International Mathematical Forum, Vol. 11, 2016, no. 12, 573 - 582
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/imf.2016.6452

On Mildly Hurewicz Spaces

Ljubiša D.R. Kočinac

University of Niš, Faculty of Sciences and Mathematics
18000 Niš, Serbia

Abstract

We define and study a version of the classical Hurewicz covering property by using cover by sets which are both open and closed. We call this property mildly Hurewicz. Game-theoretic and Ramsey-theoretic characterizations of this property are given. Basic topological properties of mildly Hurewicz spaces are considered.

Mathematics Subject Classification: Primary 54D20; Secondary 05C55, 91A44

Keywords: Selection principles, mildly Hurewicz, game theory, Ramsey theory

1 Introduction

Clopen sets in a topological space are sets which are simultaneously open and closed, and clopen covers of a space are those covers whose all elements are clopen sets. The importance of clopen sets and clopen covers in topology is well known. They are used in definitions or characterizations of many topological concepts. For example, ultraparacompact and zero-dimensional spaces are defined in terms of sets and clopen covers, the Banaschewski compactification of a zero-dimensional space X is exactly the set of all ultrafilters on the Boolean algebra of clopen subsets of X, a known characterization of strong Eberlein compact spaces is given in terms of clopen covers (a space X is strong Eberlein compact space if and only if X has a point-finite T_0-separating clopen cover), and so on. We use here clopen covers in the theory of selection principles and define and study spaces that we call mildly Hurewicz.

2 Preliminaries

We use the usual topological terminology and notation as in [2]. X and Y denote topological spaces, $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ are the closure and interior of a subset A of a space X. $\mathbf{N}$ denotes the set of natural numbers, and $\mathbf{R}$ is the set of real numbers.

Now, we mention a few facts about selection principles that we consider in this article. More information about selection principles in topological spaces the interested reader can find in the survey papers [4, 5, 10, 12]. In this paper we deal mainly with a version of the classical *Hurewicz property* [3]: For each sequence $(\mathcal{U}_n)_{n\in\mathbf{N}}$ of open covers of a space X there is a sequence $(\mathcal{V}_n)_{n\in\mathbf{N}}$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$ belongs to $\cup\mathcal{V}_n = \cup\{V : V \in \mathcal{V}_n\}$ for all but finitely many n.

In [6] it was shown that this property is of the Menger-type property $\mathsf{S}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$ for suitable collections $\mathcal{A}$ and $\mathcal{B}$. Here $\mathcal{A}$ and $\mathcal{B}$ are sets of families of subsets of a set X, and $\mathsf{S}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$ is the following property: For each sequence $(A_n)_{n\in\mathbf{N}}$ of elements of $\mathcal{A}$ there is a sequence $(B_n)_{n\in\mathbf{N}}$ of finite sets such that for each n, $B_n \subseteq A_n$, and $\bigcup_{n\in\mathbf{N}} B_n \in \mathcal{B}$.

The prototype of this selection principle is the *Menger property* $\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$, introduced in [9] in a different form, where $\mathcal{O}$ is the collection of all open covers of a space X.

3 Definitions and Basic Results

A space X is *mildly compact* (*mildly Lindelöf*) if every clopen cover of X has a finite (countable) subcover [11]. These classes of spaces play an important role in the theory of function spaces.

We consider a selective version of mildly compact (and mildly Lindelöf) spaces which is related to the classical Hurewicz covering property.

Definition 3.1 *A space X is a* mildly Hurewicz *if for each sequence $(\mathcal{U}_n)_{n\in\mathbf{N}}$ of clopen covers of X there are finite sets $\mathcal{V}_n \subset \mathcal{U}_n$, $n \in \mathbf{N}$, such that each x belongs to $\bigcup\mathcal{V}_n = \bigcup\{V : V \in \mathcal{V}_n\}$ for all but finitely many n.*

It is understood that each mildly compact space is mildly Hurewicz and each mildly Hurewicz space is mildly Lindelöf. It is also evident that each clopen subset of a mildly Hurewicz space is also mildly Hurewicz, and that any continuous image of a mildly Hurewicz space is mildly Hurewicz. Every Hurewicz space is mildly Hurewicz.

Recall that a space X is *zero-dimensional* (in the sense of small inductive dimension) if it has a base consisting of clopen sets [2].

Proposition 3.2 *A zero-dimensional space X is mildly Hurewicz if and only if it is Hurewicz.*

Proof. Let X be a mildly Hurewicz space and let $(\mathcal{U}_n)_{n\in\mathbf{N}}$ be a sequence of open covers of X. As X is zero-dimensional each $\mathcal{U}_n$ can be replaced by a clopen cover $\mathcal{V}_n$ consisting of clopen basic sets. Then apply to the sequence $(\mathcal{V}_n)_{n\in\mathbf{N}}$ the fact that X is mildly Hurewicz and find for each n a finite subset $\mathcal{W}_n$ of $\mathcal{V}_n$ such that each $x \in X$ is contained in $\bigcup \mathcal{W}_n$ for all but finitely many n. Then set $\mathcal{H}_n = \{U_W \in \mathcal{U}_n : U_W \supset W, W \in \mathcal{W}_n\}$, $n \in \mathbf{N}$. Clearly, the sequence $(\mathcal{H}_n)_{n\in\mathbf{N}}$ witnesses for $(\mathcal{U}_n)_{n\in\mathbf{N}}$ that X is Hurewicz.

Example 3.3 (1) The space $X = [0,1] \setminus \{1/n : n \in \mathbf{N}\} \subset \mathbf{R}$ is a mildly Hurewicz (being mildly compact), non-compact space.

(2) The Sorgenfrey line $\mathbf{S}$ and the space $\mathbf{P}$ od irrational numbers (with the Euklidean topology inherited from the real line $\mathbf{R}$) are not mildly Hurewicz because they are zero-dimensional spaces which are not Hurewicz as it is well known.

(3) The space $[0, \omega_1)$ of all countable ordinals is not mildly Hurewicz (because it is not mildly Lindelöf).

We discuss now the behaviour of the mildly Hurewicz property under some classes of mappings.

Recall that a mapping $f : X \to Y$ is *contra-continuous* [1] if the preimage $f^{\leftarrow}(V)$ of an open set $V \subset X$ is closed in X, and *precontinuous* [8] if $f^{\leftarrow}(V) \subset \mathrm{Int}(\mathrm{Cl}(f^{\leftarrow}(V)))$ whenever V is open in Y.

Theorem 3.4 *A contra-continuous and precontinuous image $Y = f(X)$ of a mildly Hurewicz space X is a Hurewicz space.*

Proof. Let $(\mathcal{V}_n)_{n\in\mathbf{N}}$ be a sequence of open covers of Y. Since f is contra-continuous for each $n \in \mathbf{N}$ and each $V \in \mathcal{V}_n$ the set $f^{\leftarrow}(V)$ is closed in X. On the other hand, because f is precontinuous $f^{\leftarrow}(V) \subset \mathrm{Int}(\mathrm{Cl}(f^{\leftarrow}(V)))$, so that $f^{\leftarrow}(V) \subset \mathrm{Int}(f^{\leftarrow}(V))$, i.e. $f^{\leftarrow}(V) = \mathrm{Int}(f^{\leftarrow}(V))$. Therefore, for each n, the set $\mathcal{U}_n = \{f^{\leftarrow}(V) : V \in \mathcal{V}_n\}$ is a clopen cover of X. As X is a mildly Hurewicz space there is a sequence $(\mathcal{G}_n)_{n\in\mathbf{N}}$ such that for each n, $\mathcal{G}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$ belongs to $\bigcup \mathcal{G}_n$ for all but finitely many n. Let $\mathcal{W}_n = \{f(G) : G \in \mathcal{G}_n\}$. Then for each n, $\mathcal{W}_n$ is a finite subset of $\mathcal{V}_n$. Let $y = f(x) \in Y$. As $x \in \bigcup \mathcal{G}_n$ for all but finitely many n, we have that $y \in \bigcup \mathcal{W}_n$ for all but finitely many n. This means that Y is a Hurewicz space.

Recall that a mapping $f : X \to Y$ is called *weakly continuous* [7] if for each $x \in X$ and each neighbourhood V of $f(x)$ there is a neighbourhood U of x such that $f(U) \subset \mathrm{Cl}(V)$.

Theorem 3.5 *If $f : X \to Y$ is weakly continuous mapping from a Hurewicz space X onto a space Y, then Y is mildly Hurewicz.*

Proof. Let $(\mathcal{V}_n)_{n\in\mathbf{N}}$ be a sequence of clopen covers of Y. For each $x \in X$ and each $n \in \mathbf{N}$ there is $V_{n,x} \in \mathcal{V}_n$ containing $f(x)$. Since f is weakly continuous there is an open neighbourhood $U_{n,x}$ of x such that $f(U_{n,x}) \subset \mathrm{Cl}(V_{n,x})$, i.e. $f(U_{n,x}) \subset V_{n,x}$. Put $\mathcal{U}_n = \{U_{n,x} : x \in X\}$, $n \in \mathbf{N}$. Then $(\mathcal{U}_n)_{n\in\mathbf{N}}$ is a sequence of open covers of X. Since X is Hurewicz, there is a sequence $(\mathcal{H}_n)_{n\in\mathbf{N}}$ such that for each n, $\mathcal{H}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$ belongs to $\bigcup\mathcal{H}_n$ for all but finitely many n. Set $\mathcal{W}_n = \{V_{n,x} : f(H) \subset V_{n,x}, H \in \mathcal{H}_n\}$. We get a sequence $(\mathcal{W}_n)_{n\in\mathbf{N}}$ of finite sets such that $\mathcal{W}_n \subset \mathcal{V}_n$ for each $n \in \mathbf{N}$. Let $y \in Y$ and let $x \in X$ be such that $y = f(x)$. As $x \in \bigcup\mathcal{H}_n$ for all but finitely many n, we have

$$y = f(x) \in f(\cup\mathcal{H}_n) \subset \cup\mathcal{W}_n$$

for all but finitely many n, i.e. Y is mildly Hurewicz.

Theorem 3.6 *If $f : X \to Y$ is an open, perfect mapping from a space X onto a Hurewicz space Y, then X is mildly Hurewicz.*

Proof. Let $(\mathcal{U}_n)_{n\in\mathbf{N}}$ be a sequence of clopen covers of X. For each $y \in Y$ the set $F_y := f^{\leftarrow}(y)$ is compact so that for each $n \in \mathbf{N}$ there is a finite set $\mathcal{V}_{y,n} \subset \mathcal{U}_n$ which covers F_y. Let $V_{y,n} = \bigcup\mathcal{V}_{y,n}$. As f is a closed mapping, for each $n \in \mathbf{N}$ and each $y \in Y$ there is an open set $W_{y,n} \subset Y$ such that $y \in W_{y,n}$ and $f^{\leftarrow}(W_{y,n}) \subset V_{y,n}$. For each $n \in \mathbf{N}$ set $\mathcal{W}_n = \{W_{y,n} : y \in Y\}$. Then each $\mathcal{W}_n$ is an open cover of Y. Since Y is Hurewicz, there is a sequence $(\mathcal{H}_n)_{n\in\mathbf{N}}$ such that $\mathcal{H}_n$ is a finite subset of $\mathcal{W}_n$, $n \in \mathbf{N}$, and each $y \in Y$ belongs to all but finitely many sets $\bigcup\mathcal{H}_n$. For each n and each $H \in \mathcal{H}_n$ there is a finite $\mathcal{U}_{H,n} \subset \mathcal{U}_n$ with $f^{\leftarrow}(H) \subset \bigcup\mathcal{U}_{H,n}$. If $\mathcal{G}_n = \{U \in \mathcal{U}_n : U \in \mathcal{U}_{H,n}, H \in \mathcal{H}_n\}$, then $\mathcal{G}_n$ is a finite subset of $\mathcal{U}_n$ for each n. We prove that the sequence $(\mathcal{G}_n)_{n\in\mathbf{N}}$ witnesses for the given sequence $(\mathcal{U}_n)_{n\in\mathbf{N}}$ that X is mildly Hurewicz. Indeed, let $x \in X$ and $y = f(x)$. Then $y \in \bigcup\mathcal{H}_n$ for all but finitely many n, say for all $n \geq n_0$. We have that for each $n \geq n_0$, $x \in f^{\leftarrow}(\bigcup\mathcal{H}_n) \subset \bigcup\mathcal{G}_n$.

4 Characterizing Mildly Hurewicz Spaces

In this section we characterize the mildly Hurewicz property game-theoretically and Ramsey-theoretically.

The symbol $\mathsf{H}_{\mathsf{m}}(X)$ denotes the following *mildly Hurewicz game* on X: players ONE and TWO play a round for each $n \in \mathbf{N}$. In the nth round player ONE chooses a clopen cover $\mathcal{U}$ for X and then TWO chooses a finite set $\mathcal{V}_n \subset \mathcal{U}_n$. TWO wins a play $\mathcal{U}_1, \mathcal{V}_1; \mathcal{U}_2, \mathcal{V}_2; \cdots$ if each $x \in X$ belongs to $\bigcup\mathcal{V}_n$ for all but finitely many $n \in \mathbf{N}$; otherwise ONE wins.

As we have already observed spaces X having the mildly Hurewicz property satisfy: each clopen cover of X has a countable subcover, i.e. each mildly Hurewicz space is mildly Lindelöf. Therefore, when we work with the mildly Hurewicz property, we may assume that all clopen covers of a space are countable. Note also that a space X has the mildly Hurewicz property whenever ONE does not have the winning strategy in the game $\mathsf{H_m}(X)$.

We use the following notation for a space X:

- $\mathcal{C}_{\text{clop}}$ is the family of all clopen covers of X;
- Ω_{clop} denotes the collection of all clopen covers $\mathcal{U}$ of X such that each finite subset of X is contained in a member of $\mathcal{U}$ and $X \notin \mathcal{U}$;
- $\mathcal{C}^{\text{gp}}_{\text{clop}}$ denotes the collection of all groupable clopen covers of X; a clopen cover $\mathcal{U}$ of X is *groupable* if it can be represented in the form $\mathcal{U} = \bigcup_{n\in\mathbf{N}} \mathcal{U}_n$, where $\mathcal{U}_n$'s are finite, pairwise disjoint and each $x \in X$ belongs to $\bigcup \mathcal{U}_n$ for all but finitely many n (compare with [6]).

Notice that any $\mathcal{U} \in \Omega_{\text{clop}}$ satisfies:

> For each k and each partition $\mathcal{U} = \mathcal{U}_1 \cup \cdots \mathcal{U}_k$ there is an $i \leq k$ with $\mathcal{U}_i \in \Omega_{\text{clop}}$.

A space X is called *ω-mildly Lindelöf* if each cover in Ω_{clop} has a countable subcover.

Recall the following notion in Ramsey theory, called the *Baumgartner-Taylor partition relation* (see [4, 10]). For each positive integer k,

$$\mathcal{A} \to \lceil \mathcal{B} \rceil^2_k$$

denotes the following statement:

> For each A in $\mathcal{A}$ and for each function $f : [A]^2 \to \{1, \cdots, k\}$ there are a set $B \in \mathcal{B}$ with $B \subset A$, a $j \in \{1, \cdots, k\}$, and a partition $B = \bigcup_{n\in\mathbf{N}} B_n$ of B into pairwise disjoint finite sets such that for each $\{a, b\} \in [B]^2$ for which a and b are not from the same B_n, we have $f(\{a, b\}) = j$.

(The set B is called nearly homogenous of colour j.) Here $[A]^2$ denotes the set of all two-element subsets of A.

Theorem 4.1 *For an ω-mildly Lindelöf space X the following statements are equivalent:*

(1) *X has the mildly Hurewicz property;*

(2) *X satisfies* $\mathsf{S}_{\rm fin}(\Omega_{\rm clop}, \mathcal{C}_{\rm clop}^{\rm gp})$.

(3) ONE *does not have a winning strategy in the mildly Hurewicz game* $\mathsf{H}_{\mathsf{m}}(X)$ *on* X*;*

(4) ONE *has no winning strategy in the game* $\mathsf{G}_{\rm fin}(\Omega_{\rm clop}, \mathcal{C}_{\rm clop}^{\rm gp})$*;*

(5) *For each* $k \in \mathbf{N}$ *the partition relation* $\Omega_{\rm clop} \to \lceil \mathcal{C}_{\rm clop}^{\rm gp} \rceil_k^2$ *holds;*

Proof. The proof will be given by showing (1) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (3) Let φ be a strategy for ONE.

Round 1: In the first round ONE chooses a countable cover $\varphi(\emptyset) = \mathcal{U}_1 \in \mathcal{C}_{\rm clop}$, say $\mathcal{U}_1 = \{U_n : n \in \mathbf{N}\}$. One may assume that TWO's response is $\{U_i : i < n_1\}$, a finite initial part of $\mathcal{U}_1$.

Round 2: ONE chooses a countable clopen cover $\mathcal{U}_2 = \varphi(\{U_i : i < n_1\}) = \{U_{n_1,n} : n \in \mathbf{N}\}$. Suppose TWO's response is $\{U_{n_1,i} : i < n_2\}$, a finite subset of $\mathcal{U}_2 = \{U_{n_1,n} : n \in \mathbf{N}\}$.

Round 3: ONE takes another clopen cover $\mathcal{U}_3 = \varphi(\{U_i : i < n_1\}, \{U_{n_1,j} : j < n_2\}) = \{U_{n_1,n_2,n} : n \in \mathbf{N}\} \in \mathcal{C}_{\rm clop}$, and TWO chooses $\{U_{n_1,n_2,i} : i < n_3\}$, a finite subset of $\mathcal{U}_3$.

Round k: In the k-th round ONE chooses a clopen cover $\mathcal{U}_k = \varphi((\{U_i : i < n_1\}, \{U_{n_1,i} : i < n_2\}, \cdots, \{U_{n_1,n_2,\cdots,n_{k-1}}\}) = \{U_{n_1,n_2,\cdots,n_{k-1},n} : n \in \mathbf{N}\}$, and let TWO's response be $\{U_{n_1,n_2,\cdots,n_{k-1},i} : i < n_k\}$. And so on.

By (1), for each finite sequence s of natural numbers there is an $n_s \in \mathbf{N}$ such that the finite sets

$$\mathcal{V}_s = \{U_{s\frown m} : m \leq n_s\}$$

satisfy that each $x \in X$ belongs to all but finitely many sets $\bigcup \mathcal{V}_s$.

Define now inductively a sequence $n_1, n_2, \cdots$ in $\mathbf{N}$ so that $n_{m+1} = n_{n_1,\cdots,n_m}$, and take

$$\mathcal{V}_{n_1}, \mathcal{V}_{n_1,n_2}, \cdots, \mathcal{V}_{n_1,n_2,\cdots,n_m}, \cdots.$$

Then

$$X = \bigcup_{m\in\mathbf{N}} \bigcap_{k>m} \bigcup \mathcal{V}_{n_1,\cdots,n_k},$$

i.e. φ is not a winning strategy for ONE.

(3) $\Rightarrow$ (4) Let φ be a strategy for ONE in the game $\mathsf{G}_{\rm fin}(\Omega_{\rm clop}, \mathcal{C}_{\rm clop}^{\rm gp})$. We can use φ to define a strategy ψ for ONE of the mildly Hurewicz game as follows:

Round 1: $\psi(\emptyset) = \varphi(\emptyset) = \mathcal{U}_1 \in \Omega_{\rm clop}$. Let TWO respond with $\mathcal{V}_1$, a finite subset of $\mathcal{U}_1$, in the mildly Hurewicz game.

Round 2: ONE computes $\varphi(\mathcal{V}_1)$, and then plays $\mathcal{U}_2 = \psi(\mathcal{V}_1) = \varphi(\mathcal{V}_1) \setminus \mathcal{V}_1$. Suppose that TWO responds in the mildly Hurewicz game with $\mathcal{V}_2$, a finite subset of $\mathcal{U}_2$.

Round 3: ONE computes $\varphi(\mathcal{V}_1, \mathcal{V}_2)$, and then responds with $\mathcal{U}_3 = \psi(\mathcal{V}_1, \mathcal{V}_2) = \varphi(\mathcal{V}_1, \mathcal{V}_2) \setminus (\mathcal{V}_1 \cup \mathcal{V}_2)$.

And so on. Observe that ψ is really a strategy for ONE in the mildly Hurewicz game $\mathsf{H_m}(X)$.

By (3) ψ is not a winning strategy for ONE of the game $\mathsf{H_m}(X)$. Consider a ψ–play lost by ONE:

$$\psi(\emptyset),\, \mathcal{V}_1;\, \psi(\mathcal{V}_1),\, \mathcal{V}_2;\, \psi(\mathcal{V}_1, \mathcal{V}_2),\, \mathcal{V}_3;\, \cdots.$$

The definition of ψ, implies that the sets $\mathcal{V}_n$ are pairwise disjoint. Since TWO wins, each element of X belongs to $\bigcup \mathcal{V}_n$ for all but finitely many n. Therefore, Consequently, the finite sets $\mathcal{V}_n$, $n \in \mathbf{N}$, ensure that $\bigcup_{n\in\mathbf{N}} \mathcal{V}_n$ belongs to $\mathcal{C}^{\mathrm{gp}}_{\mathrm{clop}}$.

Since

$$\varphi(\emptyset), \mathcal{V}_1; \varphi(\mathcal{V}_1), \mathcal{V}_2; \varphi(\mathcal{V}_1, \mathcal{V}_2), \mathcal{V}_3; \cdots$$

is a play of the game $\mathsf{G}_{\mathrm{fin}}(\Omega_{\mathrm{clop}}, \mathcal{C}^{\mathrm{gp}}_{\mathrm{clop}})$, we have that φ is not a winning strategy for ONE in this game, i.e. (4) is true.

(4) $\Rightarrow$ (5) Let $\mathcal{U} = \{U_1, U_2, \cdots\} \in \Omega_{\mathrm{clop}}$ and $f : [\mathcal{U}]^2 \to \{1, \cdots, k\}$ be given. Let

$$\mathcal{V}_j = \{U_i : i > 1 \text{ and } f(\{U_1, U_i\}) = j\}, j = 1, 2, \cdots, k.$$

We have a partition $\mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_k$ of $\mathcal{U} \setminus \{U_1\}$ into k many pieces. There exists a j such that $\mathcal{V}_j \in \Omega_{\mathrm{clop}}$. Fix such a j and put $i_1 = j$ and $\mathcal{U}_1 = \mathcal{V}_j$. Now define

$$\mathcal{V}_j = \{U_i : i > 2 \text{ and } f(\{U_2, U_i\}) = j\}.$$

We get a partition of $\mathcal{U} \setminus \{U_1, U_2\}$ into k many pieces. One of them, say $\mathcal{V}_m$, is in Ω_{clop}. denote this m by i_2 and put $\mathcal{U}_2 = \mathcal{V}_m$. In a similar way we choose $\mathcal{U}_n \in \Omega_{\mathrm{clop}}$ and $i_n \in \mathbf{N}$, $n \geq 3$, such that $\mathcal{U}_1 \subset \mathcal{U}_2, \subset \cdots \subset \mathcal{U}_n \subset \cdots$ and

$$\mathcal{U}_{n+1} = \{U_i \in \mathcal{U}_n : i > n+1 \text{ and } f(\{U_{n+1}, U_i\}) = i_{n+1}\}.$$

For each $j \in \{1, \cdots, k\}$ define

$$\mathcal{V}_j = \{U_n : i_n = j\}.$$

We have that for each n,

$$\mathcal{U}_n \cap \mathcal{V}_1, \cdots, \mathcal{U}_n \cap \mathcal{V}_k$$

is a partition of $\mathcal{U}_n$ into k many pieces. Thus there is a j_n with $\mathcal{U}_n \cap \mathcal{V}_{j_n} \in \Omega_{\mathrm{clop}}$. Since for each n we have $\mathcal{U}_n \supset \mathcal{U}_{n+1}$, we may assume that the same j_n, denote it again by j, works for all $\mathcal{U}_n$'s.

Define a strategy φ for ONE in the game $\mathsf{G}_{\mathrm{fin}}(\Omega_{\mathrm{clop}}, \mathcal{C}^{\mathrm{gp}}_{\mathrm{clop}})$ as follows:

One first play $\varphi(\emptyset) = \mathcal{U}_1 \cap \mathcal{V}_j$. If TWO responds by choosing the finite set $\mathcal{H}_1 \subset \sigma(\emptyset)$, then ONE finds $n_1 = \max\{n : U_n \in \mathcal{V}_1\}$, and plays $\varphi(\mathcal{H}_1) = \mathcal{U}_{n_1} \cap \mathcal{V}_j$. If TWO responds selecting the finite set $\mathcal{H}_2 \subset \varphi(\mathcal{H}_1)$, then ONE computes $n_2 = \max\{n : U_n \in \mathcal{H}_2\}$ and plays $\varphi(\mathcal{H}_1, \mathcal{H}_2) = \mathcal{U}_{n_2} \cap \mathcal{V}_j$, and so on.

Notice that by the definition of the $\mathcal{U}_j$'s, during any φ-play we will have $n_1 < n_2 < \cdots$, and that $\mathcal{H}_m$'s are pairwise disjoint.

Use now the fact that ONE has no winning strategy in $\mathsf{G}_{\rm fin}(\Omega_{\rm clop}, \mathcal{C}^{\rm gp}_{\rm clop})$. There is a φ-play

$$\varphi(\emptyset), \mathcal{H}_1; \varphi(\mathcal{H}_1), \mathcal{H}_2; \varphi(\mathcal{H}_1, \mathcal{H}_2), \mathcal{H}_3; \cdots$$

lost by ONE. It is not easy to prove that moves by TWO give the required nearly homogeneous set of colour j for f which is in $\mathcal{C}^{\rm gp}_{\rm clop}$.

(5) $\Rightarrow$ (2) Let $(\mathcal{U}_n)_{n\in\mathbf{N}}$ be a sequence of (as we can assume) countable elements of $\Omega_{\rm clop}$. Let $\mathcal{U}_n = \{U_{n,k} : k \in \mathbf{N}\}$. Define $\mathcal{V}$ to be the collection of nonempty sets of the form $U_{1,n} \cap U_{n,k}$. It is clear that $\mathcal{V} \in \Omega_{\rm clop}$. For each $V \in \mathcal{V}$ choose a representation of the form $V = U_{1,n} \cap U_{n,k}$. The define the function $f : [\mathcal{V}]^2 \to \{1, 2\}$ as follows:

$$f(\{U_{1,n_1} \cap U_{n_1,k}, U_{1,n_2} \cap U_{n_2,j}\}) = \begin{cases} 1 & \text{if } n_1 = n_2, \\ 2 & \text{otherwise.} \end{cases}$$

By (5), choose a nearly homogeneous of color j set $\mathcal{W} \subset \mathcal{V}$ with $\mathcal{W} \in \mathcal{C}^{\rm gp}_{\rm clop}$. Assume $\mathcal{W} = \bigcup_{k\in\mathbf{N}} \mathcal{W}_k$ is a sequence of finite, pairwise disjoint sets such that for A and B from distinct $\mathcal{W}_k$'s we have $f(\{A, B\}) = j$. We have the following two possibilities.

Case 1: $j = 1$. There is an n such that for all $A \in \mathcal{W}$ we have $A \subset U_{1,n}$. This implies that $\mathcal{W}$ does not belong to $\mathcal{C}^{\rm gp}_{\rm clop}$. Thus, this case does not hold.

Case 2: $j = 2$. For each $n > 1$ define

$$\mathcal{H}_n = \{U_{n,i} : U_{n,i} \text{ is the second coordinate in the representation of an } W \in \mathcal{W}\}$$

and set $\mathcal{H} = \bigcup_{n\in\mathbf{N}} \mathcal{H}_n$. Then $\mathcal{H}$ is the union of finite subsets of $\mathcal{U}_n$, $n \in \mathbf{N}$.

It is easily checked that $\mathcal{H} \in \mathcal{C}^{\rm gp}_{\rm clop}$. If for some n, $\mathcal{H}_n$ was not defined, we set $\mathcal{H}_n = \emptyset$. So we get the sequence $(\mathcal{H}_n)_{n\in\mathbf{N}}$ witnessing for $(\mathcal{U}_n)_{n\in\mathbf{N}}$ that (2) holds.

(2) $\Rightarrow$ (1) Let $(\mathcal{U}_n)_{n\in\mathbf{N}}$ be a sequence of clopen covers of X. We again assume that all these covers are countable, and else that none contains a finite subcover of X.

For each n define the set $\mathcal{V}_n$ to be the set of finite unions of elements of $\mathcal{U}_n$. Of course, each $\mathcal{V}_n$ is in $\Omega_{\rm clop}$ and is countable, say $\mathcal{V}_n = \{V_{n,k} : k \in \mathbf{N}\}$. From covers $\mathcal{V}_n$, $n \in \mathbf{N}$, we define new covers $\mathcal{W}_n \in \Omega_{\rm clop}$ in the following way:

$n = 1$: $\mathcal{W}_1 = \mathcal{V}_1$;

$n > 1$: $\mathcal{W}_n = \{V_{1,m_1} \cap V_{2,m_2} \cap \cdots \cap V_{n,m_n} : n < m_1 < m_2 < \cdots < m_n\} \setminus \{\emptyset\}$. For each element of $\mathcal{W}_n$ choose a representation of the form $V_{1,m_1} \cap V_{2,m_2} \cap \cdots \cap V_{n,m_n}$ with $n < m_1 < m_2 < \cdots < m_n$.

Apply (2) to the sequence $(\mathcal{W}_n)_{n\in\mathbf{N}}$ to find for each n a finite set $\mathcal{G}_n \subset \mathcal{W}_n$ such that $\bigcup_{n\in\mathbf{N}} \mathcal{G}_n \in \mathcal{C}^{\text{gp}}_{\text{clop}}$. Thus we can choose finite, pairwise disjoint sets $\mathcal{H}_n$, $n \in \mathbf{N}$, such that $\bigcup_{n\in\mathbf{N}} \mathcal{G}_n = \bigcup_{n\in\mathbf{N}} \mathcal{H}_n$, and each $x \in X$ belongs to $\bigcup \mathcal{H}_n$ for all but finitely many n.

Let $n_1 > 1$ be large enough so that $\mathcal{H}_{n_1} \subset \bigcup_{j>1} \mathcal{G}_j$, and let $\mathcal{F}_1$ be the set of $V_{1,k}$ that occurs in the chosen representations of elements of $\mathcal{H}_{n_1}$. Then choose $n_2 > n_1$ so large that $\mathcal{H}_{n_2} \subset \bigcup_{j>2} \mathcal{G}_j$. Denote by $\mathcal{F}_2$ the set of $V_{2,k}$ that appear in the chosen representations of elements of $\mathcal{V}_{n_2}$. Continuing in this way we obtain finite sets $\mathcal{F}_n \subset \mathcal{V}_n$ such that each element of X belongs to $\bigcup \mathcal{F}_n$ for all but finitely many n.

For each element G of $\mathcal{G}_n$ choose finitely many elements of $\mathcal{U}_n$ whose union is G and let $\mathcal{L}_n$ denote the finite set of elements of $\mathcal{U}_n$ chosen in this way. Then the sequence $(\mathcal{L}_n)_{n\in\mathbf{N}}$ witnesses for $(\mathcal{U}_n)_{n\in\mathbf{N}}$ the mildly Hurewicz property of X.

References

[1] J. Dontchev, Contra-continuous functions and strongly S-closed spaces, *International Journal of Mathematics and Mathematical Sciencis*, **19** (1996), 303 - 310. http://dx.doi.org/10.1155/s0161171296000427

[2] R. Engelking, *General Topology*, 2nd Edition, Sigma Ser. Pure Math., Vol. 6, Heldermann, Berlin, 1989.

[3] W. Hurewicz, Über eine Verallgemeinerung des Borelschen Theorems, *Mathematische Zeitschrift*, **24** (1926), 401 - 421. http://dx.doi.org/10.1007/bf01216792

[4] Lj.D.R. Kočinac, Selected results on selection principles, *Proceedings of the Third Seminar on Geometry and Topology*, July 15–17, 2004, Tabriz, Iran, 71 - 104.

[5] Lj.D.R. Kočinac, Star selection principles: A survey, *Khayyam Journal of Mathematics*, **1** (2015), no. 1, 82 - 106.

[6] Lj.D.R. Kočinac and M. Scheepers, Combinatorics of open covers (VII): Groupability, *Fundamenta Mathematicae*, **179** (2003), 131 - 155. http://dx.doi.org/10.4064/fm179-2-2

[7] N. Levine, A decpomposition of continuity in topological spaces, *American Mathematical Monthly*, **68** (1961), 44 - 46. http://dx.doi.org/10.2307/2311363

[8] A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deep, On precontinuous and weak precontinuous mappings, *Proceedings of the Mathematical and Physical Society of Egypt*, **53** (1982), 47 - 53.

[9] K. Menger, Einige Überdeckungssätze der Punktmengenlehre, *Sitzungsberischte Abt. 2a, Mathematik, Astronomie, Physik, Meteorologie und Mechanik (Wiener Akademie, Wien)*, **133** (1924), 421 - 444.

[10] M. Sakai and M. Scheepers, The combinatorics of open covers, Chapter in *Recent Progress in General Topology III*, Atlantis Press, 2014, 751 - 799. http://dx.doi.org/10.2991/978-94-6239-024-9_18

[11] R. Staum, The algebra of bounded continuous functions into a nonarchimedean field, *Pacific Journal of Mathematics*, **50** (1974), 169 - 185. http://dx.doi.org/10.2140/pjm.1974.50.169

[12] B. Tsaban, Some new directions in infinite-combinatorial topology, Chapter in *Set Theory, Trends in Mathematics*, Birkhäuser Basel, 2006, 225 - 255. http://dx.doi.org/10.1007/3-7643-7692-9_7

Received: May 7, 2016; Published: June 14, 2016