Inequalities for Certain Means in Structural Mechanics\textsuperscript{1}

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Abstract
In the article, we prove that the inequality
\[ A^\alpha(a, b) I^{1-\alpha}(a, b) \leq M_{(2+\alpha)/3}(a, b) \]
holds for all \( a, b > 0 \) if \( \alpha \in \left[\frac{(3 \log 2 - 2)}{(1 - \log 2)}, 1\right) \) and the inequality is reversed if \( \alpha \in (0, \frac{3\sqrt{145} - 35}{10}] \), where \( A(a, b) \), \( I(a, b) \) and \( M_p(a, b) \) are respectively the arithmetic, identric and \( p \)th power means of \( a \) and \( b \).

Mathematics Subject Classification: 26E60

Keywords: structural mechanics, power mean, identric mean, geometric mean, arithmetic mean

\textsuperscript{1}This research was supported by the Natural Science Foundation of China under Grant 61374086 and the Natural Science Foundation of Zhejiang Province under Grant LY13A010004.

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1. Introduction

For \( p \in \mathbb{R} \), the power mean \( M_p(a, b) \) of order \( p \) and the identric mean \( I(a, b) \) of two positive numbers \( a \) and \( b \) are defined by

\[
M_p(a, b) = \begin{cases} 
\left( \frac{a^p + b^p}{2} \right)^{1/p}, & p \neq 0, \\
\sqrt{ab}, & p = 0,
\end{cases}
\]

and

\[
I(a, b) = \begin{cases} 
\frac{1}{e} \left( \frac{b^b}{a^a} \right)^{1/(b-a)}, & a \neq b, \\
\frac{a}{a}, & a = b,
\end{cases}
\]

respectively.

It is well-known that there are many practical problems in structural mechanics need to deal the power mean \( M_p(a, b) \), identric mean \( I(a, b) \) and other bivariate means. \( M_p(a, b) \) is continuous and strictly increasing with respect to \( p \in \mathbb{R} \) for fixed \( a, b > 0 \) with \( a \neq b \). In the recent past, both mean values have been the subject of intensive research. In particular, many remarkable inequalities for \( M_p(a, b) \) and \( I(a, b) \) can be found in literature [1-14].

Let \( A(a, b) = (a + b)/2, \ L(a, b) = (b - a)/(\log b - \log a) \) \( a \neq b \) and \( L(a, a) = a, \ G(a, b) = \sqrt{ab} \) and \( H(a, b) = 2ab/(a + b) \) be the arithmetic mean, logarithmic mean, geometric mean and harmonic mean of two positive numbers \( a \) and \( b \), respectively. Then

\[
\min\{a, b\} \leq H(a, b) = M_{-1}(a, b) \leq G(a, b) = M_0(a, b) \leq L(a, b) \leq I(a, b) \leq A(a, b) = M_1(a, b) \leq \max\{a, b\},
\]

and each inequality in (1.3) holds equality if and only if \( b = a \).

In [15], Alzer and Janous established the following sharp double inequality (see also [11, p. 350])

\[
M_{\log 2/\log 3}(a, b) \leq \frac{2}{3} A(a, b) + \frac{1}{3} G(a, b) \leq M_{2/3}(a, b)
\]

for all \( a, b > 0 \).

For any \( \alpha \in (0, 1) \), Janous [16] found the greatest value \( p \) and the least value \( q \) such that

\[
M_p(a, b) \leq \alpha A(a, b) + (1 - \alpha)G(a, b) \leq M_q(a, b)
\]

for all \( a, b > 0 \).
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In [17-19], the authors presented the bounds for \( L \) and \( I \) in terms of \( A \) and \( G \) as follows

\[
G^{2/3}(a,b)A^{1/3}(a,b) \leq L(a,b) \leq \frac{2}{3}G(a,b) + \frac{1}{3}A(a,b)
\]

and

\[
\frac{1}{3}G(a,b) + \frac{2}{3}A(a,b) \leq I(a,b)
\]

for all \( a,b > 0 \).

The following companion of (1.3) provides inequalities for the geometric and arithmetic means of \( L \) and \( I \), the proof can be found in [20].

\[
G^{1/2}(a,b)A^{1/2}(a,b) \leq L^{1/2}(a,b)I^{1/2}(a,b) \leq \frac{1}{2}L(a,b) + \frac{1}{2}I(a,b) \leq \frac{1}{2}G(a,b) + \frac{1}{2}A(a,b)
\]

for all \( a,b > 0 \).

The following sharp bounds for \( L, I, (LI)^{1/2}, \) and \((L+I)/2\) in terms of power means \( M_p(a,b) \) are proved in [13, 20-25].

\[
L(a,b) \leq M_{1/3}(a,b), \quad M_{2/3}(a,b) \leq I(a,b) \leq M_{\log 2}(a,b),
\]

\[
M_0(a,b) \leq \sqrt{L(a,b)I(a,b)} \leq M_{1/2}(a,b)
\]

and

\[
\frac{1}{2}(L(a,b) + I(a,b)) < M_{1/2}(a,b)
\]

for all \( a,b > 0 \).

Alzer and Qiu [26] proved

\[
M_c(a,b) \leq \frac{1}{2}L(a,b) + \frac{1}{2}I(a,b)
\]

for all \( a, b > 0 \) with the best possible parameter \( c = \log 2/(1 + \log 2) \), and

\[
\alpha A(a,b) + (1 - \alpha)G(a,b) \leq I(a,b) \leq \beta A(a,b) + (1 - \beta)G(a,b)
\]

for \( \alpha \leq 2/3, \beta \geq 2/e = 0.73575... \) and \( a,b > 0 \).

The main purpose of this paper is to give the sharp bounds for \( A^\alpha I^{1-\alpha} \) in terms of power means for some \( \alpha \in (0, 1) \).

2. Lemmas

In order to establish our main results, we need a lemma, which we present in this section.
Lemma 2.1. Let \( g(t) = (1 - r)(t^{2 + r} + t^{2 + r} + t + 1) \log t + (2r - 1)t^{2 + r} + 2rt^{2 + r} + t^{2} + 2rt + 1 - 2r. \) Then the following statements are true:

1. If \( r \in \left[\frac{3\log 2 - 2}{1 - \log 2}, 1\right), \) then there exists \( \lambda \in (1, +\infty), \) such that \( g(t) > 0 \) for \( t \in (1, \lambda) \) and \( g(t) < 0 \) for \( t \in (\lambda, +\infty). \)

2. If \( r \in (0, \frac{3\sqrt{145} - 35}{10}], \) then \( g(t) < 0 \) for \( t \in (1, +\infty). \)

Proof. Let \( r \in (0, 1), \) \( p = \frac{2 + r}{3}, \) \( g_1(t) = t^{1-p}g'(t), \) \( g_2(t) = t^pg_1'(t), \) \( g_3(t) = t^{1-p}g_2'(t), \) \( g_4(t) = t^3g_3'(t), \) \( g_5(t) = t^{p-2}g_4'(t), \) \( g_6(t) = t^3g_5'(t), \) \( g_7(t) = t^{1-p}g_6'(t), \) and \( g_8(t) = t^pg_7'(t). \) Then simple computation leads to

\[
\begin{align*}
g(1) &= 0, \\
\lim_{t \to +\infty} g(t) &= -\infty, \\
g_1(t) &= (1 - r)[t^{1-p} + (1 + p)t + p] \log t - 2t^{2-p} + (1 + r)t^{1-p} \\
&\quad + (1 - r)t^{1-p} + (2pr - p + r)t - (1 - p)t^{-1} - 2pr - r + 1, \\
g_1(1) &= 0, \\
\lim_{t \to +\infty} g_1(t) &= -\infty, \\
g_2(t) &= (1 - r)[(1 + p)t^p + 1 - p] \log t + (pr + 1)t^p + p(1 - r)t^{p-1} \\
&\quad + (1 - p)t^{p-2} - 2(2 - p)t - p(1 - r)t^{-1} - pr - p + 2, \\
g_2(1) &= 0, \\
\lim_{t \to +\infty} g_2(t) &= -\infty. \\
g_3(t) &= p(1 + p)(1 - r) \log t - 2(2 - p)t^{1-p} + (1 + p)(1 - r)t^{-p} \\
&\quad + p(1 - r)t^{-1-p} - p(1 - p)(1 - r)t^{-1} - (1 - p)(2 - p)t^{-2} \\
&\quad + p^2r - pr + 2p - r + 1, \\
g_3(1) &= 6p - 4 - 2r = 0, \\
\lim_{t \to +\infty} g_3(t) &= -\infty, \\
g_4(t) &= p(1 - r)[(1 + p)t^2 + (1 - p)t - (1 - p)t^{2-p} - (1 + p)t^{1-p}] \\
&\quad - 2(1 - p)(2 - p)(t^{3-p} - 1),
\end{align*}
\]
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\[ g_4(1) = 0, \quad \lim_{t \to +\infty} g_4(t) = -\infty, \quad \text{(2.9)} \]

\[ g_5(t) = p(1 - r)[2(1 + p)t^{p-1} + (1 - p)t^{p-2} - (1 - p)(2 - p)t^{-1} \]
\[ - (1 - p)(1 + p)t^{-2}] - 2(1 - p)(2 - p)(3 - p), \]
\[ g_5(1) = 4(1 - r)p^2 - 2(1 - p)(2 - p)(3 - p), \]
\[ = \frac{2}{27}(1 - r)(5r^2 + 35r - 4), \quad \text{(2.11)} \]

\[ \lim_{t \to +\infty} g_5(t) = -2(1 - p)(2 - p)(3 - p) < 0, \quad \text{(2.12)} \]

\[ g_6(t) = p(1 - r)[-2(1 + p)(1 - p)t^{p+1} - (1 - p)(2 - p)t^p \]
\[ + (1 - p)(2 - p)t + 2(1 + p)(1 - p)], \]
\[ g_6(1) = 0, \quad \text{(2.13)} \]

\[ g_7(t) = p(1 - p)(1 - r)[(2 - p)t^{1-p} - 2(1 + p)^2t - (2 - p)], \]
\[ g_7(1) = -p^2(1 - p)(7 + p)(1 - r) < 0, \quad \text{(2.14)} \]

\[ g_8(t) = p(1 - p)(1 - r)[-2(1 + p)^2t^p + (1 - p)(2 - p)], \quad \text{(2.15)} \]

and

\[ g_8(1) = -p^2(1 - p)(7 + p)(1 - r) < 0. \quad \text{(2.16)} \]

(1) If \( r \in \left[ \frac{3\log 2 - 2}{1 - \log 2}, 1 \right) \), then from (2.11) and \( \frac{3\log 2 - 2}{1 - \log 2} = 0.258891... > \frac{3\sqrt{15} - 35}{10} = 0.112478... \) we get

\[ g_5(1) > 0. \quad \text{(2.17)} \]

From (2.15) we clearly see that \( g_8(t) \) is strictly decreasing in \([1, +\infty)\), then (2.16) implies that \( g_8(t) < 0 \) for \( t \in [1, +\infty) \). Hence \( g_7(t) \) is strictly decreasing in \([1, +\infty)\).

From (2.14) and the monotonicity of \( g_7(t) \), we know that \( g_7(t) < 0 \) for \( t \in [1, +\infty) \). Hence \( g_6(t) \) is strictly decreasing in \([1, +\infty)\).

(2.13) and the monotonicity of \( g_6(t) \) imply that \( g_6(t) < 0 \) for \( t \in [1, +\infty) \). Hence that \( g_6(t) \) is strictly decreasing in \([1, +\infty)\).

From (2.12) and (2.17) together with the monotonicity of \( g_5(t) \), we know that there exists \( t_0 \in (1, +\infty) \), such that \( g_5(t) > 0 \) for \( t \in (1, t_0) \), and \( g_5(t) < 0 \) for \( t \in (t_0, +\infty) \). Hence \( g_4(t) \) is strictly increasing in \([1, t_0]\), and \( g_4(t) \) is strictly decreasing in \([t_0, +\infty)\).

From (2.9), (2.10) and the monotonicity of \( g_4(t) \), we obtain that there exists \( t_1 \in (1, +\infty) \), such that \( g_4(t) > 0 \) for \( t \in (1, t_1) \), and \( g_4(t) < 0 \) for \( t \in [t_1, +\infty) \). Hence \( g_3(t) \) is strictly increasing in \([1, t_1]\), and \( g_3(t) \) is strictly decreasing in \([t_1, +\infty)\).
From (2.7) and (2.8) together with the monotonicity of $g_3(t)$ we clearly see that there exists $t_2 \in (1, +\infty)$, such that $g_3(t) > 0$ for $t \in (1, t_2)$, and $g_3(t) < 0$ for $t \in (t_2, +\infty)$. Hence $g_3(t)$ is strictly increasing in $[1, t_2]$, and $g_2(t)$ is strictly decreasing in $[t_2, +\infty)$.

From (2.5), (2.6) and the monotonicity of $g_2(t)$, we obtain that there exists $t_3 \in (1, +\infty)$, such that $g_2(t) > 0$ for $t \in (1, t_3)$, and $g_2(t) < 0$ for $t \in (t_3, +\infty)$. Hence $g_1(t)$ is strictly increasing in $[1, t_3]$, and $g_1(t)$ is strictly decreasing in $[t_3, +\infty)$.

From (2.3) and (2.4) together with the monotonicity of $g_1(t)$ we know that there exists $t_4 \in (1, +\infty)$, such that $g_1(t) > 0$ for $t \in (1, t_4)$, and $g_1(t) < 0$ for $t \in (t_4, +\infty)$. Hence $g(t)$ is strictly increasing in $[1, t_4]$, and $g(t)$ is decreasing in $[t_4, +\infty)$.

Therefore, Lemma 2.1 (1) follows from (2.1) and (2.2) together with the monotonicity of $g(t)$.

(2) If $r \in (0, \frac{3\sqrt{145}-35}{10}]$, then from (2.11) we clearly see that $g_5(1) \leq 0$. (2.18)

From (2.15) we know that $g_8(t)$ is strictly decreasing. Therefore, Lemma 2.1 (2) follows from the monotonicity of $g_8(t)$, (2.16), (2.14), (2.13), (2.18), (2.9), (2.7), (2.5), (2.3) and (2.1).  □

3. Main Results

Theorem 3.1. For all $a, b > 0$, we have

$$A^\alpha(a, b)I^{1-\alpha}(a, b) \leq M_{2^{+\alpha}}(a, b)$$

(3.1)

for $\alpha \in (\frac{3\log 2-2}{1-\log 2}, 1)$, and

$$M_{2^{+\alpha}}(a, b) \leq A^\alpha(a, b)I^{1-\alpha}(a, b)$$

(3.2)

for $\alpha \in (0, \frac{3\sqrt{145}-35}{10}]$. Inequality (3.1) or (3.2) holds equality if and only if $a = b$, and the parameter $\frac{2^{+\alpha}}{3}$ in inequalities (3.1) and (3.2) cannot be improved.

Proof. If $a = b$, then from (1.1) and (1.2) we clearly see that $A^\alpha(a, b)I^{1-\alpha}(a, b) = M_{2^{+\alpha}}(a, b) = a$ for any $\alpha \in (0, 1)$.

If $a \neq b$, without loss of generality, we assume that $a > b$. Let $t = \frac{a}{b} > 1$ and $p = \frac{2^{+\alpha}}{3}$, then (1.1) and (1.2) leads to

$$M_p(a, b) - A^\alpha(a, b)I^{1-\alpha}(a, b) = b \left[ \left( \frac{t^p + 1}{2} \right)^{\frac{1}{p}} - \left( \frac{t + 1}{2} \right)^{\alpha} \left( \frac{1}{e}, t^t \right)^{1-\alpha} \right].$$

(3.3)
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Let

\[ f(t) = \frac{1}{p} \log \frac{1 + tp}{2} - \alpha \log \frac{t + 1}{2} - (1 - \alpha) \frac{t}{t - 1} \log t + (1 - \alpha), \]

then

\[ \lim_{t \to 1} f(t) = 0, \quad (3.4) \]

\[ \lim_{t \to \infty} f(t) = (1 - \alpha) + (\alpha - \frac{1}{p}) \log 2 \quad (3.5) \]

and

\[ f'(t) = \frac{g(t)}{(t + 1)(t - 1)^2(t^p + 1)}, \quad (3.6) \]

where

\[ g(t) = (1 - \alpha)(t^{p+1} + t^p + t + 1) \log t \\
+ (2\alpha - 1)t^{p+1} - 2\alpha t^p + t^{p-1} - t^2 + 2\alpha t + 1 - 2\alpha. \]

If \( \alpha \in \left[ \frac{3 \log 2 - 2}{1 - \log 2}, 1 \right) \), then (3.5) leads to

\[ \lim_{t \to \infty} f(t) = \frac{(1 - \alpha)(\alpha + 3)}{\alpha + 2} \left( \frac{\alpha + 2}{\alpha + 3} - \log 2 \right) \geq 0. \quad (3.7) \]

Therefore, \( A^\alpha(a,b)I^{1-\alpha}(a,b) < M_{2+\alpha}^2(a,b) \) for \( a \neq b \) follows from (3.3), (3.4), (3.6), (3.7) and Lemma 2.1 (1).

If \( \alpha \in (0, \frac{3\sqrt{15} - 35}{10}] \), then \( A^\alpha(a,b)I^{1-\alpha}(a,b) > M_{2+\alpha}^2(a,b) \) for \( a \neq b \) follows from (3.3), (3.4), (3.6) and Lemma 2.1 (2).

Next, we prove that the parameter \( \frac{2+\alpha}{3} \) in inequalities (3.1) and (3.2) cannot be improved.

Case 1. If \( \alpha \in (\frac{3\log 2 - 2}{1 - \log 2}, 1] \), then for any \( 0 < \varepsilon < \frac{2+\alpha}{3} \), let \( 0 < x < 1 \) and \( x \to 0 \), making use of the Taylor expansion, we have

\[ \log \left[ A^\alpha(1,1+x)I^{1-\alpha}(1,1+x) \right] - \log M_{2+\alpha}^{2+\alpha-\varepsilon}(1,1+x) \]
\[ = \alpha \log(1 + \frac{x}{2}) + \frac{(1 - \alpha)(1+x)}{x} \log(1 + x) - (1 - \alpha) \]
\[ - \frac{3}{2 + \alpha - 3\varepsilon} \log \frac{1 + (1 + x)^{2+\alpha-3\varepsilon}}{2} \]
\[ = \frac{\varepsilon}{8} x^2 + o(x^2). \quad (3.8) \]
Equation (3.8) implies that for any $\alpha \in \left[\frac{3\log 2 - 2}{1 - \log 2}, 1\right)$ and $0 < \varepsilon < \frac{2 + \alpha}{3}$, there exists $0 < \delta_1 = \delta_1(\varepsilon, \alpha) < 1$, such that

$$A^\alpha(1, 1 + x)I^{1-\alpha}(1, 1 + x) > M_{\frac{2 + \alpha}{3} + \varepsilon}(1, 1 + x)$$

for $x \in (0, \delta_1)$.

Case 2. If $\alpha \in (0, \frac{3\sqrt{145} - 35}{10}]$, then for any $0 < \varepsilon < \frac{2 + \alpha}{3}$, let $0 < x < 1$ and $x \to 0$, making use of the Taylor expansion, we have

$$\log \left[ A^\alpha(1, 1 + x)I^{1-\alpha}(1, 1 + x) \right] - \log M_{\frac{2 + \alpha}{3} + \varepsilon}(1, 1 + x) = \alpha \log \left(1 + \frac{x}{2}\right) + \frac{(1 - \alpha)(1 + x)}{x} \log(1 + x) - (1 - \alpha)$$

$$- \frac{3}{2 + \alpha + 3\varepsilon} \log \left(1 + \frac{(1 + x)^{2 + \alpha + 3\varepsilon}}{2}\right)$$

$$= -\frac{\varepsilon}{8} x^2 + o(x^2). \quad (3.9)$$

Equation (3.9) implies that for any $\alpha \in (0, \frac{3\sqrt{145} - 35}{10}]$ and $0 < \varepsilon < \frac{2 + \alpha}{3}$, there exists $0 < \delta_2 = \delta_2(\varepsilon, \alpha) < 1$, such that

$$A^\alpha(1, 1 + x)I^{1-\alpha}(1, 1 + x) < M_{\frac{2 + \alpha}{3} + \varepsilon}(1, 1 + x)$$

for $x \in (0, \delta_2)$. □

**References**


Received: March 28, 2016; Published: May 16, 2016