Mappings of the Direct Product of BF-algebras

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Abstract
In this paper, we introduce two canonical mappings of the direct product of BF-algebras and we obtain some of their properties.

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1 Introduction
In [4], the concept of BF-algebras was introduced by A. Walendziak in 2007. A BF-algebra is an algebra $A = (A; *, 0)$ of type $(2, 0)$, that is, a nonempty set $A$ together with a binary operation $*$ and a constant 0, satisfying the following axioms for all $x, y \in A$: (B1) $x * x = 0$, (B2) $x * 0 = x$, (BF) $0 * (x * y) = y * x$.

A BF$_1$-algebra is a BF-algebra satisfying (BG) $x = (x * y) * (0 * y)$ and a BF$_2$-algebra is a BF-algebra satisying (BH) $x * y = 0$ and $y * x = 0$ imply $x = y$. A BF$_2$-algebra satisfies (BH$'$) $x * y = 0$ implies $x = y$. In [4], Walendziak also introduced the notion of commutativity of BF-algebras. A BF-algebra $A$ is commutative if $x * (0 * y) = y * (0 * x)$ for all $x, y \in A$. In 2011, J.C. Endam and J.P. Vilela [2] characterized the commutativity of BF-algebras and established the relationship of BF-algebras and groups. Walendziak also introduced the notions of subalgebras, ideals, and normality in BF-algebras,
and established their properties. A subset \( I \) of \( A \) is called an **ideal** of \( A \) if it satisfies the following for all \( x, y \in A \): (I1) \( 0 \in I \), (I2) \( x \cdot y \in I \) and \( y \in I \) imply \( x \in I \). We say that an ideal \( I \) is **normal** if for any \( x, y, z \in A \), \( x \cdot y \in I \) implies \((z \cdot x) \cdot (z \cdot y) \in I \). A nonempty subset \( N \) of \( A \) is called a **subalgebra** of \( A \) if \( x \cdot y \in N \) for any \( x, y \in A \). A. Walendziak then used the concept of normality of BF-algebras to construct quotient BF-algebras. That is, given a normal ideal \( I \) of a BF-algebra \( A \), the relation \( \sim_I \) is defined by \( x \sim_I y \) if and only if \( x \cdot y \in I \) for any \( x, y \in A \). Then \( \sim_I \) is a congruence relation of \( A \). For \( x \in A \), write \( x/I \) for the congruence class containing \( x \), that is, \( x/I = \{ y \in A : x \sim_i y \} \). We denote \( A/I = \{ x/I : x \in A \} \) and define \( *' \) by \( x/I \cdot y/I = (x \cdot y)/I \). Note that \( x/I = y/I \) if and only if \( x \sim_i y \). Then the algebra \( A/I = (A/I; *', 0/I) \) is a BF-algebra. The algebra \( A/I \) is called the **quotient BF-algebra** of \( A \) modulo \( I \). The concept of BF-homomorphism was also introduced by A. Walendziak. A map \( \varphi : A \to B \) is called a **BF-homomorphism** if \( \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) \) for any \( x, y \in A \). The kernel of \( \varphi \), denoted by \( \text{ker} \varphi \), is defined to be the set \( \{ x \in A : \varphi(x) = 0_B \} \). A BF-homomorphism \( \varphi \) is called a **BF-monomorphism**, **BF-epimorphism**, or **BF-isomorphism** if \( \varphi \) is one-one, onto, or a bijection, respectively. In [3], R.C. Teves and J.C. Endam introduced and established the direct product of BF-algebras. In this paper, we introduced and established two canonical mappings of the direct product of BF-algebras.

### 2 Direct Product of BF-algebras

The results in this section are found in [3].

**Example 2.1** Let \( A = \{0, 1, 2\} \) be a set with the following table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \( (A; *, 0) \) is a BF-algebra.

**Example 2.2** [4] Let \( A = \{0, 1, 2, 3\} \) be a set with the following table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \( (A; *, 0) \) is a BF-algebra.
Let \( A = (A; *, 0_A) \) and \( B = (B; *, 0_B) \) be BF-algebras. Define the direct product of \( A \) and \( B \) to be the structure \( A \times B = (A \times B; \oplus, (0_A, 0_B)) \), where \( A \times B \) is the set \( \{ (a, b) : a \in A \text{ and } b \in B \} \) and whose binary operation \( \oplus \) is given by \((a_1, b_1) \oplus (a_2, b_2) = (a_1 * a_2, b_1 * b_2)\).

**Theorem 2.3** [3] The direct product of two BF-algebras is also a BF-algebra.

Now, we extend this direct product to any finite family of BF-algebras. Let \( I_n = \{1, 2, \ldots, n\} \) and let \( \{A_i = (A_i; *, 0_i) : i \in I_n\} \) be a finite family of BF-algebras. Define the direct product of BF-algebras \( A_1, \ldots, A_n \) to be the structure \( \prod_{i=1}^n A_i = \left( \prod_{i=1}^n A_i; \oplus, (0_1, \ldots, 0_n) \right) \), where

\[
\prod_{i=1}^n A_i = A_1 \times \cdots \times A_n = \{ (a_1, \ldots, a_n) : a_i \in A_i, i \in I_n \}
\]

and whose operation \( \oplus \) is given by

\[
(a_1, \ldots, a_n) \oplus (b_1, \ldots, b_n) = (a_1 * b_1, \ldots, a_n * b_n).
\]

**Remark 2.4** [3] If \( \{A_i = (A_i; *, 0_i) : i \in I_n\} \) is a family of BF-algebras, then \( \prod_{i=1}^n A_i \) is a BF-algebra.

**Theorem 2.5** [3] Let \( \{\varphi_i : A_i \to B_i : i \in I_n\} \) be a family of BF-homomorphisms. If \( \varphi \) is the map \( \prod_{i=1}^n A_i \to \prod_{i=1}^n B_i \) given by \( (a_1, \ldots, a_n) \mapsto (\varphi_1(a_1), \ldots, \varphi_n(a_n)) \),

then \( \varphi \) is a BF-homomorphism with \( \ker \varphi = \prod_{i=1}^n \ker \varphi_i, \varphi(\prod_{i=1}^n A_i) = \prod_{i=1}^n \varphi_i(A_i) \).

Furthermore, \( \varphi \) is a BF-monomorphism (respectively, BF-epimorphism) if and only if \( \varphi_i \) is.

**Theorem 2.6** [3] Let \( \{A_i = (A_i; *, 0_i) : i \in I_n\} \) be a family of BF-algebras and let \( J_i \) be a normal ideal of \( A_i \) for each \( i \in I_n \). Then \( \prod_{i=1}^n J_i \) is a normal ideal of \( \prod_{i=1}^n A_i \) and \( \prod_{i=1}^n A_i / \prod_{i=1}^n J_i \cong \prod_{i=1}^n (A_i / J_i) \).
3 Canonical Mappings of the Direct Product

This section presents two canonical mappings of the direct product of any finite family of BF-algebras and provides some of their properties.

**Theorem 3.1** Let \( \{ A_i = (A_i; *, 0_i): i \in I_n \} \) be a family of BF-algebras. Then \( f_k: \prod_{i=1}^{n} A_i \rightarrow A_k \) given by \( (a_1, \ldots, a_k, \ldots, a_n) \mapsto a_k \) is a BF-epimorphism of BF-algebras for each \( k \in I_n \).

**Proof:** For each \( k \in I_n \), define \( f_k: \prod_{i=1}^{n} A_i \rightarrow A_k \) by \( f_k((a_1, \ldots, a_k, \ldots, a_n)) = a_k \) for all \( (a_1, \ldots, a_k, \ldots, a_n) \in \prod_{i=1}^{n} A_i \). Let \( (a_1, \ldots, a_k, \ldots, a_n), (b_1, \ldots, b_k, \ldots, b_n) \) be elements of \( \prod_{i=1}^{n} A_i \). If \( (a_1, \ldots, a_k, \ldots, a_n) = (b_1, \ldots, b_k, \ldots, b_n) \), then \( a_i = b_i \) for each \( i \in I_n \). Thus, \( f_k((a_1, \ldots, a_k, \ldots, a_n)) = a_k = b_k = f_k((b_1, \ldots, b_k, \ldots, b_n)) \).

Hence, \( f_k \) is well-defined. If \( (a_1, \ldots, a_k, \ldots, a_n), (b_1, \ldots, b_k, \ldots, b_n) \in \prod_{i=1}^{n} A_i \), then

\[
f_k((a_1, \ldots, a_k, \ldots, a_n) \odot (b_1, \ldots, b_k, \ldots, b_n)) = f_k((a_1 * b_1, \ldots, a_k * b_k, \ldots, a_n * b_n)) = a_k * b_k = f_k((b_1, \ldots, b_k, \ldots, b_n)).
\]

Thus, \( f_k \) is a BF-homomorphism. If \( c_k \in A_k \), then \( (0_1, \ldots, c_k, \ldots, 0_n) \in \prod_{i=1}^{n} A_i \) and \( f_k((0_1, \ldots, c_k, \ldots, 0_n)) = c_k \). Therefore, \( f_k \) is onto and so \( f_k \) is an BF-epimorphism. \( \square \)

The maps \( f_k \) in Theorem 3.1 are called the *canonical projections* of the direct product. The following theorem relates the direct product \( \prod_{i=1}^{n} A_i \) and its canonical projections.

**Theorem 3.2** Let \( \{ A_i = (A_i; *, 0_i): i \in I_n \} \) be a family of BF-algebras. Then there exists a BF-algebra \( D \), together with a family of BF-homomorphisms \( \{ f_i: D \rightarrow A_i : i \in I_n \} \) with the following property: for any BF-algebra \( C \) and a family of BF-homomorphisms \( \{ \varphi_i: C \rightarrow A_i : i \in I_n \} \), there exists a unique BF-homomorphism \( \varphi: C \rightarrow D \) such that \( f_i \circ \varphi = \varphi_i \) for all \( i \in I_n \). Furthermore, \( D \) is uniquely determined up to BF-isomorphism.
Proof: Take $D = \prod_{i=1}^{n} A_i$. Then $D$ is a BF-algebra. Let $\{f_i: D \to A_i : i \in I_n\}$ be the family of canonical projections. Suppose that $C$ is any BF-algebra and $\{\varphi_i: C \to A_i : i \in I_n\}$ a family of BF-homomorphisms. Define $\varphi: C \to D$ by $\varphi(c) = (\varphi_1(c), \ldots, \varphi_i(c), \ldots, \varphi_n(c))$ for all $c \in C$. If $c, d \in C$, then
\[
\varphi(c \ast d) = (\varphi_1(c \ast d), \ldots, \varphi_i(c \ast d), \ldots, \varphi_n(c \ast d))
\]
\[
= (\varphi_1(c) \ast \varphi_1(d), \ldots, \varphi_i(c) \ast \varphi_i(d), \ldots, \varphi_n(c) \ast \varphi_n(d))
\]
\[
= (\varphi_1(c), \ldots, \varphi_i(c), \ldots, \varphi_n(c)) \oplus (\varphi_1(d), \ldots, \varphi_i(d), \ldots, \varphi_n(d))
\]
\[
= \varphi(c) \oplus \varphi(d).
\]
Hence, $\varphi$ is a BF-homomorphism. Moreover, $f_i \circ \varphi = \varphi_i$ for all $i \in I_n$ since $(f_i \circ \varphi)(c) = f_i(\varphi(c)) = f_i((\varphi_1(c), \ldots, \varphi_i(c), \ldots, \varphi_n(c))) = \varphi_i(c)$. To show that $\varphi$ is unique, let $\varphi': C \to D$ be another BF-homomorphism such that $f_i \circ \varphi' = \varphi_i$ for all $i \in I_n$. If $c \in C$, then $(f_i \circ \varphi')(c) = \varphi_i(c) = (f_i \circ \varphi')(c)$. By the definition of $\varphi$, $\varphi(c) = (\varphi_1(c), \ldots, \varphi_i(c), \ldots, \varphi_n(c))$ and assume that $\varphi'(c) = (a_1, \ldots, a_i, \ldots, a_n)$. Thus, for each $i \in I_n,$
\[
a_i = f_i((a_1, \ldots, a_i, \ldots, a_n))
\]
\[
= f_i(\varphi'(c))
\]
\[
= (f_i \circ \varphi')(c)
\]
\[
= (f_i \circ \varphi)(c)
\]
\[
= f_i(\varphi(c))
\]
\[
= f_i((\varphi_1(c), \ldots, \varphi_i(c), \ldots, \varphi_n(c)))
\]
\[
= \varphi_i(c).
\]
Hence, $\varphi(c) = (\varphi_1(c), \ldots, \varphi_i(c), \ldots, \varphi_n(c)) = (a_1, \ldots, a_i, \ldots, a_n) = \varphi'(c)$. Therefore, $\varphi$ is unique.

Suppose that a BF-algebra $D'$ has the same property as $D$ with the family of BF-homomorphisms $\{f'_i: D' \to A_i : i \in I_n\}$. If we apply this property for $D$ to the family of BF-homomorphisms $\{f_i: D' \to A_i : i \in I_n\}$ and also apply it for $D'$ to the family of BF-homomorphisms $\{f_i: D \to A_i : i \in I_n\}$, then we obtain unique BF-homomorphisms $\alpha: D' \to D$ and $\beta: D \to D'$ such that $f_i \circ \alpha = f'_i$ and $f'_i \circ \beta = f_i$ for all $i \in I_n$. Thus, $\alpha \circ \beta: D \to D$ is a unique BF-homomorphism such that $f_i \circ (\alpha \circ \beta) = f'_i$ for all $i \in I_n$. Since the identity map $\text{id}_D: D \to D$ is a BF-homomorphism such that $f_i \circ \text{id}_D = f_i$ for all $i \in I_n$, $\alpha \circ \beta = \text{id}_D$ by uniqueness. A similar argument shows that $\beta \circ \alpha = \text{id}_{D'}$. Therefore, $\beta$ is an BF-isomorphism, that is, $D$ is uniquely determined up to BF-isomorphism. \qed
Theorem 3.3 Let \( \{ A_i = (A_i; *, 0_i): i \in I \} \) be a family of BF-algebras. Then \( g_k: A_k \to \prod_{i=1}^{n} A_i \) given by \( a_k \mapsto (0_1, \ldots, 0_{k-1}, a_k, 0_{k+1}, \ldots, 0_n) \) is a BF-monomorphism of BF-algebras for each \( k \in I \).

Proof: For each \( k \in I_n \), define \( g_k: A_k \to \prod_{i=1}^{n} A_i \) by \( g_k(a_k) = (0_1, \ldots, 0_{k-1}, a_k, 0_{k+1}, \ldots, 0_n) \) for all \( a_k \in A_k \). Let \( a_k, b_k \in A_k \). If \( a_k = b_k \), then

\[
g_k(a_k) = (0_1, \ldots, 0_{k-1}, a_k, 0_{k+1}, \ldots, 0_n) = (0_1, \ldots, 0_{k-1}, b_k, 0_{k+1}, \ldots, 0_n) = g_k(b_k).
\]

Hence, \( g_k \) is well-defined. If \( a_k, b_k \in A_k \), then

\[
g_k(a_k * b_k) = (0_1, \ldots, 0_{k-1}, a_k * b_k, 0_{k+1}, \ldots, 0_n)
\]

\[
= (0_1 * 0_1, \ldots, 0_{k-1} * 0_{k-1}, a_k * b_k, 0_{k+1} * 0_{k+1}, \ldots, 0_n * 0_n)
\]

\[
= (0_1, \ldots, 0_{k-1}, a_k, 0_{k+1}, \ldots, 0_n) \otimes (0_1, \ldots, 0_{k-1}, b_k, 0_{k+1}, \ldots, 0_n)
\]

\[
= g_k(a_k) \otimes g_k(b_k).
\]

Thus, \( g_k \) is a BF-homomorphism. If \( g_k(a_k) = g_k(b_k) \), then

\[
(0_1, \ldots, 0_{k-1}, a_k, 0_{k+1}, \ldots, 0_n) = g_k(a_k) = g_k(b_k) = (0_1, \ldots, 0_{k-1}, b_k, 0_{k+1}, \ldots, 0_n).
\]

Hence, \( a_k = b_k \). Therefore, \( g_k \) is a BF-monomorphism. \( \square \)

The maps \( g_k \) in Theorem 3.3 are called the canonical injections. The following lemma shows that the image of the canonical injection is an ideal of the direct product.

Lemma 3.4 Let \( \{ A_i = (A_i; *, 0_i): i \in I \} \) be a family of BF-algebras. For each \( k \in I_n \), if \( g_k \) is the canonical injection, then \( g_k(A_k) \) is an ideal of \( \prod_{i=1}^{n} A_i \).

Proof: Let \( g_k \) be the canonical injection for each \( k \in I_n \). Since \( 0_k \in A_k \) for each \( k \in I_n \), \( (0_1, \ldots, 0_{k-1}, 0_k, 0_{k+1}, \ldots, 0_n) \in g_k(A_k) \). Hence, \( g_k(A_k) \) is not empty. If \( (0_1, \ldots, 0_{k-1}, a_k, 0_{k+1}, \ldots, 0_n) \otimes (0_1, \ldots, 0_{k-1}, b_k, 0_{k+1}, \ldots, 0_n) \in g_k(A_k) \) and \( (0_1, \ldots, 0_{k-1}, b_k, 0_{k+1}, \ldots, 0_n) \in g_k(A_k) \), then

\[
(0_1, \ldots, 0_{k-1}, a_k * b_k, 0_{k+1}, \ldots, 0_n)
\]

\[
= (0_1 * 0_1, \ldots, 0_{k-1} * 0_{k-1}, a_k * b_k, 0_{k+1} * 0_{k+1}, \ldots, 0_n * 0_n)
\]

\[
= (0_1, \ldots, 0_{k-1}, a_k, 0_{k+1}, \ldots, 0_n) \otimes (0_1, \ldots, 0_{k-1}, b_k, 0_{k+1}, \ldots, 0_n) \in g_k(A_k).
\]

This implies that \( a_k * b_k \in A_k \) for each \( k \in I_n \). Since \( b_k \in A_k \), \( a_k \in A_k \) by (I2), so that \( (0_1, \ldots, 0_{k-1}, a_k, 0_{k+1}, \ldots, 0_n) \in g_k(A_k) \) for each \( k \in I_n \). Therefore, \( g_k(A_k) \) is an ideal of \( \prod_{i=1}^{n} A_i \). \( \square \)

The following example shows that there exists \( g_k(A_k) \) in Lemma 3.4 that is not a normal ideal.
Example 3.5 Let \( A_1 \) be the BF-algebra in Example 2.1 and \( A_2 \) be the BF-algebra in Example 2.2. Then \( A_1 \times A_2 = \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 1), (2, 2), (2, 3)\}. Let \( g_1 : A_1 \to A_1 \times A_2 \) be the canonical projection. It follows that \( g_1(A_1) = \{(0, 0), (1, 0), (2, 0)\}. Now, \( g_1(A_1) \) is not normal in \( A_1 \times A_2 \) since \((1, 1) \otimes (1, 3) = (1 \ast 1, 1 \ast 3) = (0, 0) \in g_1(A_1) \) but \((2, 2) \otimes (1, 1)) \oplus (2, 2) \otimes (1, 3) = (0, 2) \notin g_1(A_1)\).

Theorem 3.6 Let \( \{A_i = (A_i;*, \theta_i) : i \in I_0\} \) be a family of BF-algebras. For each \( k \in I_0 \), if \( g_k \) is the canonical injection, then \( g_k(A_k) \) is a normal ideal of \( \prod_{i=1}^{n} A_i \) and \( \prod_{i=1}^{n} A_i / g_k(A_k) \cong \prod_{i \neq k} A_i \).

Proof: Let \( g_k \) be the canonical injection. By Lemma 3.4, \( g_k(A_k) \) is an ideal of \( \prod_{i=1}^{n} A_i \). Let \((a_1, ..., a_k, ..., a_n), (b_1, ..., b_k, ..., b_n), (c_1, ..., c_k, ..., c_n) \in \prod_{i=1}^{n} A_i \).

If \((a_1, ..., a_k, ..., a_n) \otimes (b_1, ..., b_k, ..., b_n) \in g_k(A_k)\), then
\[
(a_1 \ast b_1, ..., a_k \ast b_k, ..., a_n \ast b_n) = (a_1, ..., a_k, ..., a_n) \otimes (b_1, ..., b_k, ..., b_n)
\]

is an element of \( g_k(A_k) \) so that \( a_i \ast b_i = 0 \) for each \( i \neq k \) and \( a_k \ast b_k \in A_k \) and so \((c_k \ast a_k) \ast (c_k \ast b_k) \in A_k \). By \( (BH') \), \( a_i = b_i \) for each \( i \neq k \). Thus,
\[
((c_1, ..., c_k, ..., c_n) \ominus (a_1, ..., a_k, ..., a_n)) \\
\otimes ((c_1, ..., c_k, ..., c_n) \ominus (b_1, ..., b_k, ..., b_n))
\]

\[
= (c_1 \ast a_1, ..., c_k \ast a_k, ..., c_n \ast a_n) \ominus (c_1 \ast b_1, ..., c_k \ast b_k, ..., c_n \ast b_n)
\]

\[
= ((c_1 \ast a_1) \ast (c_1 \ast b_1), ..., (c_k \ast a_k) \ast (c_k \ast b_k), ..., (c_n \ast a_n) \ast (c_n \ast b_n))
\]

\[
= (0_1, ..., 0_{k-1}, (c_k \ast a_k) \ast (c_k \ast b_k), 0_{k+1}, ..., 0_n) \in g_k(A_k).
\]

Therefore, \( g_k(A_k) \) is a normal ideal of \( \prod_{i=1}^{n} A_i \) for each \( k \in I_0 \).

Let \( A = \prod_{i=1}^{n} A_i \) and \( A'_k = g_k(A_k) \). Define \( \varphi_k : A/A'_k \to \prod_{i \neq k} A_i \) given by
\[
\varphi_k((a_1, ..., a_n)/A'_k) = (a_1, ..., a_{k-1}, a_{k+1}, ..., a_n) \text{ for all } (a_1, ..., a_n)/A'_k \in A/A'_k.
\]

Let \((a_1, ..., a_k, ..., a_n)/A'_k, (b_1, ..., b_k, ..., b_n)/A'_k \in A/A'_k \). Suppose that
\[
(a_1, ..., a_k, ..., a_n)/A'_k = (b_1, ..., b_k, ..., b_n)/A'_k.
\]

Then
\[
(a_1, ..., a_k, ..., a_n) \sim A'_k (b_1, ..., b_k, ..., b_n), \text{ that is,}
\]

\[
(a_1 \ast b_1, ..., a_k \ast b_k, ..., a_n \ast b_n) = (a_1, ..., a_k, ..., a_n) \otimes (b_1, ..., b_k, ..., b_n) \in A'_k
\]

so that \( a_i \ast b_i = 0 \) for all \( i \neq k \). Hence, by \( (BH') \), \( a_i = b_i \) for all \( i \neq k \) and so
\[
(a_1, ..., a_{k-1}, a_{k+1}, ..., a_n) = (b_1, ..., b_{k-1}, b_{k+1}, ..., b_n).
\]

This implies that
\[
\varphi_k((a_1, ..., a_n)/A'_k) = (a_1, ..., a_{k-1}, a_{k+1}, ..., a_n) = (b_1, ..., b_{k-1}, b_{k+1}, ..., b_n) = \varphi_k((b_1, ..., b_n)/A'_k).
\]
This shows that $\varphi_k$ is well-defined. Moreover,

\[
\varphi_k((a_1, \ldots, a_k, \ldots, a_n) / A_k^0) = \varphi_k((a_1, \ldots, a_k, \ldots, a_n) * (b_1, \ldots, b_k, \ldots, b_n) / A_k^0)
\]

\[
= \varphi_k(((a_1, \ldots, a_k, \ldots, a_n) * (b_1, \ldots, b_k, \ldots, b_n)) / A_k^0)
\]

\[
= \varphi_k((a_1 * b_1, \ldots, a_k * b_k, \ldots, a_n * b_n) / A_k^0)
\]

\[
= (a_1 * b_1, \ldots, a_k * b_k, \ldots, a_n * b_n)
\]

\[
= (a_1, \ldots, a_k, a_{k+1}, \ldots, a_n) * (b_1, \ldots, b_{k+1}, \ldots, b_n)
\]

This shows that $\varphi_k$ is a BF-homomorphism.

If $\varphi_k((a_1, \ldots, a_k, \ldots, a_n) / A_k^0) = \varphi_k((b_1, \ldots, b_k, \ldots, b_n) / A_k^0)$, then

\[
(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n) = \varphi_k((a_1, \ldots, a_k, \ldots, a_n) / A_k^0)
\]

\[
= \varphi_k((b_1, \ldots, b_k, \ldots, b_n) / A_k^0)
\]

\[
= (b_1, \ldots, b_{k-1}, b_{k+1}, \ldots, b_n).
\]

Thus, $a_i = b_i$ for all $i \neq k$ so that by (BH'), $a_i * b_i = 0_i$ for all $i \neq k$. Hence, $(a_1, \ldots, a_k, \ldots, a_n) * (b_1, \ldots, b_k, \ldots, b_n) = (a_1 * b_1, \ldots, a_k * b_k, \ldots, a_n * b_n)$ is an element of $A_k^0$, that is, $(a_1, \ldots, a_k, \ldots, a_n) \sim (b_1, \ldots, b_k, \ldots, b_n)$ so that $(a_1, \ldots, a_k, \ldots, a_n) / A_k^0 = (b_1, \ldots, b_k, \ldots, b_n) / A_k^0$. This shows that $\varphi_k$ is one-to-one.

If $(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n) \in \prod_{i \neq k} A_i$, then $a_i \in A_i$ for all $i \neq k$ so that $(a_1, \ldots, a_{k-1}, 0_k, a_{k+1}, \ldots, a_n) \in A$ since $0_k \in A_k$. It follows that $(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n) = \varphi_k((a_1, \ldots, a_{k-1}, 0_k, a_{k+1}, \ldots, a_n) / A_k^0)$, where $(a_1, \ldots, a_{k-1}, 0_k, a_{k+1}, \ldots, a_n) / A_k^0 \in A / A_k^0$. This shows that $\varphi_k$ is onto.

Therefore, $\varphi_k$ is a BF-isomorphism, that is, $\prod_{i=1}^n A_i / g_k(A_k) \cong \prod_{i \neq k} A_i$.  \qed

References

http://dx.doi.org/10.2478/s12175-013-0182-6


http://dx.doi.org/10.12988/ija.2016.614


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