International Mathematical Forum, Vol. 11, 2016, no. 12, 565 - 571
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/imf.2016.6450

On a Limit where Appear the Prime Counting Function, the n-th Prime and the Mertens's Constant

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Abstract

In this article we prove the following limits

$$\lim_{n\to\infty} \frac{\left(\prod_{i=2}^{n}\left(\frac{p_i}{\pi(i)}\right)^{\frac{1}{p_i}}\right)^{\frac{1}{\sum_{i=1}^{n}\frac{1}{p_i}}}}{\sqrt{\frac{p_n}{\pi(n)}}} = e^{-M}$$

$$\lim_{n\to\infty} \frac{\left(\prod_{i=1}^{n} p_i^{\frac{1}{i}}\right)^{\frac{1}{\sum_{i=1}^{n}\frac{1}{i}}}}{\frac{p_n}{\sqrt{n}}} = \frac{1}{e^{1+\frac{\gamma}{2}}}$$

where $\pi(n)$ is the prime counting function, p_n is the n-th prime, M is the Mertens's constant and γ is the Euler's constant.

Mathematics Subject Classification: 11A99, 11B99

Keywords: Prime numbers, prime counting function, limits, Mertens's constant, Euler's constant

1 Main Results

In a previous article [2] we have proved limits where appear the n-th prime p_n, in this article we prove limits where also appear the prime counting function.

Let p_n be the n-th prime. In the sequel we need the following theorem from [1].

Theorem 1.1 *The following formula holds*

$$\sum_{i=1}^{n} \frac{1}{p_i} = \log\log n + M + o(1) \tag{1}$$

where M is called Mertens's constant.

Let $\pi(n)$ be the prime counting function. We have the following theorem.

Theorem 1.2 *The following limit holds*

$$\lim_{n\to\infty} \frac{\left(\prod_{i=2}^{n}\left(\frac{p_i}{\pi(i)}\right)^{\frac{1}{p_i}}\right)^{\frac{1}{\sum_{i=1}^{n}\frac{1}{p_i}}}}{\sqrt{\frac{p_n}{\pi(n)}}} = e^{-M} \tag{2}$$

Proof. We have (L'Hospital's rule)

$$\frac{1}{1+x} = 1 - x + f(x)x \tag{3}$$

where $\lim_{x\to 0} f(x) = 0$.

Equations (1) and (3) give

$$\begin{aligned}
\frac{1}{\sum_{i=1}^{n}\frac{1}{p_i}} &= \frac{1}{\log\log n + M + o(1)} = \frac{1}{\log\log n}\frac{1}{1+\frac{M+o(1)}{\log\log n}} \\
&= \frac{1}{\log\log n}\left(1 - \frac{M+o(1)}{\log\log n} + f\left(\frac{M+o(1)}{\log\log n}\right)\frac{M+o(1)}{\log\log n}\right) \\
&= \frac{1}{\log\log n} - \frac{M}{(\log\log n)^2} + o\left(\frac{1}{(\log\log n)^2}\right)
\end{aligned} \tag{4}$$

We have (Prime Number Theorem)

$$p_i \sim i\log i \tag{5}$$

Equation (5) gives

$$\log p_i = \log i + \log\log i + o(1) \tag{6}$$

We also have (Prime Number Theorem)

$$\pi(i) \sim \frac{i}{\log i} \tag{7}$$

Equation (7) gives

$$\log \pi(i) = \log i - \log\log i + o(1) \tag{8}$$

Equations (6) and (8) give $(i = n)$

$$\log \sqrt{\frac{p_n}{\pi(n)}} = \frac{1}{2}\left(\log p_n - \log \pi(n)\right) = \log\log n + o(1) \tag{9}$$

On the other hand, equations (6) and (8) give

$$\log \left(\prod_{i=s}^{n} \left(\frac{p_i}{\pi(i)} \right)^{\frac{1}{p_i}} \right) = \sum_{i=s}^{n} \frac{\log p_i - \log \pi(i)}{p_i} = \sum_{i=s}^{n} \frac{2\log\log i + h(i)}{p_i}$$
$$= \sum_{i=s}^{n} \frac{2\log\log i}{p_i} + \sum_{i=s}^{n} h(i)\frac{1}{p_i} \tag{10}$$

where $h(i) \to 0$ and s is a fixed and convenient positive integer.

We have following formula from [1]

$$\frac{1}{p_i} = \frac{1}{i\log i} - \frac{\log\log i - 1}{i\log^2 i} + o\left(\frac{1}{i\log^2 i}\right) = \frac{1}{i\log i} - g(i)\frac{\log\log i}{i\log^2 i} \tag{11}$$

where $g(i) \to 1$.

Equation (11) gives

$$\sum_{i=s}^{n} \frac{2\log\log i}{p_i} = \sum_{i=s}^{n} \frac{2\log\log i}{i\log i} - \sum_{i=s}^{n} g(i)\frac{2(\log\log i)^2}{i\log^2 i} \tag{12}$$

Now, we have

$$\sum_{i=s}^{n} \frac{2\log\log i}{i\log i} = \int_{s}^{n} \frac{2\log\log x}{x\log x}\, dx + O(1) = (\log\log n)^2 + O(1) \tag{13}$$

Note that the left side of (13) is a sum of rectangles of basis 1 and height $\frac{2\log\log i}{i\log i}$. Besides, the function $\frac{2\log\log x}{x\log x}$ is strictly decreasing in the interval $[s, n]$. Therefore the integral aproximate the sum with error $O(1)$.

We have

$$\sum g(i)\frac{2(\log\log i)^2}{i\log^2 i} = \sum g(i)\frac{2(\log\log i)^2}{\log^{1/2} i}\frac{1}{i\log^{3/2} i} < \sum \frac{1}{i\log^{3/2} i}$$

where the last series converges (Integral criterion). Therefore (Comparison criterion) the first series also converges. That is, we have

$$\sum_{i=s}^{\infty} g(i)\frac{2(\log\log i)^2}{i\log^2 i} = C$$

and consequently we have

$$\sum_{i=s}^{n} g(i)\frac{2(\log\log i)^2}{i\log^2 i} = C + o(1) \tag{14}$$

Substituting (13) and (14) into (12) we obtain

$$\sum_{i=s}^{n} \frac{2\log\log i}{p_i} = (\log\log n)^2 + O(1) \tag{15}$$

Let $\epsilon > 0$, there exists n_0 such that if $n \geq n_0$ we have $|h(i)| < \epsilon$. Hence, we obtain (see equation (1))

$$\begin{aligned}\left|\sum_{i=s}^{n} h(i)\frac{1}{p_i}\right| &\leq \sum_{i=s}^{n} |h(i)|\frac{1}{p_i} \leq \sum_{i=s}^{n_0-1} |h(i)|\frac{1}{p_i} + \epsilon\sum_{i=n_0}^{n}\frac{1}{p_i} \\ &\leq \sum_{i=s}^{n_0-1} |h(i)|\frac{1}{p_i} + \epsilon\sum_{i=1}^{n}\frac{1}{p_i} \leq 2\epsilon\log\log n\end{aligned}$$

That is

$$\sum_{i=s}^{n} h(i)\frac{1}{p_i} = o(\log\log n) \tag{16}$$

since $\epsilon > 0$ can be arbitrarily small.

Substituting (15) and (16) into (10) we find that

$$\log\left(\prod_{i=s}^{n}\left(\frac{p_i}{\pi(i)}\right)^{\frac{1}{p_i}}\right) = (\log\log n)^2 + o(\log\log n) \tag{17}$$

Equations (4), (9) and (17) give

$$\begin{aligned}\log\left(\frac{\left(\prod_{i=2}^{n}\left(\frac{p_i}{\pi(i)}\right)^{\frac{1}{p_i}}\right)^{\frac{1}{\sum_{i=1}^{n}\frac{1}{p_i}}}}{\sqrt{\frac{p_n}{\pi(n)}}}\right) &= \frac{1}{\sum_{i=1}^{n}\frac{1}{p_i}}\log\left(\prod_{i=2}^{n}\left(\frac{p_i}{\pi(i)}\right)^{\frac{1}{p_i}}\right) \\ - \log\left(\sqrt{\frac{p_n}{\pi(n)}}\right) &= -M + o(1)\end{aligned}$$

That is, limit (2). The theorem is proved.

Corollary 1.3 *The following limit holds*

$$\lim_{n\to\infty}\frac{\left(\prod_{i=2}^{n}\left(\frac{\pi(i)}{p_i}\right)^{\frac{1}{p_i}}\right)^{\frac{1}{\sum_{i=1}^{n}\frac{1}{p_i}}}}{\sqrt{\frac{\pi(n)}{p_n}}} = e^M$$

Proof. We have (see (2))

$$\frac{\left(\prod_{i=2}^{n}\left(\frac{\pi(i)}{p_i}\right)^{\frac{1}{p_i}}\right)^{\frac{1}{\sum_{i=1}^{n}\frac{1}{p_i}}}}{\sqrt{\frac{\pi(n)}{p_n}}} = \frac{\left(\prod_{i=2}^{n}\left((\frac{p_i}{\pi(i)})^{-1}\right)^{\frac{1}{p_i}}\right)^{\frac{1}{\sum_{i=1}^{n}\frac{1}{p_i}}}}{\sqrt{(\frac{p_n}{\pi(n)})^{-1}}}$$

$$= \left(\frac{\left(\prod_{i=2}^{n}\left(\frac{p_i}{\pi(i)}\right)^{\frac{1}{p_i}}\right)^{\frac{1}{\sum_{i=1}^{n}\frac{1}{p_i}}}}{\sqrt{\frac{p_n}{\pi(n)}}}\right)^{-1}$$

The corollary is proved.

We have the following theorem.

Theorem 1.4 *The following limit holds*

$$\lim_{n\to\infty}\frac{\left(\prod_{i=1}^{n}p_i^{\frac{1}{i}}\right)^{\frac{1}{\sum_{i=1}^{n}\frac{1}{i}}}}{\sqrt{n}\log n} = \lim_{n\to\infty}\frac{\left(\prod_{i=1}^{n}p_i^{\frac{1}{i}}\right)^{\frac{1}{\sum_{i=1}^{n}\frac{1}{i}}}}{\frac{p_n}{\sqrt{n}}} = \frac{1}{e^{1+\frac{\gamma}{2}}} \tag{18}$$

where γ is the Euler's constant.

Proof. It is well-known the formula

$$\sum_{i=1}^{n}\frac{1}{i} = \log n + \gamma + o(1) \tag{19}$$

We have (L'Hospital's rule)

$$\frac{1}{1+x} = 1 - x + f(x)x \tag{20}$$

where $\lim_{x\to 0} f(x) = 0$.

Equations (19) and (20) give

$$\frac{1}{\sum_{i=1}^{n}\frac{1}{i}} = \frac{1}{\log n + \gamma + o(1)} = \frac{1}{\log n}\frac{1}{1+\frac{\gamma+o(1)}{\log n}}$$

$$= \frac{1}{\log n}\left(1 - \frac{\gamma+o(1)}{\log n} + f\left(\frac{\gamma+o(1)}{\log n}\right)\frac{\gamma+o(1)}{\log n}\right)$$

$$= \frac{1}{\log n} - \frac{\gamma}{(\log n)^2} + o\left(\frac{1}{(\log n)^2}\right) \tag{21}$$

We have (Prime Number Theorem)

$$p_i = f(i) i \log i$$

where $f(i) \to 1$. Consequently

$$\log p_i = \log i + \log\log i + h(i)$$

where $h(i) \to 0$. Therefore

$$\sum_{i=s}^{n} \frac{1}{i} \log p_i = \sum_{i=s}^{n} \frac{\log i}{i} + \sum_{i=s}^{n} \frac{\log\log i}{i} + \sum_{i=s}^{n} h(i)\frac{1}{i} \tag{22}$$

Now, we have

$$\sum_{i=s}^{n} \frac{\log i}{i} = \int_s^n \frac{\log x}{x}\, dx + O(1) = \frac{\log^2 n}{2} + O(1) \tag{23}$$

since the left side is a sum of rectangles of basis 1 and height $\frac{\log i}{i}$ and the function $\frac{\log x}{x}$ is strictly decreasing in the interval $[s, n]$.

We also have

$$\begin{aligned} \sum_{i=s}^{n} \frac{\log\log i}{i} &= \int_s^n \frac{\log\log x}{x}\, dx + O(1) = [\log x \log\log x - \log x]_s^n + O(1) \\ &= \log n \log\log n - \log n + O(1) \end{aligned} \tag{24}$$

Let $\epsilon > 0$. There exists n_0 such that if $i \geq n_0$ then $|h(i)| < \epsilon$. Hence

$$\begin{aligned} \left|\sum_{i=s}^{n} h(i)\frac{1}{i}\right| &\leq \sum_{i=s}^{n} |h(i)|\, \frac{1}{i} \leq \sum_{i=s}^{n_0-1} |h(i)|\, \frac{1}{i} + \epsilon \sum_{i=n_0}^{n} \frac{1}{i} \leq \sum_{i=s}^{n_0-1} |h(i)|\, \frac{1}{i} + \epsilon \sum_{i=1}^{n} \frac{1}{i} \\ &\leq 2\epsilon \log n \end{aligned} \tag{25}$$

since

$$\sum_{i=1}^{n} \frac{1}{i} = \int_1^n \frac{1}{x}\, dx + O(1) = \log n + O(1) \sim \log n$$

Equation (25) gives

$$\sum_{i=s}^{n} h(i)\frac{1}{i} = o(\log n) \tag{26}$$

since ϵ is arbitrarily small.

Substituting (23), (24) and (26) into (22) we obtain

$$\sum_{i=s}^{n} \frac{1}{i} \log p_i = \frac{\log^2 n}{2} + \log\log n \log n - \log n + o(\log n) \tag{27}$$

Now

$$\log\left(\frac{\left(\prod_{i=1}^{n} p_i^{\frac{1}{i}}\right)^{\frac{1}{\sum_{i=1}^{n}\frac{1}{i}}}}{\sqrt{n}\log n}\right) = \frac{1}{\sum_{i=1}^{n}\frac{1}{i}}\left(\sum_{i=1}^{n}\frac{1}{i}\log p_i\right) - \frac{\log n}{2} - \log\log n \tag{28}$$

Substituting (21) and (27) into (28) we obtain (18). The theorem is proved.

Acknowledgements. The author is very grateful to Universidad Nacional de Luján.

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Received: May 3, 2016; Published: June 14, 2016