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Gaps between Consecutive Perfect Powers

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Abstract

Let P_n be the n-th perfect power and $d_n = P_{n+1} - P_n$ the difference between the two consecutive perfect powers P_n and P_{n+1}. In a previous article of the author the following conjecture was established, $d_n \sim 2n$. In this article we prove that this conjecture is false, since we prove that

$$\limsup \frac{d_n}{2n} = 1, \qquad \liminf \frac{d_n}{2n} = 0.$$

Therefore there exist small gaps between consecutive perfect powers.

We also prove the stronger result

$$\liminf \frac{d_n}{(2n)^{(2/3)+\epsilon}} = 0,$$

where ϵ is a fixed but arbitrary positive real number.

Besides, using the ideas of this article, we obtain a shorter proof of a theorem proved in another article of the author.

Mathematics Subject Classification: 11A99, 11B99

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1 Introduction

A natural number of the form m^n where m and $n \geq 2$ are positive integer is called a perfect power. The first few terms of the integer sequence of perfect powers are

$$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, \ldots$$

Let $A(n)$ be the number of perfect powers in the open interval $((n-1)^2, n^2)$, where $n \geq 1$ is a positive integer. It is well-known that $A(n) = 0$ for almost all intervals $((n-1)^2, n^2)$, since we have the theorem (see [3])

Theorem 1.1 *Let us consider the n open intervals $(0, 1^2), (1^2, 2^2), \ldots, ((n-1)^2, n^2)$. Let $S(n)$ be the number of these n open intervals that contain some perfect power. The following limit holds*

$$\lim_{n\to\infty} \frac{S(n)}{n} = 0$$

Therefore, if $S_1(n)$ is the number of these n open intervals that do not contain perfect powers then the following limit holds

$$\lim_{n\to\infty} \frac{S_1(n)}{n} = 1 \tag{1}$$

since $S(n) + S_1(n) = n$.

Clearly

$$A(n) \geq 1 \tag{2}$$

infinite times, since there are infinite perfect powers not a square.

Let P_n be the n-th perfect power and $d_n = P_{n+1} - P_n$ the difference between the two consecutive perfect powers P_n and P_{n+1}. We have the inequality (see [1])

$$d_n = P_{n+1} - P_n < 2n \qquad (n \geq 3). \tag{3}$$

Let P_n be the n-th perfect power. We have (see [4])

$$P_n \sim n^2 \tag{4}$$

Therefore

$$P_{n+1} \sim P_n \sim n^2$$

2 Main Results

In a previous article of the author [2] the following conjecture was established

$$d_n \sim 2n$$

Now, we give a proof that this conjecture is false.

Theorem 2.1 *The conjecture $d_n \sim 2n$ is false*

Proof. Let us consider the perfect powers P_n such that

$$k^2 \le P_n < (k+1)^2 \tag{5}$$

The number of perfect powers in this interval is $A(k+1)+1$. Note that there is always a perfect power P_n that satisfies inequality (5), namely $P_n = k^2$. We denote the sum of the corresponding $A(k+1)+1$ differences d_n in the form

$$\sum d_n = (k+1)^2 - k^2 = 2k+1 \tag{6}$$

Inequality (5) gives

$$1 \le \frac{P_n}{k^2} < \frac{(k+1)^2}{k^2} \tag{7}$$

Therefore, since both sides in (7) have limit 1, we have

$$\lim_{n\to\infty} \frac{P_n}{k^2} = 1 \tag{8}$$

Now, equations (8) and (4) give

$$\lim_{n\to\infty} \frac{P_n}{k^2} = \lim_{n\to\infty} \frac{k(n)n^2}{k^2} = \lim_{n\to\infty} k(n)\left(\frac{n}{k}\right)^2 = 1 \tag{9}$$

where $k(n) \to 1$. Therefore equation (9) gives

$$\lim_{n\to\infty} \frac{n}{k} = 1$$

and consequently

$$\lim_{n\to\infty} \frac{2n}{2k+1} = 1 \tag{10}$$

Note that $2k+1 = (k+1)^2 - k^2$ (see (6))

Suppose that

$$\lim_{n\to\infty} \frac{d_n}{2n} = 1 \tag{11}$$

Therefore (see (10) and (11))

$$\lim_{n\to\infty} \frac{d_n}{2k+1} = \lim_{n\to\infty} \frac{d_n}{2n}\frac{2n}{2k+1} = 1.1 = 1$$

Consequently, from a certain value of n we have

$$\frac{d_n}{2k+1} > \frac{2}{3} \tag{12}$$

Since $A(k+1) \geq 1$ infinite times (see (2)) we have that the number of d_n in the sum $\sum d_n$ is at least 2 infinite times. Hence (see (6) and (12))

$$1 = \frac{\sum d_n}{2k+1} \geq \frac{d_n}{2k+1} + \frac{d_n}{2k+1} > \frac{2}{3} + \frac{2}{3} = \frac{4}{3} > 1$$

That is, an evident contradiction. The theorem is proved.

Theorem 2.2 *We have*

$$\limsup \frac{d_n}{2n} = \limsup \frac{d_n}{2k+1} = 1 \tag{13}$$

$$0 \leq \liminf \frac{d_n}{2n} = \liminf \frac{d_n}{2k+1} < 1 \tag{14}$$

Proof. We have (see (3))

$$0 \leq \frac{d_n}{2n} \leq 1$$

Therefore

$$0 \leq \liminf \frac{d_n}{2n} \leq \limsup \frac{d_n}{2n} \leq 1 \tag{15}$$

On the other hand, we have (see (6))

$$0 \leq \frac{d_n}{2k+1} \leq \frac{\sum d_n}{2k+1} = \frac{2k+1}{2k+1} = 1$$

Therefore

$$0 \leq \liminf \frac{d_n}{2k+1} \leq \limsup \frac{d_n}{2k+1} \leq 1 \tag{16}$$

Now

$$\frac{d_n}{2n} = \frac{d_n}{2k+1}\frac{2k+1}{2n} \qquad \frac{d_n}{2k+1} = \frac{d_n}{2n}\frac{2n}{2k+1} \tag{17}$$

Equations (15), (16) and (17) imply

$$\liminf \frac{d_n}{2n} = \liminf \frac{d_n}{2k+1}$$

$$\limsup \frac{d_n}{2n} = \limsup \frac{d_n}{2k+1}$$

There are infinite values of k such that $A(k+1)=0$ (see (1)). Therefore in the interval $[k^2,(k+1)^2)$ there is an unique perfect power P_n, namely k^2, and consequently an unique difference d_n, namely $(k+1)^2-k^2=2k+1$. Hence, for these infinite values of k we have $\frac{d_n}{2k+1}=\frac{2k+1}{2k+1}=1$. That is

$$\limsup \frac{d_n}{2n} = \limsup \frac{d_n}{2k+1} = 1$$

Therefore (13) is proved.

On the other hand, suppose that

$$\liminf \frac{d_n}{2n} = \liminf \frac{d_n}{2k+1} = 1$$

Then

$$\lim_{n\to\infty} \frac{d_n}{2n} = 1$$

This is impossible by Theorem 2.1. Consequently

$$0 \le \liminf \frac{d_n}{2n} = \liminf \frac{d_n}{2k+1} < 1$$

and (14) is proved. The theorem is proved.

The following theorem was proved in [1, Theorem 3.1 and Corollaries 3.2 and 3.3]. We have obtained the following proof shorter using the ideas of this article.

Theorem 2.3 *Let $\epsilon > 0$ an arbitrary but fixed real number. Let us consider the first n consecutive differences*

$$d_1=(P_2-P_1), d_2=(P_3-P_2), \ldots, d_n=(P_{n+1}-P_n).$$

Let $v(n)$ be the number of these differences such that $(2-2\epsilon)i < d_i < 2i$. We have the following limit

$$\lim_{n\to\infty} \frac{v(n)}{n} = 1.$$

Proof. There are infinite values of k such that $A(k+1)=0$ (see (1)). In this proof we work as this sequence of values of k. Therefore in the interval $[k^2,(k+1)^2)$ there is an unique perfect power P_i, namely k^2, and consequently an unique difference d_i, namely, $(k+1)^2-k^2=2k+1$. Hence for these infinite values of k we have $\frac{d_i}{2k+1}=\frac{2k+1}{2k+1}=1$, and therefore

$$\lim_{k\to\infty} \frac{d_i}{2k+1} = \lim_{k\to\infty} 1 = 1 \tag{18}$$

Now (see (18) and (10))

$$\lim_{k\to\infty}\frac{d_i}{2i}=\lim_{k\to\infty}\frac{d_i}{2k+1}\frac{2k+1}{2i}=1.1=1 \tag{19}$$

Let $\epsilon>0$ a fixed but arbitrary real number. There exists k_ϵ such that if $k\geq k_\epsilon+1$ and $A(k+1)=0$ we have (see (19) and (3))

$$1-\epsilon<\frac{d_i}{2i}<1$$

That is

$$(2-2\epsilon)i<d_i<2i$$

The inequality $s^2\leq P_{n+1}$ has the solutions $s=1,2,\ldots,\left\lfloor\sqrt{P_{n+1}}\right\rfloor$ and consequently $\left\lfloor\sqrt{P_{n+1}}\right\rfloor$ solutions. Here (as usual) $\lfloor . \rfloor$ is the integer part function.

The number of k such that $A(k+1)=0$ and $(k+1)^2\leq P_{n+1}$ will be (see (1) and (4))

$$\begin{aligned} & S_1\left(\left\lfloor\sqrt{P_{n+1}}\right\rfloor\right)=a(n)\left\lfloor\sqrt{P_{n+1}}\right\rfloor=a(n)\left(\sqrt{P_{n+1}}-\alpha(n)\right)\\ = {} & a(n)(b(n)n-\alpha(n))=c(n)n\end{aligned}$$

where $a(n)\to 1$, $b(n)\to 1$, $c(n)\to 1$ and $0\leq\alpha(n)<1$.

Now

$$c(n)n-q_\epsilon=S_1\left(\left\lfloor\sqrt{P_{n+1}}\right\rfloor\right)-q_\epsilon\leq v(n)\leq n$$

where q_ϵ is the number of k such that $k\leq k_\epsilon$ and $A(k+1)=0$.

Consequently

$$v(n)\sim n$$

The theorem is proved.

In the following theorem we prove that there exist small gaps in the sequence of perfect powers.

Theorem 2.4 *We have*

$$\liminf\frac{d_n}{2n}=\liminf\frac{d_n}{2k+1}=0 \tag{20}$$

Proof. We shall need the following Taylor's formulae.

$$\frac{1}{1-x}=1+x+f(x)x \tag{21}$$

where $\lim_{x\to 0} f(x) = 0$. This is the Taylor's formula of the geometric power series.

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + g(x)x^2 \tag{22}$$

where $\lim_{x\to 0} g(x) = 0$. This is the Taylor's formula of the binomial power series.

Suppose that n is not a square. Consequently

$$\left\lfloor n^{3/2} \right\rfloor < n^{3/2} < \left\lfloor n^{3/2} \right\rfloor + 1 \tag{23}$$

That is

$$\left\lfloor n^{3/2} \right\rfloor^2 < n^3 < \left(\left\lfloor n^{3/2} \right\rfloor + 1\right)^2 \tag{24}$$

Therefore the perfect power n^3 is in the interval

$$\left[k^2, (k+1)^2\right) = \left[\left\lfloor n^{3/2} \right\rfloor^2, \left(\left\lfloor n^{3/2} \right\rfloor + 1\right)^2\right) \tag{25}$$

Let P_i be the first perfect power greater than $\left\lfloor n^{3/2} \right\rfloor^2$. Hence we have $P_i \leq n^3$. The first difference d_i in interval (25) will be

$$d_i = P_i - \left\lfloor n^{3/2} \right\rfloor^2 \leq n^3 - \left\lfloor n^{3/2} \right\rfloor^2 \tag{26}$$

Equations (26) and (21) give

$$\begin{aligned}
0 &< \frac{d_i}{2\left\lfloor n^{3/2} \right\rfloor + 1} \leq \frac{n^3 - \left\lfloor n^{3/2} \right\rfloor^2}{2\left\lfloor n^{3/2} \right\rfloor} = \frac{n^3}{2\left\lfloor n^{3/2} \right\rfloor} - \frac{\left\lfloor n^{3/2} \right\rfloor}{2} \\
&= \frac{n^3}{2\left(n^{3/2} - \epsilon(n)\right)} - \frac{\left(n^{3/2} - \epsilon(n)\right)}{2} = \frac{n^{3/2}}{2}\frac{1}{1 - \frac{\epsilon(n)}{n^{3/2}}} - \frac{n^{3/2}}{2} + \frac{\epsilon(n)}{2} \\
&= \frac{n^{3/2}}{2}\left(1 + \frac{\epsilon(n)}{n^{3/2}} + f\left(\frac{\epsilon(n)}{n^{3/2}}\right)\frac{\epsilon(n)}{n^{3/2}}\right) - \frac{n^{3/2}}{2} + \frac{\epsilon(n)}{2} = \epsilon(n) \\
&+ f\left(\frac{\epsilon(n)}{n^{3/2}}\right)\frac{\epsilon(n)}{2} = \epsilon(n) + o(1) \qquad (n \to \infty)
\end{aligned}$$

That is

$$0 < \frac{d_i}{2\left\lfloor n^{3/2} \right\rfloor + 1} \leq \epsilon(n) + o(1) \qquad (n \to \infty) \tag{27}$$

where $\epsilon(n)$ is the fractional part

$$\epsilon(n) = n^{3/2} - \left\lfloor n^{3/2} \right\rfloor \tag{28}$$

and therefore (see (23))

$$0 < \epsilon(n) < 1$$

Suppose that n is of the form $n = 4s^2 + 1$. Consequently (see equation (22))

$$\begin{aligned} n^{3/2} &= \left(4s^2+1\right)^{3/2} = \left(4s^2\right)^{3/2}\left(1+\frac{1}{4s^2}\right)^{3/2} \\ &= 8s^3\left(1+\frac{3}{2}\frac{1}{4s^2}+\frac{(3/2)((3/2)-1)}{2}\frac{1}{(4s^2)^2}+g\left(\frac{1}{4s^2}\right)\frac{1}{(4s^2)^2}\right) \\ &= 8s^3+3s+\left(\frac{3}{8}+g\left(\frac{1}{4s^2}\right)\right)\frac{1}{2s} \qquad (29) \end{aligned}$$

Therefore (see (29) and (28))

$$\left\lfloor n^{3/2} \right\rfloor = 8s^3 + 3s \qquad (s \to \infty)$$

and

$$\epsilon(n) = \left(\frac{3}{8}+g\left(\frac{1}{4s^2}\right)\right)\frac{1}{2s} = o(1) \qquad (s \to \infty) \tag{30}$$

Therefore (see (27) and (30)) if $n = 4s^2 + 1$ then

$$\lim_{s\to\infty} \frac{d_i}{2\left\lfloor n^{3/2}\right\rfloor + 1} = 0 \tag{31}$$

Equation (31) implies (20). The theorem is proved.

From the proof of this theorem we can deduce the following stronger result.

Theorem 2.5 *Let ϵ be a fixed but arbitrary positive real number. We have*

$$\liminf \frac{d_n}{(2k+1)^{(2/3)+\epsilon}} = \liminf \frac{d_n}{(2n)^{(2/3)+\epsilon}} = 0 \tag{32}$$

Proof. We have (see the proof of theorem 2.1)

$$\lim_{n\to\infty} \frac{n}{k} = 1$$

Therefore (see (10))

$$\lim_{n\to\infty} \frac{(2n)^{(2/3)+\epsilon}}{(2k+1)^{(2/3)+\epsilon}} = 1$$

and consequently (see the proof of theorem 2.2)

$$\liminf \frac{d_n}{(2k+1)^{(2/3)+\epsilon}} = \liminf \frac{d_n}{(2n)^{(2/3)+\epsilon}}$$

If we consider the numbers of the form $4s^2+1$, from the proof of theorem 2.4 we obtain the inequality

$$(8s^3+3s)^2 < (4s^2+1)^3 < (8s^3+3s+1)^2$$

Consequently (see the proof of theorem 2.4)

$$0 < \frac{d_i}{(2(8s^3+3s)+1)^{(2/3)+\epsilon}} \leq \frac{(4s^2+1)^3-(8s^3+3s)^2}{(2(8s^3+3s)+1)^{(2/3)+\epsilon}}$$
$$= \frac{3s^2+1}{s^{2+3\epsilon}\left(16+\frac{6}{s^2}+\frac{1}{s^3}\right)^{(2/3)+\epsilon}} \to 0 \qquad (s \to \infty)$$

and hence

$$\lim_{s\to\infty} \frac{d_i}{(2(8s^3+3s)+1)^{(2/3)+\epsilon}} = 0 \tag{33}$$

Limit (33) implies (32). The theorem is proved.

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