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Cardinality of Accumulation Points of Infinite Sets

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Abstract

One of the fundamental theorems in real analysis is the Bolzano-Weierstrass property according to which every bounded infinite set of real numbers has an accumulation point. Since this theorem essentially asserts the completeness of the real numbers, the notion of accumulation point becomes substantial. This work provides an efficient number of examples which cover every possible case in the study of accumulation points, classifying the different sizes of the derived set A' and of the sets $A \cap A'$, $A' \setminus A$, for an infinite set A.

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1 Introduction

The "accumulation point" is a mathematical notion due to Cantor ([2]) and although it is fundamental in real analysis, it is also important in other areas of pure mathematics, such as the study of metric or topological spaces. Following the usual notation for a metric space (X, d), we denote by $V(x_0, \varepsilon)$ $= \{x \in X \mid d(x, x_0) < \varepsilon\}$ the *open sphere of center* x_0 *and radius* ε and by $D(x_0, \varepsilon)$ the set $V(x_0, \varepsilon) \setminus \{x_0\}$. Let $A \subseteq X$. The point $x_0 \in X$ is called *accumulation point of* A if for every $\varepsilon > 0$ the set $A \cap D(x_0, \varepsilon)$ is non-empty.

The set of all accumulation points of A is denoted by A' and it is called *derived set of* A. It is obvious that $x_0 \in X$ is an accumulation point of A if and only if for every $\varepsilon > 0$, $V(x_0, \varepsilon)$ contains infinitely many points of A. The point $x_0 \in A$ is called *isolated point of* A if it is not an accumulation point of A.

It is clear that a finite set does not have any accumulation point. The determination of all accumulation points of an infinite set A can be simplified by examining its subsets since, if A is a finite union $A = \bigcup_{i=1}^{k} A_i$ then $A' = \bigcup_{i=1}^{k} A_i'$ and if $A = \bigcup_{i \in I} A_i$ where I is countable, then $\bigcup_{i \in I} A_i' \subseteq A'$.

In this paper, we study the question "How many accumulation points could an infinite set have?" using a variety of examples that cover the whole range of possible answers, simultaneously clarifying the distinction between accumulation points which are contained in the original set and those which are not. For an infinite set A, with derived set A', we distinguish three general cases in order to classify $card(A')$, $card(A \cap A')$ and $card(A' \setminus A)$:

- *A and A' both being countable sets.* If A' is finite, we arrive at a model that covers all possible demands: for given numbers $k, l \in \{0, 1, 2, \ldots\}$, with $l \leq k$, we specify a set A having k accumulation points, l of which are contained in A. If A' is a countable infinite set, we focus on the cardinality of $A \cap A'$, arriving at examples that represent four study groups: The set A does not contain any of its accumulation points, A contains a finite number of its accumulation points, A contains all of its accumulation points, A contains a countable infinite subset of its accumulation points while the remaining accumulation points form a finite or infinite set.

- *A being a countable set and A' being an uncountable set.* In this case, we present examples that illustrate two anti-diametric possibilities: all elements of A are isolated points or all elements of A are accumulation points.

- *A being an uncountable set.* It is known that, in this case, the derived set A' is uncountable and moreover A contains uncountably many of its accumulation points. Thus, we can only study the cardinality of $A' \setminus A$, arriving at examples that represent four study groups: The set A contains all of its accumulation points, A has a finite number of accumulation points which are not contained in it, $A' \setminus A$ is an infinite countable or uncountable set.

We restrict our examples mainly in subsets of real numbers and only for one example we use a subset of $\mathbb{R}^2$ endowed with the euclidean distance. Many of the examples used in this work can be found either as examples or exercises in various books. At the end of the paper, we cite indicative bibliography.

2 Countable infinite sets with countable derived set

We present, at first, examples of countable infinite subsets A of $\mathbb{R}$, for which A' is a finite set, covering all possible cases. Proofs are omitted, since they follow easily.

The most well known example of a set without accumulation points is the set $\mathbb{N}$ of natural numbers. A basic example for the case $card(A') = 1$ and $A \cap A' = \emptyset$ is the set $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$ which has, but not contains, the point 0 as its only accumulation point. Using the structure of this set we can construct examples for any case where A' is finite and $A \cap A' = \emptyset$. Two characteristic constructions, for $k \in \mathbb{N}$, are the set $A = \{k + \frac{1}{n} \mid n \in \mathbb{N}\}$ which has, but not contains, the natural number k as its only accumulation point and the set $B = \{1+\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{3+\frac{1}{n} \mid n \in \mathbb{N}\} \cup \cdots \cup \{(2k-1)+\frac{1}{n} \mid n \in \mathbb{N}\}$ which has exactly k accumulation points, none of which is contained in it.

If we extend these sets, we can cover the case where $card(A \cap A') = card(A') > 0$. Two extreme cases can be presented by the set $A = \{k, k + \frac{1}{n} \mid n \in \mathbb{N}\}$ which contains the natural number k as its only accumulation point and the set $B = \{1\} \cup \{1 + \frac{1}{n} \mid n \in \mathbb{N}\} \cup \{2 + \frac{1}{n} \mid n \in \mathbb{N}\} \cup \{3 + \frac{1}{n} \mid n \in \mathbb{N}\} \cup \cdots \cup \{k + \frac{1}{n} \mid n \in \mathbb{N}\}$ which has exactly k accumulation points, all of which are contained in it.

All other cases, with $0 < card(A \cap A') < card(A')$ are covered by the following example.

Example 2.1. *Let $k, l \in \mathbb{N}$ with $l < k$. The set*

$$\begin{aligned} A &= \{1, 1 + \tfrac{1}{n} \mid n \in \mathbb{N}\} \cup \{3, 3 + \tfrac{1}{n} \mid n \in \mathbb{N}\} \cup \cdots \\ &\quad \cdots \cup \{2l-1, (2l-1) + \tfrac{1}{n} \mid n \in \mathbb{N}\} \cup \{(2l+1) + \tfrac{1}{n} \mid n \in \mathbb{N}\} \cup \cdots \\ &\quad \cdots \cup \{(2k-1) + \tfrac{1}{n} \mid n \in \mathbb{N}\} \end{aligned}$$

has exactly k accumulation points, l of which are contained in A.

For the case where A' is a countable infinite set, we can use as basic example the set $A = \{1\} \cup (\bigcup_{i \in \mathbb{N}} \{i + \frac{1}{n} \mid n \in \mathbb{N}\})$ for which $A' = \mathbb{N}$, that is A has a countable infinite set of accumulation points all of which are contained in it. The following example presents the exact opposite situation, that is a countable infinite set A having a countable infinite set of accumulation points with $A \cap A' = \emptyset$.

Example 2.2. *The set*

$$A = \{1 + \frac{1}{n} \mid n \in \mathbb{N}\} \cup \{3 + \frac{1}{n} \mid n \in \mathbb{N}\} \cup \cdots = \bigcup_{i \in \mathbb{N}} \{(2i-1) + \frac{1}{n} \mid n \in \mathbb{N}\}$$

has a countable infinite set of accumulation points none of which is contained in A.

Proof. Obviously, $A' \subseteq \mathbb{N}$. Let k be an odd natural number, so there exists $i \in \mathbb{N}$ such that $k = 2i - 1$, thus k is an accumulation point of the set $\{k + \frac{1}{n} \mid n \in \mathbb{N}\} = \{(2i-1) + \frac{1}{n} \mid n \in \mathbb{N}\}$, therefore $k \in A'$.

Let now m be an even natural number, so there exists $i \in \mathbb{N}$, such that $m = 2i$. The interval $(2i, 2i+1)$ does not contain any element of A, while the interval $(2i-1, 2i)$ contains infinitely many elements of A. For $\varepsilon < \frac{1}{2}$, we have $D(m, \varepsilon) \cap A = \emptyset$ thus m is not an accumulation point of A. Hence, $A' = \{1, 3, 5, \ldots\}$ which is a countable infinite set, but A does not contain any odd natural number, thus it does not contain any of its accumulation points. $\square$

We can now construct examples of countable infinite sets that comprise some of their accumulation points.

Example 2.3. *Let $k \in \mathbb{N}$. The set*

$$A = (\bigcup_{i=1}^{k} \{(2i-1), (2i-1) + \frac{1}{n} \mid n \in \mathbb{N}\}) \cup (\bigcup_{i=k}^{\infty} \{(2i+1) + \frac{1}{n} \mid n \in \mathbb{N}\})$$

has a countable infinite set of accumulation points and it contains a finite number k of them, since $A' = \{2i-1 \mid i \in \mathbb{N}\}$ and $A \cap A' = \{1, 3, \ldots, 2k-1\}$.

Example 2.4. *Let $k \in \mathbb{N}$. The set*

$$A = (\bigcup_{i=1}^{k} \{(2i-1) + \frac{1}{n} \mid n \in \mathbb{N}\}) \cup (\bigcup_{i=k}^{\infty} \{(2i+1), (2i+1) + \frac{1}{n} \mid n \in \mathbb{N}\})$$

has a countable infinite set of accumulation points, a finite number k of which is not contained in A, since $A' = \{2i-1 \mid i \in \mathbb{N}\}$ and $A \cap A' = \{2k+2i-1 \mid i \in \mathbb{N}\}$, that means that only a finite number k of accumulation points of A, namely numbers $1, 3, 5, \ldots, 2k-1$, are not contained in it.

Example 2.5. *The set*

$$A = (\bigcup_{k=1}^{\infty} \{(4k-3), (4k-3) + \frac{1}{n} \mid n \in \mathbb{N}\}) \cup (\bigcup_{k=1}^{\infty} \{(4k-1) + \frac{1}{n} \mid n \in \mathbb{N}\})$$

has a countable infinite set of accumulation points and both sets $A \cap A'$ and $A' \setminus A$ are countable infinite sets.

Proof. The set A can be rewritten in the form

$$A = \{4k - 3 \mid k \in \mathbb{N}\} \cup (\bigcup_{k=1}^{\infty} \{(2k-1) + \frac{1}{n} \mid n \in \mathbb{N}\})$$

Clearly, $A' = \{2i - 1 \mid i \in \mathbb{N}\}$ and $A \cap A' = \{4i - 3 \mid i \in \mathbb{N}\}$. Thus, the set A has as accumulation points the odd natural numbers which form a countable infinite set and contains those of the form $4i - 3, i \in \mathbb{N}$, which also form a countable infinite set, while it does not contain the remaining odd natural numbers that is the set $\{4i - 1 \mid i \in \mathbb{N}\}$, which is also a countable infinite set. □

3 Countable infinite sets with uncountable derived set

The set $\mathbb{Q}$ of rational numbers with $\mathbb{Q}' = \mathbb{R}$, is a well known example of a countable infinite set A having an uncountable set of accumulation points, with $A \cap A'$ being a countable infinite set while $A' \setminus A$ is an uncountable set. In this case, every element of A is an accumulation point. We can construct many other examples that correspond to the same case, just by considering all rational numbers belonging to a specific interval onto the real line or an intersection of $\mathbb{Q}$ with a union of real intervals.

The following example ([7]) covers the antidiametric case of a countable infinite set which has uncountably many accumulation points without containing any of them. We denote by $gcd\{m, n\}$ the greatest common divisor of natural numbers m, n.

Example 3.1. *For the set*

$$S = \{(\frac{p}{q}, \frac{1}{q}) \mid p, q \in \mathbb{N}, \frac{p}{q} < 1, gcd\{p, q\} = 1\}$$

the following propositions hold:

(1) *The set S is countable.*

(2) *Every element of S is isolated.*

(3) *The set S has uncountably many accumulation points.*

Proof. (1) Obviously $S \subseteq \mathbb{Q}^2$, thus S is a countable set. Moreover, it is easy to see that S is a countable union of finite sets, description that gives a clear image of its distribution onto the real plane. In fact, let $n \in \mathbb{N}$. We consider the segment S_n of the real plane, where

$$S_n = \{(x, y) \in \mathbb{R}^2 \mid x \in (0, 1), y = \frac{1}{n}\}.$$

Clearly, all segments $S_n, n \in \mathbb{N}$, are parallel to the horizontal axe and S_1 does not contain any element of S while the segment $S_n, n \in \mathbb{N}, n > 1$, contains a finite number of elements of S since

$$S \cap S_n = \{(\frac{p}{n}, \frac{1}{n}) \mid p \in \mathbb{N}, p < n, gcd\{p, n\} = 1\}$$

with $card(S \cap S_n) \leq n - 1$. It is clear also that any element of S belongs only to one segment $S_n, n \in \mathbb{N}, n > 1$, and the distance between those segments progressively decreases. Thus, the set S is a countable union of finite sets, since $S = \bigcup_{n \in \mathbb{N}, n>1}(S \cap S_n)$.

(2) Let $A = (\frac{p_0}{q_0}, \frac{1}{q_0}) \in S$. Then, $A \in S_{q_0}$ and since $S \cap S_{q_0}$ is finite, we can calculate $d_1 = min_i\{d(A, A_i) \mid A_i \in S \cap S_{q_0}, A_i \neq A\} > 0$. Put d_2 the distance of segment S_{q_0} from its nearest one, that is $d_2 = \frac{1}{q_0} - \frac{1}{q_0 + 1}$.

Consider $\varepsilon < min\{d_1, d_2\}$. Obviously, $D(A, \varepsilon) \cap S = \emptyset$, thus A is an isolated point of S.

(3) We shall prove that every point of the segment $S_0 = \{(x, y) \in \mathbb{R}^2 \mid x \in (0, 1), y = 0\}$ is an accumulation point of S. Clearly, this segment is the part of the real plane that comprises only the interval $(0, 1)$ of the horizontal axe and does not include any element of S.

Let $A = (x_0, 0)$ be an element of S_0 and $\varepsilon > 0$. There exists a prime number q_0 such that $\frac{1}{q_0} < \frac{\varepsilon}{2}$. Then, $S \cap S_{q_0} = \bigcup_{i=1}^{q_0-1}\{A_i\}$ with $A_i = (\frac{i}{q_0}, \frac{1}{q_0}), 1 \leq i \leq q_0 - 1$. Thus, all points of $S \cap S_{q_0}$ are equally distributed on S_{q_0} since $d(A_i, A_{i+1}) = \frac{1}{q_0}$, $i = 1, 2, \ldots, q_0 - 2$. The segment

$$K = \{(x, y) \in \mathbb{R}^2 \mid x \in (x_0 - \frac{\varepsilon}{2}, x_0 + \frac{\varepsilon}{2}), \, y = \frac{1}{q_0}\}$$

is a subset of S_{q_0} and it is also a subset of the open sphere $V(A, \varepsilon)$. Since K has length $\varepsilon > \frac{2}{q_0}$, there exists $A_k \in S \cap S_{q_0}$ with $A_k \in K$. Then, $A_k \in S \cap V(A, \varepsilon)$ and obviously $A_k \neq A$. Thus, $D(A, \varepsilon) \cap S \neq \emptyset$ which means that A is an accumulation point of S. Hence, every point of S_0 is an accumulation point of S and clearly they are uncountably many. □

4 Accumulation points of uncountable sets

Intervals in $\mathbb{R}$ can be used to construct many simple examples of uncountable sets that do not contain a countable number of their accumulation points.

A basic example is the closed interval $A = [a, b], a, b \in \mathbb{R}, a < b$, that contains all of its accumulation points, so $card(A' \setminus A) = 0$. The intervals $A_1 =$

$[a, b)$ and $A_2 = (a, b)$ are obviously uncountable sets which do not contain only one or two of their accumulation points, respectively, since $A_1' = A_2' = [a, b]$. Finite unions of pairwise disjoint open intervals form uncountable sets having k accumulation points not belonging to them, where $k \in \mathbb{N}, k \geq 3$. For example, for any $k \in \mathbb{N}, k \geq 3$, let $A = (1, 2) \cup (2, 3) \cup \cdots \cup (k-1, k) = \bigcup_{i=1}^{k-1}(i, i+1)$. Obviously, $A' = [1, k]$ and $A' \setminus A = \{1, 2, \ldots, k\}$. The case $A' \setminus A$ being a countable infinite set can be easily covered using intervals, just by considering the countable union $A = (1, 2) \cup (2, 3) \cup \cdots = \bigcup_{i=1}^{\infty}(i, i+1)$ which is an uncountable set with $A' = [1, +\infty)$ and $A' \setminus A = \mathbb{N}$.

Apart from intervals, there are many characteristic subsets of $\mathbb{R}$ that give examples of uncountable sets having countable or uncountable number of accumulation points not belonging to them. We present two such examples, derived from the analytic and topological approach of real numbers.

Example 4.1. *Let A be the set of irrational numbers. A is uncountable with $A' \setminus A$ being a countable infinite set.*

Proof. The field of real numbers can be constructed using Cauchy sequences as the completion of rational numbers. From this procedure, it follows that A is an uncountable and dense subset of $\mathbb{R}$, thus $A' \setminus A = \mathbb{R} \setminus (\mathbb{R} \setminus \mathbb{Q}) = \mathbb{Q}$, that is A has a countable infinite set of accumulation points not belonging to A. □

The Cantor set ([3]) is a subset of the interval $[0, 1]$ with remarkable analytic and topological properties that help us understand the "completeness" of real numbers. An easy way to illustrate the Cantor set C is starting from the interval $[0, 1]$ and repeatedly removing the open middle third from each remaining segment. This process is continued ad infinitum and C is the set of all remaining points. The intuitive image created from this procedure can easily drive to the false conclusion that only the endpoints of the removed intervals finally remain in the Cantor set. In fact, C possesses some very interesting properties that indicate its complexity. First of all, C is uncountable, it contains as many numbers as the interval $[0, 1]$ but it does not contain any interval of non-zero length.

Example 4.2. *(i) The Cantor set C is an uncountable set with $C' \setminus C = \emptyset$.*

(ii) The complement A of C in $[0, 1]$ *is an uncountable set having uncountably many accumulation points not belonging to A.*

Proof. (i) C is formed as the complement in $[0, 1]$ of a countable union of open intervals, thus it is a closed subset of $\mathbb{R}$. Therefore, it contains all of its accumulation points.

(ii) A is a countable union of pairwise disjoint intervals, so it is an uncountable set. Obviously, $A \subseteq A'$. Let $x \in C$ and $0 < \varepsilon < min\{x, 1-x\}$. If

$D(x,\varepsilon) \cap A = \emptyset$ then $(x, x+\varepsilon) \cap A = \emptyset$, thus $(x, x+\varepsilon) \subseteq [0,1] \setminus A = C$ which is impossible since C does not contain any interval of non-zero length. Therefore $x \in A'$, that is $A' = [0,1]$ and $A' \setminus A = C$ which is an uncountable set.

□

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