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Inequalities for Certain Means in Structural Mechanics[1]

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Abstract

In the article, we prove that the inequality

$$A^{\alpha}(a,b)I^{1-\alpha}(a,b) \le M_{(2+\alpha)/3}(a,b)$$

holds for all $a, b > 0$ if $\alpha \in [(3\log 2-2)/(1-\log 2), 1)$ and the inequality is reversed if $\alpha \in (0, (3\sqrt{145}-35)/10]$, where $A(a,b)$, $I(a,b)$ and $M_p(a,b)$ are respectively the arithmetic, identric and pth power means of a and b.

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1. Introduction

For $p \in \mathbb{R}$, the power mean $M_p(a,b)$ of order p and the identric mean $I(a,b)$ of two positive numbers a and b are defined by

$$M_p(a,b) = \begin{cases} \left(\frac{a^p + b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases} \tag{1.1}$$

and

$$I(a,b) = \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, & a \neq b, \\ a, & a = b, \end{cases} \tag{1.2}$$

respectively.

It is well-known that there are many practical problems in structural mechanics need to deal the power mean $M_p(a,b)$, identric mean $I(a,b)$ and other bivariate means. $M_p(a,b)$ is continuous and strictly increasing with respect to $p \in R$ for fixed $a, b > 0$ with $a \neq b$. In the recent past, both mean values have been the subject of intensive research. In particular, many remarkable inequalities for $M_p(a,b)$ and $I(a,b)$ can be found in literature [1-14].

Let $A(a,b) = (a+b)/2$, $L(a,b) = (b-a)/(\log b - \log a)$ $(a \neq b)$ and $L(a,a) = a$, $G(a,b) = \sqrt{ab}$ and $H(a,b) = 2ab/(a+b)$ be the arithmetic mean, logarithmic mean, geometric mean and harmonic mean of two positive numbers a and b, respectively. Then

$$\begin{aligned} \min\{a,b\} &\leq H(a,b) = M_{-1}(a,b) \leq G(a,b) = M_0(a,b) \leq L(a,b) \\ &\leq I(a,b) \leq A(a,b) = M_1(a,b) \leq \max\{a,b\}, \end{aligned} \tag{1.3}$$

and each inequality in (1.3) holds equality if and only if $b = a$.

In [15], Alzer and Janous established the following sharp double inequality (see also [11, p. 350])

$$M_{\log 2/\log 3}(a,b) \leq \frac{2}{3}A(a,b) + \frac{1}{3}G(a,b) \leq M_{2/3}(a,b)$$

for all $a, b > 0$.

For any $\alpha \in (0,1)$, Janous [16] found the greatest value p and the least value q such that

$$M_p(a,b) \leq \alpha A(a,b) + (1-\alpha)G(a,b) \leq M_q(a,b)$$

for all $a, b > 0$.

In [17-19], the authors presented the bounds for L and I in terms of A and G as follows

$$G^{2/3}(a,b)A^{1/3}(a,b) \le L(a,b) \le \frac{2}{3}G(a,b) + \frac{1}{3}A(a,b)$$

and

$$\frac{1}{3}G(a,b) + \frac{2}{3}A(a,b) \le I(a,b)$$

for all $a, b > 0$.

The following companion of (1.3) provides inequalities for the geometric and arithmetic means of L and I, the proof can be found in [20].

$$G^{\frac{1}{2}}(a,b)A^{\frac{1}{2}}(a,b) \le L^{\frac{1}{2}}(a,b)I^{\frac{1}{2}}(a,b) \le \frac{1}{2}L(a,b) + \frac{1}{2}I(a,b)$$
$$\le \frac{1}{2}G(a,b) + \frac{1}{2}A(a,b)$$

for all $a, b > 0$.

The following sharp bounds for L, I, $(LI)^{1/2}$, and $(L+I)/2$ in terms of power means $M_p(a,b)$ are proved in [13, 20-25].

$$L(a,b) \le M_{1/3}(a,b), \quad M_{2/3}(a,b) \le I(a,b) \le M_{\log 2}(a,b),$$

$$M_0(a,b) \le \sqrt{L(a,b)I(a,b)} \le M_{1/2}(a,b)$$

and

$$\frac{1}{2}\left(L(a,b) + I(a,b)\right) < M_{1/2}(a,b)$$

for all $a, b > 0$.

Alzer and Qiu [26] proved

$$M_c(a,b) \le \frac{1}{2}L(a,b) + \frac{1}{2}I(a,b)$$

for all a, $b > 0$ with the best possible parameter $c = \log 2/(1 + \log 2)$, and

$$\alpha A(a,b) + (1-\alpha)G(a,b) \le I(a,b) \le \beta A(a,b) + (1-\beta)G(a,b)$$

for $\alpha \le 2/3$, $\beta \ge 2/e = 0.73575...$ and $a, b > 0$.

The main purpose of this paper is to give the sharp bounds for $A^{\alpha}I^{1-\alpha}$ in terms of power means for some $\alpha \in (0,1)$.

2. Lemmas

In order to establish our main results, we need a lemma, which we present in this section.

Lemma 2.1. Let $g(t) = (1-r)(t^{\frac{2+r}{3}+1} + t^{\frac{2+r}{3}} + t + 1)\log t + (2r-1)t^{\frac{2+r}{3}+1} - 2rt^{\frac{2+r}{3}} + t^{\frac{2+r}{3}-1} - t^2 + 2rt + 1 - 2r$. Then the following statements are true:

(1) If $r \in [\frac{3\log 2-2}{1-\log 2}, 1)$, then there exists $\lambda \in (1,+\infty)$, such that $g(t) > 0$ for $t \in (1,\lambda)$ and $g(t) < 0$ for $t \in (\lambda,+\infty)$.

(2) If $r \in (0, \frac{3\sqrt{145}-35}{10}]$, then $g(t) < 0$ for $t \in (1,+\infty)$.

Proof. Let $r \in (0,1)$, $p = \frac{2+r}{3}$, $g_1(t) = t^{1-p}g'(t)$, $g_2(t) = t^p g_1'(t)$, $g_3(t) = t^{1-p}g_2'(t)$, $g_4(t) = t^3 g_3'(t)$, $g_5(t) = t^{p-2}g_4'(t)$, $g_6(t) = t^3 g_5'(t)$, $g_7(t) = t^{1-p}g_6'(t)$, and $g_8(t) = t^p g_7'(t)$. Then simple computation leads to

$$g(1) = 0, \tag{2.1}$$

$$\lim_{t\to+\infty} g(t) = -\infty, \tag{2.2}$$

$$g_1(t) = (1-r)[t^{1-p} + (1+p)t + p]\log t - 2t^{2-p} + (1+r)t^{1-p} \\ +(1-r)t^{-p} + (2pr-p+r)t - (1-p)t^{-1} - 2pr - r + 1,$$

$$g_1(1) = 0, \tag{2.3}$$

$$\lim_{t\to+\infty} g_1(t) = -\infty, \tag{2.4}$$

$$g_2(t) = (1-r)[(1+p)t^p + 1 - p]\log t + (pr+1)t^p + p(1-r)t^{p-1} \\ +(1-p)t^{p-2} - 2(2-p)t - p(1-r)t^{-1} - pr - p + 2,$$

$$g_2(1) = 0, \tag{2.5}$$

$$\lim_{t\to+\infty} g_2(t) = -\infty. \tag{2.6}$$

$$g_3(t) = p(1+p)(1-r)\log t - 2(2-p)t^{1-p} + (1+p)(1-r)t^{-p} \\ +p(1-r)t^{-1-p} - p(1-p)(1-r)t^{-1} - (1-p)(2-p)t^{-2} \\ +p^2r - pr + 2p - r + 1,$$

$$g_3(1) = 6p - 4 - 2r = 0, \tag{2.7}$$

$$\lim_{t\to+\infty} g_3(t) = -\infty, \tag{2.8}$$

$$g_4(t) = p(1-r)[(1+p)t^2 + (1-p)t - (1-p)t^{2-p} - (1+p)t^{1-p}] \\ -2(1-p)(2-p)(t^{3-p} - 1),$$

$$g_4(1) = 0, \tag{2.9}$$

$$\lim_{t\to+\infty} g_4(t) = -\infty, \tag{2.10}$$

$$g_5(t) = p(1-r)[2(1+p)t^{p-1} + (1-p)t^{p-2} - (1-p)(2-p)t^{-1} \\ -(1-p)(1+p)t^{-2}] - 2(1-p)(2-p)(3-p),$$

$$g_5(1) = 4(1-r)p^2 - 2(1-p)(2-p)(3-p),$$

$$= \frac{2}{27}(1-r)(5r^2+35r-4), \tag{2.11}$$

$$\lim_{t\to+\infty} g_5(t) = -2(1-p)(2-p)(3-p) < 0, \tag{2.12}$$

$$g_6(t) = p(1-r)[-2(1+p)(1-p)t^{p+1} - (1-p)(2-p)t^p \\ +(1-p)(2-p)t + 2(1+p)(1-p)],$$

$$g_6(1) = 0, \tag{2.13}$$

$$g_7(t) = p(1-p)(1-r)[(2-p)t^{1-p} - 2(1+p)^2 t - p(2-p)],$$

$$g_7(1) = -p^2(1-p)(7+p)(1-r) < 0, \tag{2.14}$$

$$g_8(t) = p(1-p)(1-r)[-2(1+p)^2 t^p + (1-p)(2-p)], \tag{2.15}$$

and

$$g_8(1) = -p^2(1-p)(7+p)(1-r) < 0. \tag{2.16}$$

(1) If $r \in [\frac{3\log 2-2}{1-\log 2}, 1)$, then from (2.11) and $\frac{3\log 2-2}{1-\log 2} = 0.258891... > \frac{3\sqrt{145}-35}{10} = 0.112478...$ we get

$$g_5(1) > 0. \tag{2.17}$$

From (2.15) we clearly see that $g_8(t)$ is strictly decreasing in $[1, +\infty)$, then (2.16) implies that $g_8(t) < 0$ for $t \in [1, +\infty)$. Hence $g_7(t)$ is strictly decreasing in $[1, +\infty)$.

From (2.14) and the monotonicity of $g_7(t)$, we know that $g_7(t) < 0$ for $t \in [1, +\infty)$. Hence $g_6(t)$ is strictly decreasing in $[1, +\infty)$.

(2.13) and the monotonicity of $g_6(t)$ imply that $g_6(t) < 0$ for $t \in [1, +\infty)$. Hence that $g_5(t)$ is strictly decreasing in $[1, +\infty)$.

From (2.12) and (2.17) together with the monotonicity of $g_5(t)$, we know that there exists $t_0 \in (1, +\infty)$, such that $g_5(t) > 0$ for $t \in (1, t_0)$, and $g_5(t) < 0$ for $t \in (t_0, +\infty)$. Hence $g_4(t)$ is strictly increasing in $[1, t_0]$, and $g_4(t)$ is strictly decreasing in $[t_0, +\infty)$.

From (2.9), (2.10) and the monotonicity of $g_4(t)$, we obtain that there exists $t_1 \in (1, +\infty)$, such that $g_4(t) > 0$ for $t \in (1, t_1)$, and $g_4(t) < 0$ for $t \in [t_1, +\infty)$. Hence $g_3(t)$ is strictly increasing in $[1, t_1]$, and $g_3(t)$ is strictly decreasing in $[t_1, +\infty)$.

From (2.7) and (2.8) together with the monotonicity of $g_3(t)$ we clearly see that there exists $t_2 \in (1, +\infty)$, such that $g_3(t) > 0$ for $t \in (1, t_2)$, and $g_3(t) < 0$ for $t \in (t_2, +\infty)$. Hence $g_2(t)$ is strictly increasing in $[1, t_2]$, and $g_2(t)$ is strictly decreasing in $[t_2, +\infty)$.

From (2.5), (2.6) and the monotonicity of $g_2(t)$, we obtain that there exists $t_3 \in (1, +\infty)$, such that $g_2(t) > 0$ for $t \in (1, t_3)$, and $g_2(t) < 0$ for $t \in (t_3, +\infty)$. Hence $g_1(t)$ is strictly increasing in $[1, t_3]$, and $g_1(t)$ is strictly decreasing in $[t_3, +\infty)$.

From (2.3) and (2.4) together with the monotonicity of $g_1(t)$ we know that there exists $t_4 \in (1, +\infty)$, such that $g_1(t) > 0$ for $t \in (1, t_4)$, and $g_1(t) < 0$ for $t \in (t_4, +\infty)$. Hence $g(t)$ is strictly increasing in $[1, t_4]$, and $g(t)$ is decreasing in $[t_4, +\infty)$.

Therefore, Lemma 2.1 (1) follows from (2.1) and (2.2) together with the monotonicity of $g(t)$.

(2) If $r \in (0, \frac{3\sqrt{145}-35}{10}]$, then from (2.11) we clearly see that

$$g_5(1) \leq 0. \tag{2.18}$$

From (2.15) we know that $g_8(t)$ is strictly decreasing. Therefore, Lemma 2.1 (2) follows from the monotonicity of $g_8(t)$, (2.16), (2.14), (2.13), (2.18), (2.9), (2.7), (2.5), (2.3) and (2.1). □

3. Main Results

Theorem 3.1. For all $a, b > 0$, we have

$$A^{\alpha}(a,b)I^{1-\alpha}(a,b) \leq M_{\frac{2+\alpha}{3}}(a,b) \tag{3.1}$$

for $\alpha \in [\frac{3\log 2-2}{1-\log 2}, 1)$, and

$$M_{\frac{2+\alpha}{3}}(a,b) \leq A^{\alpha}(a,b)I^{1-\alpha}(a,b) \tag{3.2}$$

for $\alpha \in (0, \frac{3\sqrt{145}-35}{10}]$. Inequality (3.1) or (3.2) holds equality if and only if $a = b$, and the parameter $\frac{2+\alpha}{3}$ in inequalities (3.1) and (3.2) cannot be improved.

Proof. If $a = b$, then from (1.1) and (1.2) we clearly see that $A^{\alpha}(a,b)I^{1-\alpha}(a,b) = M_{\frac{2+\alpha}{3}}(a,b) = a$ for any $\alpha \in (0,1)$.

If $a \neq b$, without loss of generality, we assume that $a > b$. Let $t = \frac{a}{b} > 1$ and $p = \frac{2+\alpha}{3}$, then (1.1) and (1.2) leads to

$$\begin{aligned} & M_p(a,b) - A^{\alpha}(a,b)I^{1-\alpha}(a,b) \\ = \quad & b\left[\left(\frac{t^p+1}{2}\right)^{\frac{1}{p}} - \left(\frac{t+1}{2}\right)^{\alpha}\left(\frac{1}{e}\cdot t^{\frac{t}{t-1}}\right)^{1-\alpha}\right]. \end{aligned} \tag{3.3}$$

Let

$$f(t)=\frac{1}{p}\log\frac{1+t^p}{2}-\alpha\log\frac{t+1}{2}-(1-\alpha)\frac{t}{t-1}\log t+(1-\alpha),$$

then

$$\lim_{t\to 1} f(t)=0, \tag{3.4}$$

$$\lim_{t\to\infty} f(t)=(1-\alpha)+(\alpha-\frac{1}{p})\log 2 \tag{3.5}$$

and

$$f'(t) \quad = \frac{g(t)}{(t+1)(t-1)^2(t^p+1)}, \tag{3.6}$$

where

$$\begin{aligned} g(t)&=(1-\alpha)(t^{p+1}+t^p+t+1)\log t\\ &\quad+(2\alpha-1)t^{p+1}-2\alpha t^p+t^{p-1}-t^2+2\alpha t+1-2\alpha. \end{aligned}$$

If $\alpha\in[\frac{3\log 2-2}{1-\log 2},1)$, then (3.5) leads to

$$\lim_{t\to\infty} f(t)=\frac{(1-\alpha)(\alpha+3)}{\alpha+2}\left(\frac{\alpha+2}{\alpha+3}-\log 2\right)\geq 0. \tag{3.7}$$

Therefore, $A^{\alpha}(a,b)I^{1-\alpha}(a,b)<M_{\frac{2+\alpha}{3}}(a,b)$ for $a\neq b$ follows from (3.3), (3.4), (3.6), (3.7) and Lemma 2.1 (1).

If $\alpha\in(0,\frac{3\sqrt{145}-35}{10}]$, then $A^{\alpha}(a,b)I^{1-\alpha}(a,b)>M_{\frac{2+\alpha}{3}}(a,b)$ for $a\neq b$ follows from (3.3), (3.4), (3.6) and Lemma 2.1 (2).

Next, we prove that the parameter $\frac{2+\alpha}{3}$ in inequalities (3.1) and (3.2) cannot be improved.

Case 1. If $\alpha\in[\frac{3\log 2-2}{1-\log 2},1)$, then for any $0<\varepsilon<\frac{2+\alpha}{3}$, let $0<x<1$ and $x\to 0$, making use of the Taylor expansion, we have

$$\begin{aligned} &\log\left[A^{\alpha}(1,1+x)I^{1-\alpha}(1,1+x)\right]-\log M_{\frac{2+\alpha}{3}-\varepsilon}(1,1+x)\\ =\;&\alpha\log(1+\frac{x}{2})+\frac{(1-\alpha)(1+x)}{x}\log(1+x)-(1-\alpha)\\ &-\frac{3}{2+\alpha-3\varepsilon}\log\frac{1+(1+x)^{\frac{2+\alpha-3\varepsilon}{3}}}{2}\\ =\;&\frac{\varepsilon}{8}x^2+o(x^2). \end{aligned} \tag{3.8}$$

Equation (3.8) implies that for any $\alpha \in [\frac{3\log 2-2}{1-\log 2}, 1)$ and $0 < \varepsilon < \frac{2+\alpha}{3}$, there exists $0 < \delta_1 = \delta_1(\varepsilon, \alpha) < 1$, such that

$$A^{\alpha}(1, 1+x)I^{1-\alpha}(1, 1+x) > M_{\frac{2+\alpha}{3}-\varepsilon}(1, 1+x)$$

for $x \in (0, \delta_1)$.

Case 2. If $\alpha \in (0, \frac{3\sqrt{145}-35}{10}]$, then for any $0 < \varepsilon < \frac{2+\alpha}{3}$, let $0 < x < 1$ and $x \to 0$, making use of the Taylor expansion, we have

$$\begin{aligned} & \log\left[A^{\alpha}(1, 1+x)I^{1-\alpha}(1, 1+x)\right] - \log M_{\frac{2+\alpha}{3}+\varepsilon}(1, 1+x) \\ = \; & \alpha\log(1+\frac{x}{2}) + \frac{(1-\alpha)(1+x)}{x}\log(1+x) - (1-\alpha) \\ & -\frac{3}{2+\alpha+3\varepsilon}\log\frac{1+(1+x)^{\frac{2+\alpha+3\varepsilon}{3}}}{2} \\ = \; & -\frac{\varepsilon}{8}x^2 + o(x^2). \end{aligned} \tag{3.9}$$

Equation (3.9) implies that for any $\alpha \in (0, \frac{3\sqrt{145}-35}{10}]$ and $0 < \varepsilon < \frac{2+\alpha}{3}$, there exists $0 < \delta_2 = \delta_2(\varepsilon, \alpha) < 1$, such that

$$A^{\alpha}(1, 1+x)I^{1-\alpha}(1, 1+x) < M_{\frac{2+\alpha}{3}+\varepsilon}(1, 1+x)$$

for $x \in (0, \delta_2)$. $\square$

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