the identities involving these numbers derived by the matrices $F^{n/2}$ and $S^{n/2}$. Further we show that the matrices F^n and S^n are generalized to rational powers by using the Abel's functional equation.

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1 Introduction

We summarize basic information about the definition and the properties associated with Jacobsthal and Jacobsthal-Lucas numbers. The Jacobsthal $\{J_n\}_{n=0}^{\infty}$ and Jacobsthal-Lucas $\{j_n\}_{n=0}^{\infty}$ sequences are defined by the recurrence relations

$$J_{n+2} = J_{n+1} + 2J_n, \ j_{n+2} = j_{n+1} + 2j_n, \ n \geq 0,$$

where $J_0 = 0$, $J_1 = 1$, $j_0 = 2$ and $j_1 = 1$. Also, The Jacobsthal and Jacobsthal-Lucas numbers can easily be computed using the closed-forms alike $J_n = (2^n - (-1)^n)/3$ and $j = 2^n + (-1)^n$ [1],[2]. And so, the J_n and j_n numbers can be derived by matrices since by taking successive powers of matrices F and S [3],[4];

$$F^n = \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix}, \; S^n = \frac{1}{2}\begin{bmatrix} j_n & 9J_n \\ J_n & j_n \end{bmatrix}.$$

There are less theory related to complex value Jacobsthal and Jacobsthal-Lucas sequences from other integer sequences. In [1] study, Horadam derived from cartesian coordinates x, y of a point in the plane,

$$x = \left(2^\theta - \cos\theta\pi\right)/3 \; and \; y = -2^\theta \sin\theta\pi/3 \; for \; J_n, \tag{1}$$

$$x = 2^\theta + \cos\theta\pi \; and \; y = 2^\theta \sin\theta\pi \; for \; j_n, \; \theta \in \mathbb{R}. \tag{2}$$

The author denominated equations (1) and (2) as the modified Binet forms for J_n and j_n, respectively. Setting $\theta = n \in \mathbb{Z}$, $x = J_n$ in (1) and $x = j_n$ in (2) are the ordinary Binet form for J_n and j_n. The equations (1) and (2) yield the Jacobsthal and Jacobsthal-Lucas curves. Theirs stationary points lie on the suitable branches of the rectangular hyperbolas

$$y\left(x \pm \frac{k\log 2}{3}\right) = \mp\frac{k\pi}{9} \; and \; y\left(x \pm k\log 2\right) = \mp k\pi.$$

Also, for $J_\theta + j_\theta = 2J_{\theta+1}$, a composite curve is given to being equivalent to the Jacobsthal curve [1].

It is the object of this article to reveal the corresponding complex value sequences associated with real index $\{J_x\}$ and $\{j_x\}$, for $x \in \mathbb{R}$ via computing the square roots of the matrices $F^n$and S^n.

2 Main Results

Our aim is not to calculate the square roots of a $2x2$ matrix. The certain methods have been studied for computing the square roots of arbitrary $2x2$ matrices [5],[6], and a matrix may be have several square roots or no square roots. If a matrix generate any sequence and have several square roots, we investigate properties which new sequences generated by square roots matrices have, and the main results of the paper are given the following.

Theorem 2.1 *Let $F_i^{n/2}$ $(i = 1, 2, 3, 4)$ denote the square roots of the matrix F^n. Then,*

$$F_{1,2}^{n/2} = \pm\begin{bmatrix} J_{(n+2)/2} & 2J_{n/2} \\ J_{n/2} & 2J_{(n-2)/2} \end{bmatrix}, \; F_{3,4}^{n/2} = \frac{\pm 1}{3}\begin{bmatrix} j_{(n+2)/2} & 2j_{n/2} \\ j_{n/2} & 2j_{(n-2)/2} \end{bmatrix}. \tag{3}$$

Proof. When square roots of the matrix F^n are computed by the Cayley-Hamilton method [5], it is seen that the matrix F^n has four square roots in the forms

$$\sqrt{F^n} = \frac{\pm 1}{\sqrt{T \pm 2\sqrt{\det(F^n)}}}\left[F^n \pm \sqrt{\det(F^n)} I\right], \tag{4}$$

where I is the identity matrix. $T = j_n$ is the trace of the matrix F^n and as $\det(F^n) = 2^n e^{n\pi i}$, we have $\sqrt{\det(F^n)} = 2^{n/2} e^{n\pi i/2}$. Thus, it is obtained that $\sqrt{T \pm 2\sqrt{\det(F^n)}} = 2^{n/2} \pm e^{n\pi i/2}$. We select firstly as

$$\sqrt{F^n} = \frac{\pm 1}{2^{n/2} + e^{n\pi i/2}} \begin{bmatrix} J_{n+1} + 2^{n/2} e^{n\pi i/2} & 2J_n \\ J_n & 2J_{n-1} + 2^{n/2} e^{n\pi i/2} \end{bmatrix}.$$

Using the modified Binet form in (1), we can write the values of elements (1,1) and (2,1) for the right side matrix as

$$\frac{\pm\left(2^{n/2} - e^{n\pi i/2}\right)\left(J_{n+1} + 2^{n/2} e^{n\pi i/2}\right)}{2^n - e^{n\pi i}} = \frac{\pm\left(2^{(n+2)/2} - e^{(n+2)\pi i/2}\right)}{3} = \pm J_{(n+2)/2},$$

and

$$\frac{\pm J_n\left(2^{n/2} - e_n^{n\pi i/2}\right)}{2^n - e^{n\pi i}} = \frac{\pm J_n\left(2^{n/2} - e^{n\pi i/2}\right)}{3J_n} = \pm J_{n/2}.$$

If the elements (1,2) and (2,2) are founded in the similar way, we get two square roots of the matrix F^n

$$F_1^{n/2} = \begin{bmatrix} J_{(n+2)/2} & 2J_{n/2} \\ J_{n/2} & 2J_{(n-2)/2} \end{bmatrix},\ F_2^{n/2} = -\begin{bmatrix} J_{(n+2)/2} & 2J_{n/2} \\ J_{n/2} & 2J_{(n-2)/2} \end{bmatrix}.$$

When we choose the other state of the matrix equation (4), we have

$$\sqrt{F^n} = \frac{\pm\left(2^{n/2} + e^{n\pi i/2}\right)}{2^n - e^{n\pi i}} \begin{bmatrix} J_{n+1} - 2^{n/2} e^{n\pi i/2} & 2J_n \\ J_n & 2J_{n-1} - 2^{n/2} e^{n\pi i/2} \end{bmatrix}.$$

If all elements are given in the same way, then the other two square roots matrices are possessed. ■

Moreover, it is $F^n = F_i^{n/2} F_i^{n/2}$ because the square roots of the matrix F^n are the matrices $F_i^{n/2}$ $(i = 1, 2, 3, 4)$. If we admit the matrices $F_i^{n/2}$ $(i = 1, 2)$, then the equations

$$J_{(n+2)/2}^2 + 2J_{n/2}^2 = J_{n+1},\ J_{(n+2)/2} J_{n/2} + 2J_{n/2} J_{(n-2)/2} = 2J_n$$

are determined by equating of corresponding elements for equal matrices. In addition, considering the elements of the matrices for $F_i^{n/2}$ $(i = 3, 4)$ cause

$$j_{(n+2)/2}^2 + 2j_{n/2}^2 = 9J_{n+1},\ j_{(n+2)/2} j_{n/2} + 2j_{n/2} j_{(n-2)/2} = 9J_n.$$

And also, taking the determinant of the matrices $F_i^{n/2}$ $(i = 1, 2, 3, 4)$ give

$$J_{(n+2)/2}J_{(n-2)/2} - J_{n/2}^2 = 2^{(n-2)/2}e^{n\pi i/2},\ j_{(n+2)/2}j_{(n-2)/2} - j_{n/2}^2 = -9e^{n\pi i/2}2^{(n-2)/2},$$

which are a general case of the Cassini-like formula for the Jacobsthal and Jacobsthal-Lucas numbers with specialized rational subscripts, respectively. Hence, the matrices $F_i^{n/2}$ $(i = 1, 2, 3, 4)$ in (3) are nonsingular, the inverses of the matrices $F_i^{n/2}$ are denominated notation $F_i^{-n/2}$ by given;

$$F_{1,2}^{-n/2} = \frac{\pm e^{-n\pi i/2}}{2^{n/2}} \begin{bmatrix} 2J_{(n-2)/2} & -2J_{n/2} \\ -J_{n/2} & J_{(n+2)/2} \end{bmatrix},\ F_{3,4}^{-n/2} = \frac{\mp e^{-n\pi i/2}}{3.2^{n/2}} \begin{bmatrix} 2j_{(n-2)/2} & -2j_{n/2} \\ -j_{n/2} & j_{(n+2)/2} \end{bmatrix}.$$

A lot of elementary properties for these numbers can be found by equating of corresponding elements for the equal matrices such as $F^{k/2}F^{(n+1)/2} = F^{(k+n+1)/2}$, $F^nF^{1/2} = F^{(2n+1)/2}$ and $F^{n/2}F^{(n+1)/2} = F^{(2n+1)/2}$ for all integer n and k. For equal matrices $F_i^{k/2}F_i^{(n+1)/2} = F_1^{(k+n+1)/2}$ $(i = 1, 2)$ or $(i = 3, 4)$, equating of corresponding elements gives following equalities

$$J_{(k+n+1)/2} = J_{k/2}J_{(n+3)/2} + 2J_{(k-2)/2}J_{(n+1)/2} = \frac{1}{9}\left(j_{(k+2)/2}j_{(n+1)/2} + 2j_{k/2}j_{(n-1)/2}\right).$$

Considering the equation $F^nF_i^{1/2} = F_1^{(2n+1)/2}$ $(i = 1, 3)$ and $F_i^{n/2}F_i^{(n+1)/2} = F_1^{(2n+1)/2}$ $(i = 1, 2, 3, 4)$ derive

$$\begin{aligned} J_{(2n+1)/2} &= J_nJ_{3/2} + 2J_{n-1}J_{1/2} = \frac{1}{3}\left(J_{n+1}j_{1/2} + 2J_nj_{-1/2}\right), \\ J_{(2n+1)/2} &= J_{(n+2)/2}J_{(n+1)/2} + 2J_{n/2}J_{(n-1)/2} = \frac{1}{9}\left(j_{n/2}j_{(n+3)/2} + 2j_{(n-2)/2}j_{(n+1)/2}\right). \end{aligned}$$

Computing the equalities $F_i^{(k-n)/2} = F_i^{k/2}F_i^{-n/2}$ $(i = 1, 2, 3, 4)$, we have the equalities

$$J_{(k-n)/2} = \frac{2^{-n/2}}{e^{n\pi i/2}}\left(J_{k/2}J_{(n+2)/2} - J_{(k+2)/2}J_{n/2}\right) = \frac{e^{-n\pi i/2}}{9.2^{n/2}}\left(j_{(k+2)/2}j_{n/2} - j_{k/2}j_{(n+2)/2}\right).$$

The next goal is to find different relations between the Jacobsthal and Jacobsthal-Lucas numbers with specialized rational subscripts by using the square roots of the matrix S^n.

Theorem 2.2 *Let $S_i^{n/2}$ $(i = 1, 2, 3, 4)$ denote a square root of the matrix S^n. Then,*

$$S_{1,2}^{n/2} = \frac{\pm 1}{2}\begin{bmatrix} j_{n/2} & 9J_{n/2} \\ J_{n/2} & j_{n/2} \end{bmatrix},\ S_{3,4}^{n/2} = \frac{\pm 3}{2}\begin{bmatrix} J_{n/2} & j_{n/2} \\ j_{n/2}/9 & J_{n/2} \end{bmatrix}.$$

Proof. If we apply for the Cayley-Hamilton method for computing the square roots of the matrix S^n, square root matrices are obtained in form

$$\sqrt{S^n} = \frac{\pm 1}{\sqrt{T \pm 2\sqrt{\det(S^n)}}} \left[S^n \pm \sqrt{\det(S^n)} I \right]. \tag{5}$$

where I is the identity matrix. $T = j_n$ is the trace of the matrix S^n and since $\det(S^n) = 2^n e^{n\pi i}$, we have $\sqrt{\det(S^n)} = 2^{n/2} e^{n\pi i/2}$. Therefore, it is obtained that $\sqrt{T \pm 2\sqrt{\det(S^n)}} = 2^{n/2} \pm e^{n\pi i/2}$. From this point of view, we select as

$$S^{n/2} = \frac{\pm 1}{2\left(2^{n/2} + e^{n\pi i/2}\right)} \begin{bmatrix} j_n + 2^{(n+2)/2} e^{n\pi i/2 2^{n/2}} & 9J_n \\ J_n & j_n + 2^{(n+2)/2} e^{n\pi i/2} \end{bmatrix}.$$

Using the modified Binet forms in (1) and (2), we write down the values of elements $(1,1)$ and $(1,2)$ for the right side matrix

$$\frac{\pm\left(2^{n/2} - e^{n\pi i/2}\right)\left(j_n + 2^{(n+2)/2} e^{n\pi i/2}\right)}{2\left(2^n - e^{n\pi i}\right)} = \frac{\pm\left(2^n - e^{n\pi i}\right)\left(2^{n/2} + e^{n\pi i/2}\right)}{6J_n} = \frac{\pm j_{n/2}}{2},$$

and

$$\frac{\pm 9J_n\left(2^{n/2} - e^{n\pi i/2}\right)}{2\left(2^n - e^{n\pi i}\right)} = \frac{\pm 9J_n\left(2^{n/2} - e^{n\pi i/2}\right)}{3J_n} = \pm 9J_{n/2}.$$

When other elements are given in the same way, we determine two square root matrices

$$S_1^{n/2} = \frac{1}{2} \begin{bmatrix} j_{n/2} & 9J_{n/2} \\ J_{n/2} & j_{n/2} \end{bmatrix}, \quad S_2^{n/2} = \frac{-1}{2} \begin{bmatrix} j_{n/2} & 9J_{n/2} \\ J_{n/2} & j_{n/2} \end{bmatrix}.$$

If we prefer the other circumstance in the matrix equation (5), then the other two matrices are procured by computing the above mentioned values of all elements for the right side matrix. ∎

We suppose that the matrix $S_i^{n/2}$ $(i = 1, 2, 3, 4)$ is one of the square roots of the matrix S^n. Equating corresponding elements of the equal matrices $S_i^{n/2} S_i^{n/2} = S^n$ gives following identities

$$j_{n/2}^2 + 9J_{n/2}^2 = 2j_n, \;\; J_{n/2} j_{n/2} = J_n,$$

and so taking the determinant of the matrices $S_i^{n/2}$ $(i = 1, 2, 3, 4)$ yields

$$j_{n/2}^2 - 9J_{n/2}^2 = 2^{(n+4)/2} e^{n\pi i/2}.$$

In addition to above mentioned equalities, different equalities for the Jacobsthal and Jacobsthal-Lucas numbers with specialized rational subscripts can be found by equating of corresponding elements for the equal matrices

$S^{k/2}S^{(n+1)/2} = S^{(k+n+1)/2}$, $S^nS^{1/2} = S^{(2n+1)/2}$ and $S^{n/2}S^{(n+1)/2} = S^{(2n+1)/2}$. Also, since the matrices $S_i^{n/2}$ $(i = 1, 2, 3, 4)$ are nonsingular, if the inverses of the matrices $S_i^{n/2}$ are shown with $S_i^{-n/2}$, we get

$$S_{1,2}^{-n/2} = \frac{\pm e^{-n\pi i/2}}{2^{(n+2)/2}} \begin{bmatrix} j_{n/2} & -9J_{n/2} \\ -J_{n/2} & j_{n/2} \end{bmatrix}, \; S_{3,4}^{-n/2} = \frac{\pm 3e^{-n\pi i/2}}{2^{(n+2)/2}} \begin{bmatrix} -J_{n/2} & j_{n/2} \\ j_{n/2}/9 & -J_{n/2} \end{bmatrix}.$$

For all integers n and k, equating of corresponding elements for the equal matrices $S_i^{(n+1)/2}S_i^{k/2} = S_1^{(n+k+1)/2}$ $(i = 1, 2, 3, 4)$, we get

$$\begin{aligned} 2j_{(n+k+1)/2} &= j_{(n+1)/2}j_{k/2} + 9J_{(n+1)/2}J_{k/2}, \\ 2J_{(n+k+1)/2} &= J_{(n+1)/2}j_{k/2} + j_{(n+1)/2}J_{k/2}. \end{aligned}$$

The equations of the matrices $S^nS_i^{1/2} = S_1^{(2n+1)/2}$ $(i = 1, 3)$ and $S_i^{n/2}S_i^{(n+1)/2} = S_1^{(2n+1)/2}$ $(i = 1, 2, 3, 4)$ give to equalities

$$\begin{aligned} 2J_{(2n+1)/2} &= J_nj_{1/2} + j_nJ_{1/2}, \; 2j_{(2n+1)/2} = j_nj_{1/2} + 9J_nJ_{1/2}, \\ 2j_{(2n+1)/2} &= j_{n/2}j_{(n+1)/2} + 9J_{n/2}J_{(n+1)/2}, \\ 2J_{(2n+1)/2} &= J_{n/2}j_{(n+1)/2} + j_{n/2}J_{(n+1)/2}. \end{aligned}$$

The equations $S_i^{k/2}S_i^{-n/2} = S_1^{(k-n)/2}$ $(i = 1, 2, 3, 4)$ generate to equalities

$$\begin{aligned} 2e^{n\pi i/2}j_{(k-n)/2} &= j_{k/2}j_{n/2} - 9J_{k/2}J_{n/2}, \\ 2e^{n\pi i/2}J_{(k-n)/2} &= J_{k/2}j_{n/2} - j_{k/2}J_{n/2}. \end{aligned}$$

Also, the computing square roots of different matrix generators of the Jacobsthal and Jacobsthal-Lucas numbers [3],[4] can be carried out in the above mentioned fashion.

Lastly, we establish extended matrices $F^{r/q}$ and $S^{r/q}$ for $q \in \mathbb{Z}^+$ and $r \in \mathbb{Z}$, which generate the Jacobsthal and Jacobsthal-Lucas number with rational subscripts. To do this, we use the connection between the equation $p(x) = J_rx^2 + (2J_{r-1} - J_{r+1})x - 2J_r$ $\left(or\ p(x) = \frac{J_r}{2}x^2 + \left(\frac{j_r - j_r}{2}\right)x - \frac{9J_r}{2}\right)$ and the roots matrices $F^{r/q}$ $\left(or\ S^{r/q}\right)$ via the Abel's functional equation [5]. The polynomial $p(x) = J_r(x^2 - x - 2)$ is related to the matrices F^r. Let $A(x) = \int \frac{dx}{p(x)}$ be, then

$$A(x) = \int \frac{dx}{J_r(x-2)(x+1)} = \frac{1}{3J_r} \ln\left(\frac{x-2}{x+1}\right).$$

We define a function $\Phi_{F^r}(x) = \frac{J_{r+1}x + 2J_r}{J_rx + 2J_{r-1}}$ for the matrix F^r. The Abel's functional equation $A(\Phi_{F^r}(x)) = A(x) + k$ is satisfied for the certain real constant k. Then,

$$\begin{aligned} A\left(\frac{J_{r+1}x + 2J_r}{J_rx + 2J_{r-1}}\right) &= \frac{1}{3J_r} \ln\left(\frac{x(J_{r+1} - 2J_r) + J_r - 2J_{r-1}}{x(J_{r+1} + J_r) + J_r + J_{r-1}}\right) \\ &= A(x) + \frac{1}{3J_r} \ln\left(\frac{e^{r\pi i}}{2^r}\right). \end{aligned}$$

A closed form for the q^{th} roots of the matrix F^r is derived from the functional equation $\Phi_{F^{r/q}}(x) = A^{-1}\left(A(x) + \frac{k}{q}\right)$, where the inverse function of $A(x)$ is shown with $A^{-1}(x)$ given by

$$A^{-1}(x) = \frac{e^{3xJ_r} + 2}{1 - e^{3xJ_r}}.$$

It follows that

$$\begin{aligned} \Phi_{F^{r/q}}(x) &= A^{-1}\left(\ln\left(\frac{x-2}{x+1}\right)^{\frac{1}{3J_r}} + \frac{1}{3J_r}\ln\left(\frac{e^{r\pi i/q}}{2^{r/q}}\right)\right) \\ &= \frac{e^{r\pi i/q}(x-2) + 2(x+1)2^{r/q}}{2^{r/q}(x+1) - e^{r\pi i/q}(x-2)} \\ &= \frac{x\left(2^{r/q}.2 + e^{r\pi i/q}\right) + 2\left(2^{r/q} - e^{r\pi i/q}\right)}{x\left(2^{r/q} - e^{r\pi i/q}\right) + 2.e^{r\pi i/q} + 2^{r/q}} = \frac{xJ_{r/q+1} + 2J_{r/q}}{xJ_{r/q} + 2J_{r/q-1}}. \end{aligned}$$

Hence the roots matrices $F^{r/q}$ is related to the function $\Phi_{F^{r/q}}(x)$, we have

$$F^{r/q} = \pm\begin{bmatrix} J_{(r+q)/q} & 2J_{r/q} \\ J_{r/q} & 2J_{(r-q)/q} \end{bmatrix}. \tag{6}$$

If we make out above mentioned calculation for the matrix S^r, that is, if the function $\Phi_{S^{r/q}}(x)$ is computed, we have

$$S^{r/q} = \frac{\pm 1}{2}\begin{bmatrix} j_{r/q} & 9J_{r/q} \\ J_{r/q} & j_{r/q} \end{bmatrix}. \tag{7}$$

Taking the determinant of the matrix equations (6) and (7) yields

$$\begin{aligned} J_{(r+q)/q}J_{(r-q)/q} - J_{r/q}^2 &= 2^{(r-q)/q}e^{r\pi i/q}, \\ j_{r/q}^2 - 9J_{r/q}^2 &= 4e^{r\pi i/q}2^{r/q}. \end{aligned}$$

Considering different rational powers of the matrix F and S, and using the following matrix equations,

$$\begin{aligned} F^{(rs+qt)/qs} &= F^{r/q}F^{t/s}, \left(r, t \in \mathbb{Z} \text{ and } q, s \in \mathbb{Z}^+\right), \\ S^{(rs+qt)/qs} &= S^{r/q}S^{t/s}, \end{aligned}$$

we determine the identities involving terms of the Jacobsthal and Jacobsthal-Lucas numbers with rational subscripts by given

$$\begin{aligned} J_{(rs+qt)/qs} &= J_{(r/q)+1}J_{t/s} + 2J_{r/q}J_{(t/s)-1} = J_{r/q}J_{(t/s)+1} + 2J_{(r/q)-1}J_{t/s}, \\ 2J_{(rs+qt)/qs} &= J_{r/q}j_{t/s} + j_{r/q}J_{t/s},\ 2j_{(rs+qt)/qs} = j_{r/q}j_{t/s} + 8J_{r/q}J_{t/s}. \end{aligned}$$

It shows that, the Jacobsthal and Jacobsthal-Lucas numbers with rational subscripts hold analogous identities for the classical Jacobsthal and Jacobsthal-Lucas numbers.

References

[1] A.F. Horadam, Jacobsthal and Pell Curves, *The Fib. Quart.*, **26** (1988), no. 1, 77 - 83.

[2] A.F. Horadam, Jacobsthal representation numbers, *The Fib. Quart.*, **34** (1996), no. 1, 40 - 54.

[3] F. Koken, D. Bozkurt, On the Jacobsthal Numbers by Matrix Methods, *Int. Journal Contemp. Math. Sciences,* **3** (2008), no. 13, 605 - 614.

[4] F. Köken, D. Bozkurt, On the Jacobsthal-Lucas Numbers by Matrix Method, *Int. Journal Contemp. Math. Sciences,* **3** (2008), no. 33, 1629 - 1633.

[5] S. Northshield, Square Roots of 2x2 Matrices, *Contemporary Mathematics,* **517** (2010), 289 - 304. http://dx.doi.org/10.1090/conm/517/10147

[6] D. Sullivan, The Square Roots of 2x2 Matrices, *Math. Magazine,* **66** (1993), no. 5, 314 - 316. http://dx.doi.org/10.2307/2690509

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