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Mappings of the Direct Product of BF-algebras

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Abstract

In this paper, we introduce two canonical mappings of the direct product of BF-algebras and we obtain some of their properties.

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1 Introduction

In [4], the concept of BF-algebras was introduced by A. Walendziak in 2007. A BF-*algebra* is an algebra $\mathbf{A} = (A; *, 0)$ of type (2, 0), that is, a nonempty set A together with a binary operation $*$ and a constant 0, satisfying the following axioms for all $x, y \in A$: (B1) $x * x = 0$, (B2) $x * 0 = x$, (BF) $0 * (x * y) = y * x$. A BF_1-*algebra* is a BF-algebra satisfying (BG) $x = (x * y) * (0 * y)$ and a BF_2-*algebra* is a BF-algebra satisying (BH) $x * y = 0$ and $y * x = 0$ imply $x = y$. A BF_2-algebra satisfies (BH$'$) $x * y = 0$ implies $x = y$. In [4], Walendziak also introduced the notion of commutativity of BF-algebras. A BF-algebra $\mathbf{A}$ is *commutative* if $x * (0 * y) = y * (0 * x)$ for all $x, y \in A$. In 2011, J.C. Endam and J.P. Vilela [2] characterized the commutativity of BF-algebras and established the relationship of BF-algebras and groups. Walendziak also introduced the notions of subalgebras, ideals, and normality in BF-algebras, and established their properties. A subset I of A is called an *ideal* of $\mathbf{A}$ if it satisfies the following for all $x, y \in A$: (I1) $0 \in I$, (I2) $x * y \in I$ and $y \in I$ imply $x \in I$. We say that an ideal I is *normal* if for any $x, y, z \in A$, $x * y \in I$ implies $(z * x) * (z * y) \in I$. A nonempty subset N of A is called a *subalgebra* of $\mathbf{A}$ if $x * y \in N$ for any $x, y \in N$. A. Walendziak then used the concept of normality of BF-algebras to construct quotient BF-algebras. That is, given a normal ideal I of a BF-algebra $\mathbf{A}$, the relation $\sim_I$ is defined by $x \sim_I y$ if and only if $x * y \in I$ for any $x, y \in A$. Then $\sim_I$ is a congruence relation of $\mathbf{A}$. For $x \in A$, write x/I for the congruence class containing x, that is, $x/I = \{y \in A\colon\ x \sim_I y\}$. We denote $A/I = \{x/I\colon\ x \in A\}$ and define $*'$ by $x/I *' y/I = (x * y)/I$. Note that $x/I = y/I$ if and only if $x \sim_I y$. Then the algebra $\mathbf{A}/I = (A/I; *', 0/\mathrm{I})$ is a BF-*algebra*. The algebra $\mathbf{A}/I$ is called the *quotient* BF-*algebra* of $\mathbf{A}$ modulo I. The concept of BF-homomorphism was also introduced by A. Walendziak. A map $\varphi : A \to B$ is called a *BF-homomorphism* if $\varphi(x * y) = \varphi(x) * \varphi(y)$ for any $x, y \in A$. The *kernel of* φ, denoted by ker φ, is defined to be the set $\{x \in A : \varphi(x) = 0_B\}$. A BF-homomorphism φ is called a *BF-monomorphism*, *BF-epimorphism*, or *BF-isomorphism* if φ is one-one, onto, or a bijection, respectively. In [3], R.C. Teves and J.C. Endam introduced and established the direct product of BF-algebras. In this paper, we introduced and established two canonical mappings of the direct product of BF-algebras.

2 Direct Product of BF-algebras

The results in this section are found in [3].

Example 2.1 Let $A = \{0, 1, 2\}$ be a set with the following table:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(A; *, 0)$ is a BF-algebra.

Example 2.2 [4] Let $A = \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	0
2	2	3	0	2
3	3	0	2	0

Then $(A; *, 0)$ is a BF-algebra.

Let $\mathbf{A} = (A; *, 0_A)$ and $\mathbf{B} = (B; *, 0_B)$ be BF-algebras. Define the direct product of $\mathbf{A}$ and $\mathbf{B}$ to be the structure $\mathbf{A} \times \mathbf{B} = (A \times B; \circledast, (0_A, 0_B))$, where $A \times B$ is the set $\{(a, b) : a \in A \text{ and } b \in B\}$ and whose binary operation $\circledast$ is given by $(a_1, b_1) \circledast (a_2, b_2) = (a_1 * a_2, b_1 * b_2)$.

Theorem 2.3 *[3] The direct product of two BF-algebras is also a BF-algebra.*

Now, we extend this direct product to any finite family of BF-algebras. Let $\mathrm{I}_n = \{1, 2, \ldots, n\}$ and let $\{\mathbf{A}_i = (A_i; *, 0_i) : i \in \mathrm{I}_n\}$ be a finite family of BF-algebras. Define the direct product of BF-algebras $\mathbf{A}_1, \ldots, \mathbf{A}_n$ to be the structure $\prod_{i=1}^{n} \mathbf{A}_i = \left(\prod_{i=1}^{n} A_i; \circledast, (0_1, \ldots, 0_n) \right)$, where

$$\prod_{i=1}^{n} A_i = A_1 \times \cdots \times A_n = \{(a_1, \ldots, a_n) : a_i \in A_i, i \in \mathrm{I}_n\}$$

and whose operation $\circledast$ is given by

$$(a_1, \ldots, a_n) \circledast (b_1, \ldots, b_n) = (a_1 * b_1, \ldots, a_n * b_n).$$

Remark 2.4 *[3] If $\{\boldsymbol{A}_i = (A_i; *, 0_i)$: $i \in I_n\}$ is a family of BF-algebras, then $\prod_{i=1}^{n} \boldsymbol{A}_i$ is a BF-algebra.*

Theorem 2.5 *[3] Let $\{\varphi_i : A_i \to B_i$: $i \in I_n\}$ be a family of BF-homomorphisms. If φ is the map $\prod_{i=1}^{n} A_i \to \prod_{i=1}^{n} B_i$ given by $(a_1, \ldots, a_n) \mapsto (\varphi_1(a_1), \ldots, \varphi_n(a_n))$, then φ is a BF-homomorphism with $\ker \varphi = \prod_{i=1}^{n} \ker \varphi_i$, $\varphi(\prod_{i=1}^{n} A_i) = \prod_{i=1}^{n} \varphi_i(A_i)$. Furthermore, φ is a BF-monomorphism (respectively, BF-epimorphism) if and only if φ_i is.*

Theorem 2.6 *[3] Let $\{\boldsymbol{A}_i = (A_i; *, 0_i)$: $i \in I_n\}$ be a family of BF-algebras and let J_i be a normal ideal of $\boldsymbol{A}_i$ for each $i \in I_n$. Then $\prod_{i=1}^{n} J_i$ is a normal ideal of $\prod_{i=1}^{n} \boldsymbol{A}_i$ and $\prod_{i=1}^{n} A_i \,/ \prod_{i=1}^{n} J_i \cong \prod_{i=1}^{n} (A_i / J_i)$.*

3 Canonical Mappings of the Direct Product

This section presents two canonical mappings of the direct product of any finite family of BF-algebras and provides some of their properties.

Theorem 3.1 *Let $\{\boldsymbol{A}_i = (A_i;\, *,\, 0_i)\colon i \in I_n\}$ be a family of BF-algebras. Then $f_k\colon \prod_{i=1}^{n} A_i \to A_k$ given by $(a_1, \ldots, a_k, \ldots, a_n) \mapsto a_k$ is a BF-epimorphism of BF-algebras for each $k \in I_n$.*

Proof: For each $k \in \mathrm{I}_n$, define $f_k\colon \prod_{i=1}^{n} A_i \to A_k$ by $f_k((a_1, \ldots, a_k, \ldots, a_n)) = a_k$ for all $(a_1, \ldots, a_k, \ldots, a_n) \in \prod_{i=1}^{n} A_i$. Let $(a_1, \ldots, a_k, \ldots, a_n)$, $(b_1, \ldots, b_k, \ldots, b_n)$ be elements of $\prod_{i=1}^{n} A_i$. If $(a_1, \ldots, a_k, \ldots, a_n) = (b_1, \ldots, b_k, \ldots, b_n)$, then $a_i = b_i$ for each $i \in \mathrm{I}_n$. Thus, $f_k((a_1, \ldots, a_k, \ldots, a_n)) = a_k = b_k = f_k((b_1, \ldots, b_k, \ldots, b_n))$. Hence, f_k is well-defined. If $(a_1, \ldots, a_k, \ldots, a_n)$, $(b_1, \ldots, b_k, \ldots, b_n) \in \prod_{i=1}^{n} A_i$, then

$$\begin{aligned} f_k((a_1, \ldots, a_k, \ldots, a_n) \circledast (b_1, \ldots, b_k, \ldots, b_n)) &= f_k((a_1 * b_1, \ldots, a_k * b_k, \ldots, a_n * b_n)) \\ &= a_k * b_k \\ &= f_k((a_1, \ldots, a_k, \ldots, a_n)) * f_k((b_1, \ldots, b_k, \ldots, b_n)). \end{aligned}$$

Thus, f_k is a BF-homomorphism. If $c_k \in A_k$, then $(0_1, \ldots, c_k, \ldots, 0_n) \in \prod_{i=1}^{n} A_i$ and $f_k((0_1, \ldots, c_k, \ldots, 0_n)) = c_k$. Therefore, f_k is onto and so f_k is an BF-epimorphism. □

The maps f_k in Theorem 3.1 are called the *canonical projections* of the direct product. The following theorem relates the direct product $\prod_{i=1}^{n} \mathbf{A}_i$ and its canonical projections.

Theorem 3.2 *Let $\{\boldsymbol{A}_i = (A_i;\, *,\, 0_i)\colon i \in I_n\}$ be a family of BF-algebras. Then there exists a BF-algebra $\boldsymbol{D}$, together with a family of BF-homomorphisms $\{f_i\colon D \to A_i : i \in I_n\}$ with the following property: for any BF-algebra $\boldsymbol{C}$ and a family of BF-homomorphisms $\{\varphi_i\colon C \to A_i : i \in I_n\}$, there exists a unique BF-homomorphism $\varphi\colon C \to D$ such that $f_i \circ \varphi = \varphi_i$ for all $i \in I_n$. Furthermore, D is uniquely determined up to BF-isomorphism.*

Proof: Take $\mathbf{D} = \prod_{i=1}^{n} \mathbf{A}_i$. Then $\mathbf{D}$ is a BF-algebra. Let $\{f_i\colon D \to A_i : i \in \mathrm{I}_n\}$ be the family of canonical projections. Suppose that $\mathbf{C}$ is any BF-algebra and $\{\varphi_i\colon C \to A_i : i \in \mathrm{I}_n\}$ a family of BF-homomorphisms. Define $\varphi\colon C \to D$ by $\varphi(c) = (\varphi_1(c), \dots, \varphi_i(c), \dots, \varphi_n(c))$ for all $c \in C$. If $c, d \in C$, then

$$\begin{aligned}
\varphi(c * d) &= (\varphi_1(c * d), \dots, \varphi_i(c * d), \dots, \varphi_n(c * d)) \\
&= (\varphi_1(c) * \varphi_1(d), \dots, \varphi_i(c) * \varphi_i(d), \dots, \varphi_n(c) * \varphi_n(d)) \\
&= (\varphi_1(c), \dots, \varphi_i(c), \dots, \varphi_n(c)) \circledast (\varphi_1(d), \dots, \varphi_i(d), \dots, \varphi_n(d)) \\
&= \varphi(c) \circledast \varphi(d).
\end{aligned}$$

Hence, φ is a BF-homomorphism. Moreover, $f_i \circ \varphi = \varphi_i$ for all $i \in \mathrm{I}_n$ since $(f_i \circ \varphi)(c) = f_i(\varphi(c)) = f_i((\varphi_1(c), \dots, \varphi_i(c), \dots, \varphi_n(c))) = \varphi_i(c)$. To show that φ is unique, let $\varphi' : C \to D$ be another BF-homomorphism such that $f_i \circ \varphi' = \varphi_i$ for all $i \in \mathrm{I}_n$. If $c \in C$, then $(f_i \circ \varphi)(c) = \varphi_i(c) = (f_i \circ \varphi')(c)$. By the definition of φ, $\varphi(c) = (\varphi_1(c), \dots, \varphi_i(c), \dots, \varphi_n(c))$ and assume that $\varphi'(c) = (a_1, \dots, a_i, \dots, a_n)$. Thus, for each $i \in \mathrm{I}_n$,

$$\begin{aligned}
a_i &= f_i((a_1, \dots, a_i, \dots, a_n)) \\
&= f_i(\varphi'(c)) \\
&= (f_i \circ \varphi')(c) \\
&= (f_i \circ \varphi)(c) \\
&= f_i(\varphi(c)) \\
&= f_i((\varphi_1(c), \dots, \varphi_i(c), \dots, \varphi_n(c))) \\
&= \varphi_i(c).
\end{aligned}$$

Hence, $\varphi(c) = (\varphi_1(c), \dots, \varphi_i(c), \dots, \varphi_n(c)) = (a_1, \dots, a_i, \dots, a_n) = \varphi'(c)$. Therefore, φ is unique.

Suppose that a BF-algebra $\mathbf{D}'$ has the same property as $\mathbf{D}$ with the family of BF-homomorphisms $\{f_i'\colon D' \to A_i\colon i \in \mathrm{I}_n\}$. If we apply this property for $\mathbf{D}$ to the family of BF-homomorphisms $\{f_i'\colon D' \to A_i\colon i \in \mathrm{I}_n\}$ and also apply it for $\mathbf{D}'$ to the family of BF-homorphisms $\{f_i\colon D \to A_i\colon i \in \mathrm{I}_n\}$, then we obtain unique BF-homomorphisms $\alpha\colon D' \to D$ and $\beta\colon D \to D'$ such that $f_i \circ \alpha = f_i'$ and $f_i' \circ \beta = f_i$ for all $i \in \mathrm{I}_n$. Thus, $\alpha \circ \beta : D \to D$ is a unique BF-homomorphism such that $f_i \circ (\alpha \circ \beta) = f_i$ for all $i \in \mathrm{I}_n$. Since the identity map $\mathrm{id}_D\colon D \to D$ is a BF-homomorphism such that $f_i \circ \mathrm{id}_D = f_i$ for all $i \in \mathrm{I}_n$, $\alpha \circ \beta = \mathrm{id}_D$ by uniqueness. A similar argument shows that $\beta \circ \alpha = \mathrm{id}_{D'}$. Therefore, β is an BF-isomorphism, that is, D is uniquely determined up to BF-isomorphism. □

Theorem 3.3 *Let $\{\boldsymbol{A}_i = (A_i; *, 0_i): i \in I_n\}$ be a family of BF-algebras. Then $g_k\colon A_k \to \prod_{i=1}^{n} A_i$ given by $a_k \mapsto (0_1, \ldots, 0_{k-1}, a_k, 0_{k+1}, \ldots, 0_n)$ is a BF-monomorphism of BF-algebras for each $k \in I_n$.*

Proof: For each $k \in \mathrm{I}_n$, define $g_k\colon A_k \to \prod_{i=1}^{n} A_i$ by $g_k(a_k) = (0_1, \ldots, 0_{k-1}, a_k, 0_{k+1}, \ldots, 0_n)$ for all $a_k \in A_k$. Let $a_k, b_k \in A_k$. If $a_k = b_k$, then
$g_k(a_k) = (0_1, \ldots, 0_{k-1}, a_k, 0_{k+1}, \ldots, 0_n) = (0_1, \ldots, 0_{k-1}, b_k, 0_{k+1}, \ldots, 0_n) = g_k(b_k)$.
Hence, g_k is well-defined. If $a_k, b_k \in A_k$, then

$$\begin{aligned}
g_k(a_k * b_k) &= (0_1, \ldots, 0_{k-1}, a_k * b_k, 0_{k+1}, \ldots, 0_n)\\
&= (0_1 * 0_1, \ldots, 0_{k-1} * 0_{k-1}, a_k * b_k, 0_{k+1} * 0_{k+1}, \ldots, 0_n * 0_n)\\
&= (0_1, \ldots, 0_{k-1}, a_k, 0_{k+1}, \ldots, 0_n) \circledast (0_1, \ldots, 0_{k-1}, b_k, 0_{k+1}, \ldots, 0_n)\\
&= g_k(a_k) \circledast g_k(b_k).
\end{aligned}$$

Thus, g_k is a BF-homomorphism. If $g_k(a_k) = g_k(b_k)$, then
$(0_1, \ldots, 0_{k-1}, a_k, 0_{k+1}, \ldots, 0_n) = g_k(a_k) = g_k(b_k) = (0_1, \ldots, 0_{k-1}, b_k, 0_{k+1}, \ldots, 0_n)$.
Hence, $a_k = b_k$. Therefore, g_k is a BF-monomorphism. □

The maps g_k in Theorem 3.3 are called the *canonical injections*. The following lemma shows that the image of the canonical injection is an ideal of the direct product.

Lemma 3.4 *Let $\{\boldsymbol{A}_i = (A_i; *, 0_i): i \in I_n\}$ be a family of BF-algebras. For each $k \in I_n$, if g_k is the canonical injection, then $g_k(A_k)$ is an ideal of $\prod_{i=1}^{n} \boldsymbol{A}_i$.*

Proof: Let g_k be the canonical injection for each $k \in \mathrm{I}_n$. Since $0_k \in A_k$ for each $k \in \mathrm{I}_n$, $(0_1, \ldots, 0_{k-1}, 0_k, 0_{k+1}, \ldots, 0_n) \in g_k(A_k)$. Hence, $g_k(A_k)$ is not empty. If $(0_1, \ldots, 0_{k-1}, a_k, 0_{k+1}, \ldots, 0_n) \circledast (0_1, \ldots, 0_{k-1}, b_k, 0_{k+1}, \ldots, 0_n) \in g_k(A_k)$ and $(0_1, \ldots, 0_{k-1}, b_k, 0_{k+1}, \ldots, 0_n) \in g_k(A_k)$, then
$(0_1, \ldots, 0_{k-1}, a_k * b_k, 0_{k+1}, \ldots, 0_n)$
$= (0_1 * 0_1, \ldots, 0_{k-1} * 0_{k-1}, a_k * b_k, 0_{k+1} * 0_{k+1}, \ldots, 0_n * 0_n)$
$= (0_1, \ldots, 0_{k-1}, a_k, 0_{k+1}, \ldots, 0_n) \circledast (0_1, \ldots, 0_{k-1}, b_k, 0_{k+1}, \ldots, 0_n) \in g_k(A_k)$.
This implies that $a_k * b_k \in A_k$ for each $k \in \mathrm{I}_n$. Since $b_k \in A_k$, $a_k \in A_k$ by (I2), so that $(0_1, \ldots, 0_{k-1}, a_k, 0_{k+1}, \ldots, 0_n) \in g_k(A_k)$ for each $k \in \mathrm{I}_n$. Therefore, $g_k(A_k)$ is an ideal of $\prod_{i=1}^{n} \mathbf{A}_i$. □

The following example shows that there exists $g_k(A_k)$ in Lemma 3.4 that is not a normal ideal.

Example 3.5 Let $\mathbf{A}_1$ be the BF-algebra in Example 2.1 and $\mathbf{A}_2$ be the BF-algebra in Example 2.2. Then $A_1 \times A_2 = \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 1), (2, 2), (2, 3)\}$. Let $g_1 : A_1 \to A_1 \times A_2$ be the canonical projection. It follows that $g_1(A_1) = \{(0, 0), (1, 0), (2, 0)\}$. Now, $g_1(A_1)$ is not normal in $\mathbf{A}_1 \times \mathbf{A}_2$ since $(1,1) \circledast (1,3) = (1*1, 1*3) = (0,0) \in g_1(A_1)$ but $((2,2) \circledast (1,1)) \circledast ((2,2) \circledast (1,3)) = (0,2) \notin g_1(A_1)$.

Theorem 3.6 *Let $\{\boldsymbol{A}_i = (A_i; *, 0_i): i \in I_n\}$ be a family of BF$_2$-algebras. For each $k \in I_n$, if g_k is the canonical injection, then $g_k(A_k)$ is a normal ideal of $\prod_{i=1}^{n} \boldsymbol{A}_i$ and $\prod_{i=1}^{n} A_i \,/\, g_k(A_k) \cong \prod_{i\neq k} A_i$.*

Proof: Let g_k be the canonical injection. By Lemma 3.4, $g_k(A_k)$ is an ideal of $\prod_{i=1}^{n} \mathbf{A}_i$. Let $(a_1, \dots, a_k, \dots, a_n), (b_1, \dots, b_k, \dots, b_n), (c_1, \dots, c_k, \dots, c_n) \in \prod_{i=1}^{n} A_i$. If $(a_1, \dots, a_k, \dots, a_n) \circledast (b_1, \dots, b_k, \dots, b_n) \in g_k(A_k)$, then $(a_1 * b_1, \dots, a_k * b_k, \dots, a_n * b_n) = (a_1, \dots, a_k, \dots, a_n) \circledast (b_1, \dots, b_k, \dots, b_n)$ is an element of $g_k(A_k)$ so that $a_i * b_i = 0_i$ for each $i \neq k$ and $a_k * b_k \in A_k$ and so $(c_k * a_k) * (c_k * b_k) \in A_k$. By (BH′), $a_i = b_i$ for each $i \neq k$. Thus,

$$\begin{aligned}
&((c_1, \dots, c_k, \dots, c_n) \circledast (a_1, \dots, a_k, \dots, a_n)) \\
&\quad \circledast ((c_1, \dots, c_k, \dots, c_n) \circledast (b_1, \dots, b_k, \dots, b_n)) \\
&= (c_1 * a_1, \dots, c_k * a_k, \dots, c_n * a_n) \circledast (c_1 * b_1, \dots, c_k * b_k, \dots, c_n * b_n) \\
&= ((c_1 * a_1) * (c_1 * b_1), \dots, (c_k * a_k) * (c_k * b_k), \dots, (c_n * a_n) * (c_n * b_n)) \\
&= ((c_1 * a_1) * (c_1 * a_1), \dots, (c_k * a_k) * (c_k * b_k), \dots, (c_n * a_n) * (c_n * a_n)) \\
&= (0_1, \dots, 0_{k-1}, (c_k * a_k) * (c_k * b_k), 0_{k+1}, \dots, 0_n) \in g_k(A_k).
\end{aligned}$$

Therefore, $g_k(A_k)$ is a normal ideal of $\prod_{i=1}^{n} \mathbf{A}_i$ for each $k \in \mathrm{I}_n$.

Let $A = \prod_{i=1}^{n} A_i$ and $A'_k = g_k(A_k)$. Define $\varphi_k : A/A'_k \to \prod_{i\neq k} A_i$ given by $\varphi_k((a_1, \dots, a_n)/A'_k) = (a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n)$ for all $(a_1, \dots, a_n)/A'_k \in A/A'_k$. Let $(a_1, \dots, a_k, \dots, a_n)/A'_k,\ (b_1, \dots, b_k, \dots, b_n)/A'_k \in A/A'_k$. Suppose that $(a_1, \dots, a_k, \dots, a_n)/A'_k = (b_1, \dots, b_k, \dots, b_n)/A'_k$. Then $(a_1, \dots, a_k, \dots, a_n) \sim_{A'_k} (b_1, \dots, b_k, \dots, b_n)$, that is,

$$(a_1 * b_1, \dots, a_k * b_k, \dots, a_n * b_n) = (a_1, \dots, a_k, \dots, a_n) \circledast (b_1, \dots, b_k, \dots, b_n) \in A'_k$$

so that $a_i * b_i = 0_i$ for all $i \neq k$. Hence, by (BH′), $a_i = b_i$ for all $i \neq k$ and so $(a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n) = (b_1, \dots, b_{k-1}, b_{k+1}, \dots, b_n)$. This implies that

$$\begin{aligned}
\varphi_k((a_1, \dots, a_k, \dots, a_n)/A'_k) &= (a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n) \\
&= (b_1, \dots, b_{k-1}, b_{k+1}, \dots, b_n) \\
&= \varphi_k((b_1, \dots, b_k, \dots, b_n)/A'_k).
\end{aligned}$$

This shows that φ_k is well-defined. Moreover,

$$\begin{aligned}
&\varphi_k((a_1,\ldots,\, a_k,\ldots,\, a_n)/A'_k *' (b_1,\ldots,\, b_k,\ldots,\, b_n)/A'_k)\\
&\qquad = \varphi_k(((a_1,\ldots,\, a_k,\ldots,\, a_n) \circledast (b_1,\ldots,\, b_k,\ldots,\, b_n))/A'_k)\\
&\qquad = \varphi_k((a_1 * b_1,\ldots,\, a_k * b_k,\ldots,\, a_n * b_n)/A'_k)\\
&\qquad = (a_1 * b_1,\ldots,\, a_{k-1} * b_{k-1},\, a_{k+1} * b_{k+1},\ldots,\, a_n * b_n)\\
&\qquad = (a_1,\ldots,\, a_{k-1},\, a_{k+1},\ldots,\, a_n) \circledast (b_1,\ldots,\, b_{k-1},\, b_{k+1},\ldots,\, b_n)\\
&\qquad = \varphi_k((a_1,\ldots,\, a_k,\ldots,\, a_n)/A'_k) \circledast \varphi_k((b_1,\ldots,\, b_k,\ldots,\, b_n)/A'_k).
\end{aligned}$$

This shows that φ_k is a BF-homomorphism.
If $\varphi_k((a_1,\ldots,\, a_k,\ldots,\, a_n)/A'_k) = \varphi_k((b_1,\ldots,\, b_k,\ldots,\, b_n)/A'_k)$, then

$$\begin{aligned}
(a_1,\ldots,a_{k-1},a_{k+1},\ldots,a_n) &= \varphi_k((a_1,\ldots,a_k,\ldots,a_n)/A'_k)\\
&= \varphi_k((b_1,\ldots,b_k,\ldots,b_n)/A'_k)\\
&= (b_1,\ldots,b_{k-1},b_{k+1},\ldots,b_n).
\end{aligned}$$

Thus, $a_i = b_i$ for all $i \neq k$ so that by (BH′), $a_i * b_i = 0_i$ for all $i \neq k$. Hence, $(a_1,\ldots,a_k,\ldots,a_n) \circledast (b_1,\ldots,\, b_k,\ldots,\, b_n) = (a_1 * b_1,\ldots,\, a_k * b_k,\ldots,\, a_n * b_n)$ is an element of A'_k, that is, $(a_1,\ldots,\, a_k,\ldots,\, a_n) \sim_{A'_k} (b_1,\ldots,\, b_k,\ldots,\, b_n)$ so that $(a_1,\ldots,\, a_k,\ldots,\, a_n)/A'_k = (b_1,\ldots,\, b_k,\ldots,\, b_n)/A'_k$. This shows that φ_k is one-to-one.

If $(a_1,\ldots,\, a_{k-1},\, a_{k+1},\ldots,\, a_n) \in \prod_{i\neq k} A_i$, then $a_i \in A_i$ for all $i \neq k$ so that $(a_1,\ldots,\, a_{k-1},\, 0_k,\, a_{k+1},\ldots,\, a_n) \in A$ since $0_k \in A_k$. It follows that $(a_1,\ldots,\, a_{k-1},\, a_{k+1},\ldots,\, a_n) = \varphi_k((a_1,\ldots,\, a_{k-1},\, 0_k,\, a_{k+1},\ldots,\, a_n)/A'_k)$, where $(a_1,\ldots,\, a_{k-1},\, 0_k,\, a_{k+1},\ldots,\, a_n)/A'_k \in A/A'_k$. This shows that φ_k is onto. Therefore, φ_k is a BF-isomorphism, that is, $\prod_{i=1}^{n} A_i \;/\; g_k(A_k) \cong \prod_{i\neq k} A_i$. □

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