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Distance Two Labeling of Certain Snake Graphs

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Abstract

An L(2,1) labeling (or) distance two labeling of a graph G is a function f from the vertex set V(G) to the set of all non-negative integers such that $|f(x) - f(y)| \geq 2$ if d(x,y) =1 and $|f(x) - f(y)| \geq 1$ if d(x, y) = 2. The L(2, 1) labeling number $\lambda(G)$ of G is the smallest number k such that G has an L(2,1) labeling with $\max\{f(v), v \in V(G)\} = k$. In this paper we determine the L(2,1) labeling number $\lambda(G)$ for the line graph of triangular snake graph and spiked snake graph.

Keywords: L(2,1) labeling, Snake graph, triangular snake graph, spiked snake graph

1 Introduction

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. Labeled graphs have a broad range of applications such as coding theory, X-ray crystallography, radar, astronomy, circuit design, communication networks.

The concept of graph labeling was introduced by Rosa in 1967[9]. Hence in the intervening years various labeling of graphs such as graceful labeling, harmonious labeling, magic labeling, antimagic labeling, bimagic labeling, prime labeling, cordial labeling, total cordial labeling, k-graceful labeling and odd graceful labeling etc., have been studied in over 2100 papers. Hale [6] introduced the graph theory model of assignment of channels in 1980. A channel assignment problem was designed in such a way that the vertices of distance two are considered to be close and vertices which are adjacent, are considered to be very close which paved way for distance two labeling of graphs. Labeling with a condition of distance two was introduced by J.R. Griggs and R.K. Yeh [5] who proved that every graph with maximum degree k has an $L(2,1)$-labeling with span at most k^2+2k and proved the conjecture for 2-regular graphs. G.J. Chang and D. Kuo [1] improved this upper bound to k^2+k. Chang et al. [2] generalized this to obtain $k^2+(d-1)k$ as an upper bound on the minimum span of an $L(d,1)$-labeling. Z. Fredi and J.H. Kang [3] proved it for 3-regular Hamiltonian graphs and for the incidence graphs of projective planes. Tight bounds on the maximum span have been obtained for special classes of graphs like paths, cycles, wheels, complete k-partite graphs and graphs with diameter 2, trees [1,2], etc. Bounds have also been obtained for various other graph families like chordal graphs and unit interval graphs and planar graphs [8] and hyper cubes [5, 10, 11].

2 Preliminaries

In this section we give the basic notations relevant to this paper. In this paper, the graphs considered are all finite, undirected and simple. $V(G)$ and $E(G)$ denote the vertex set and the edge set of G.

Definition 2.1.

An $L(2,1)$ labeling of a graph is a labeling of the vertices with non-negative integers such that the labels on adjacent vertices differ by at least 2 and the labels on vertices at distance 2 differ by at least 1.

Definition 2.2.

An L(2,1) labeling (or) distance two labeling of a graph G is a function f from the vertex set V(G) to the set of all non-negative integers such that $|f(x) - f(y)| \geq 2$ if $d(x,y)=1$ and $|f(x) - f(y)| \geq 1$ if $d(x,y) = 2$. The L(2, 1) labeling number $\lambda(G)$ of G is the smallest number k such that G has an L(2, 1) labeling with $\max\{f(v), v\in V(G)\} = k$.

Definition 2.3.

A triangular snake T_n is obtained from a path $a_1, a_2 \ldots a_n$ by joining a_i and a_{i+1} to a new vertex b_i for i=1,2,…..n-1. A triangular snake has $2n$+1 vertices and $3n$ edges, where n is the number of vertices in the path.
A triangular snake T_n is shown in figure 1.

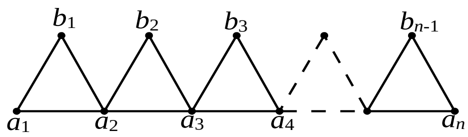

Figure 1

Definition 2.4.
For a given graph *G*, its line graph *L*(*G*) is a graph such that (i) each vertex of *L*(*G*) represents an edge of *G* and (ii) two vertices of *L*(*G*) are adjacent if and only if their corresponding edges share a common end point ("are adjacent") in *G*. A triangular snake T_5 is shown in figure 2 and the line graph of a triangular snake L(T_5) is shown in figure 3.

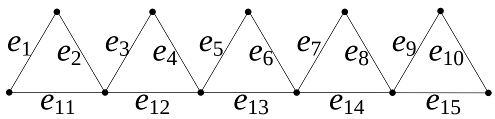

Figure 2

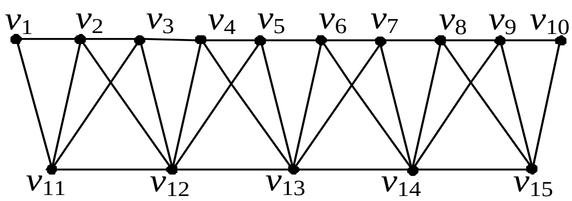

Figure 3

Definition 2.5.
A spiked snake graph *SS*(4,*n*) is a graph obtained from a cyclic snake *S*(4,*n*) with additional edges . It is a graph with vertex set $V(SS(4,n)) = \{v_1, v_2, \dots v_{n+1}\} \cup \{x_i, w_i, u_i, y_i : 1 \leq i \leq n\}$ and edge set $E(SS(4,n)) = \{v_iw_i, w_iv_{i+1}, v_{i+1}u_i, u_iv_i, w_ix_i, u_iy_i : 1 \leq i \leq n\}$. A spiked snake graph *SS*(4,4) is shown is figure 4.

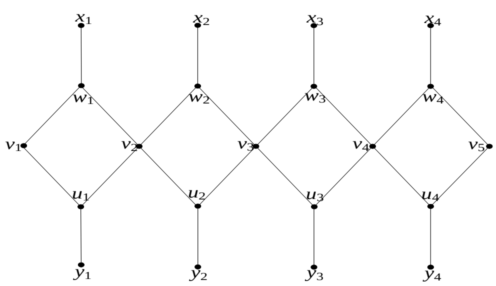

Figure 4

Definition 2.6.
Any function $f : V(G) \to N \cup \{0\}$ is said to be a valid $L(2,1)$ labeling if and only if it satisfies the condition $d(u, v) + |f(u) - f(v)| \geq 3$.

3. Main Results

Theorem 3.1.
The L(2,1) labeling number $\lambda(G)$ of a line graph of a triangular snake $L(T_n)$, $n \geq 3$. is10.

Proof.
Denote the vertices of the line graph of a triangular snake graph as follows:
$V(G) = V_1 \cup V_2$ where $V_1 = \{v_i / 1 \leq i \leq 2n\}$ and $V_2 = \{v_i / 2n+1 \leq i \leq 3n\}$
For $1 \leq i \leq 2n$
Define a mapping $f : V_1(G) \to N \cup \{0\}$ by

$$f(v_{3i-2}) = 0; \quad 1 \leq i \leq \left\lceil \frac{2n+1}{3} \right\rceil$$

$$f(v_{3i-1}) = 2; \quad 1 \leq i \leq \left\lceil \frac{2n\text{-}1}{3} \right\rceil$$

$$f(\mathrm{v}_{3i}) = 4; \quad 1 \leq i \leq \left\lfloor \frac{2n}{3} \right\rfloor$$

For $2n+1 \leq i \leq 3n$
Define a mapping $f: V_2(G) \to N \cup \{0\}$ by

$$f(v_{2n+3i-2}) = 6; \quad 1 \leq i \leq \left\lceil \frac{n}{3} \right\rceil$$

$$f(v_{2n+3i-1}) = 8; \quad 1 \leq i \leq \left\lceil \frac{n}{3} \right\rceil$$

$$f(v_{2n+3i}) = 10; \quad 1 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor$$

Claim:
The $L(2,1)$ labeling number of the line graph of a triangular graph $L(T_n) = 10$.

Case (i):
Let x, y be any two vertices in $V_1(G)$.

Subcase (i).
Let x, y be any two adjacent vertices on $V_1(G)$, such that $x = v_{3i-2}$ and $y = v_{3i-1}$.
Then $f(x) = 0$, $f(y) = 2$ and $d(x, y) = 1$.
Therefore $d(x, y) + |f(x) - f(y)| = 1 + |0-2| = 3$.

***Subcase* (ii).**
Let x, y be any two non–adjacent vertices on $V_1(G)$., such that $x = v_{3i-2}$and $y = v_{3i}$.Then $f(x) = 0$, $f(y) = 4$ and $d(x, y) \geq 2$.Therefore $d(x, y) + |f(x) - f(y)| \geq 2 + |0-4| \geq 3$.

Case (ii):
Let x, y be any two vertices in $V_2(G)$.

Subcase (i).
Let x, y be any two adjacent vertices on the $V_2(G)$,such that $x = v_{2n+3i-2}$and$y = v_{2n+3i-1}$.Then $f(x) = 6$ and $f(y) = 8$ and $d(x, y) = 1$.
Therefore $d(x, y) + |f(x) - f(y)| = 1 + |6-8| = 3$

Subcase (ii).
Let x, y be any two non–adjacent vertices on $V_2(G)$, such that $x = v_{2n+3i-2}$ and $y = v_{2n+3i}$.Then $f(x) = 6$and $f(y) = 10$ and $d(x, y) \geq 2$.Therefore $d(x, y) + |f(x) - f(y)| \geq 2 + |6-10| \geq 3$.

***Case (iii)*:**
Let x, y be any two vertices, on the $V_1(G)$ and $V_2(G)$ respectively

Subcase (i).
Let x, y be any two adjacent vertices, on $V_1(G)$ and $V_2(G)$ respectively. Let x be a vertex on $V_1(G)$and y be a vertex on $V_2(G)$such that $x = v_{3i-2}$ and $y = v_{2n+3i-2}$Then $f(x) = 0$and $f(y) = 6$and $d(x, y) \geq 1$.Therefore $d(x, y) + |f(x) - f(y)| \geq 1 + |0-6| \geq 3$.
***Subcase* (*ii*).**
Let x, y be any two non-adjacent vertices, on $V_1(G)$ and $V_2(G)$ respectively.
Let x be a vertex on $V_1(G)$ and y be a vertex on $V_2(G)$ such that $x = v_{3i-2}$ and $y = v_{2n+3i-1}$.Then $f(x) = 0$ and $f(y) = 8$, then $d(x, y) \geq 1$.Therefore $d(x, y) + |f(x) - f(y)| \geq 1 + |0-8| \geq 3$.
Similarly, for all the other possibilities of $x$and$y$ we find that$d(x, y) + |f(x) - f(y)| \geq 3$. Therefore the $L(2, 1)$ labeling number of $L(T_n)$ is $\lambda(G) = 10$.

Example 3.3.
The $L(2,1)$ labeling of line graph of a triangular snake $L(T_5)$ is shown in Figure 5.

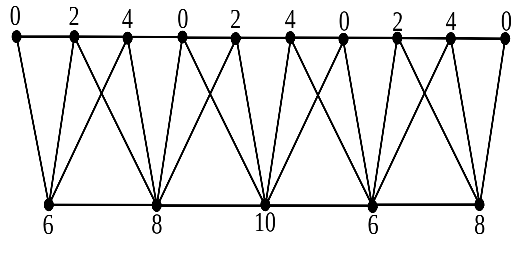

Figure 5

Hence the $L(2,1)$ labeling of line graph of a triangular snake $L(T_n)$ is 10

Theorem 3.2.
The $L(2, 1)$ labeling number $\lambda(G)$ of a spiked snake graph $SS(4,n)$, $n \geq 2$ is 6.

Proof.
Denote the vertices of a spiked snake graph $SS(4,n)$ as follows:
$V(SS(4,n)) = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$ where
$V_1 = \{x_i / 1 \leq i \leq n\}$
$V_2 = \{w_i / 1 \leq i \leq n\}$
$V_3 = \{v_i / 1 \leq i \leq n+1\}$
$V_4 = \{u_i / 1 \leq i \leq n\}$
$V_5 = \{y_i / 1 \leq i \leq n\}$

Define a mapping $f: V(G) \rightarrow N \cup \{0\}$ such that

$$f(x_i) = 0, \text{ for } 1 \leq i \leq n$$

$$f(\mathrm{y}_i) = 3, \text{ for } 1 \leq i \leq n$$

$$f(w_{2i}) = 3, \text{ for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$$

$$f(w_{2i-1}) = 2, \text{ for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$$

$$f(u_{2i}) = 1, \text{ for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$$

$$f(u_{2i-1}) = 0, \text{ for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$$

$$f(v_{2i}) = 6, \text{ for } 1 \leq i \leq \left\lfloor \frac{n+1}{2} \right\rfloor$$

$$f(v_{2i-1}) = 5, \text{ for } 1 \leq i \leq \left\lceil \frac{n+1}{2} \right\rceil$$

Claim.
The $L(2,1)$ labeling number $\lambda(G)$ of $SS(4,n)$ is 6.

***Case (i)*:**
Let s, t be any two vertices in $V_1(G)$.Clearly $d(s, t) \geq 4$.Hence the $L(2,1)$ labeling condition is satisfied.

***Case (ii)*:**
Let s, t be any two vertices in $V_2(G)$. such that $s = w_{2i}$ and $t = w_{2i-1}$.Then $f(s) = 3$, $f(t) = 2$ and $d(s, t) = 2$.Therefore $d(s, t) + |f(s) - f(t)| \geq 2 + |3-2| \geq 3$.

***Case (iii)*:**
Let s, t be any two vertices in $V_3(G)$, such that $s = v_{2i}$ and $t = v_{2i-1}$.Then $f(s) = 6$, $f(t) = 5$ and $d(s, t) = 2$.Therefore $d(s, t) + |f(s) - f(t)| \geq 2 + |6-5| \geq 3$.

***Case (iv)*:**
Let s, t be any two vertices in $V_4(G)$,such that $s = u_{2i}$ and $t = u_{2i-1}$.Then $f(s) = 1$, $f(t) = 0$ and $d(s, t) = 2$.Therefore $d(s, t) + |f(s) - f(t)| \geq 2 + |1-0| = 3$.

***Case (v)*:**
Let s, t be any two vertices in $V_5(G)$. Clearly $d(s, t) \geq 4$.Hence the $L(2,1)$ labeling condition is satisfied.

Case (vi):
Let s, t be any two vertices in $V_1(G)$ and $V_2(G)$ respectively

Subcase (i).
Let s, t be any two adjacent vertices in $V_1(G)$ and $V_2(G)$ respectively such that $s = x_{2i-1}$and $t = w_{2i-1}$then $f(s) = 0$, $f(t) = 2$ and $d(s, t) = 1$. Therefore $d(s, t) + |f(s) - f(t)| = 1 + |0-2| \geq 3$.

Subcase (ii).
Let s, t be any two non-adjacent vertices in $V_1(G)$ and $V_2(G)$ respectively such that $s = x_{2i-1}$and $t = w_{2i}$ then $d(s, t) \geq 3$. Therefore the $L(2,1)$ labeling condition is satisfied.

***Case (vii)*:**
Let s, t be any two vertices in $V_1(G)$ and $V_3(G)$ respectively

***Subcase* (i).**
Let s, t be any two vertices in $V_1(G)$ and $V_3(G)$ respectively such that $s = x_i$and $t = v_{2i}$ then$f(s) = 0$, $f(t) = 6$ and $d(s, t) \geq 2$. Therefore $d(s, t) + |f(s) - f(t)| \geq 2 + |0-6| \geq 3$.

***Case (viii)*:**
Let s, t be any two vertices in $V_2(G)$ and $V_3(G)$ respectively

Subcase (i).
Let s, t be any two adjacent vertices in $V_2(G)$ and $V_3(G)$ respectively such that $s = w_{2i}$and $t = v_{2i}$ then $f(s) = 3$, $f(t) = 6$ and $d(s, t) = 1$. Therefore $d(s, t) + |f(s) - f(t)| = 1 + |3-6| \geq 3$.

Subcase (ii).
Let s, t be any two non-adjacent vertices in $V_2(G)$ and $V_3(G)$ respectively such that $s = w_{2i}$and $t = v_{2i-1}$ then $f(s) = 3$, $f(t) = 5$ and $d(s, t) \geq 1$. Therefore $d(s, t) + |f(s) - f(t)| \geq 1 + |3-5| \geq 3$.

***Case (ix)*:**
Let s, t be any two vertices in $V_2(G)$ and $V_4(G)$ respectively
***Subcase (i)*.**
Let s, t be any two vertices in V_2(G) and$V_4(G)$ respectively such that $s = w_{2i}$and $t = u_{2i}$ then $f(s) = 3$, $f(t) = 1$ and $d(s, t) \geq 2$. Therefore $d(s, t) + |f(s) - f(t)| \geq 2 + |3-1| \geq 3$.
***Case (x)*:**
Let s, t be e any two vertices in $V_3(G)$ and $V_4(G)$respectively

***Subcase* (i).**
Let s, t be any two adjacent vertices in $V_3(G)$ and $V_4(G)$ respectively such that $s = v_{2i}$and $t = u_{2i}$ then $f(s) = 6$, $f(t) = 1$ and $d(s, t) = 1$. Therefore $d(s, t) + |f(s) - f(t)| = 1+ |6-1| \geq 3$.

***Subcase (ii)*.**
Let s, t be any two non-adjacent vertices in $V_3(G)$ and $V_4(G)$ respectively then $d(s, t) \geq 3$. Hence the $L(2,1)$ labeling condition is satisfied.

***Case (xi)*:**
Let s, t be any two vertices in $V_4(G)$ and $V_5(G)$ respectively

Subcase (i).
Let s, t be any two adjacent vertices in $V_4(G)$ and $V_5(G)$ respectively such that $s = u_{2i}$and $t = y_{2i}$ then $f(s) =1$, $f(t) = 3$ and $d(s, t) = 1$. Therefore $d(s, t) + |f(s) - f(t)| = 1+ |1-3| \geq 3$.

***Subcase (ii)*.**
Let s, t be any two non-adjacent vertices in $V_4(G)$ and $V_5(G)$ respectively, then $d(s, t) \geq 3$. Hence the $L(2,1)$ labeling condition is satisfied.
Similarly, for all the possibilities of s and t we find $d(s, t) + |f(s) - f(t)| \geq 3$. Therefore the $L(2, 1)$ labeling of $SS(4,n)$ is $\lambda(G) = 6$

Example 3.2.
$L(2,1)$ labeling of spiked snake graph $SS(4,n)$ is shown in Figure 6.

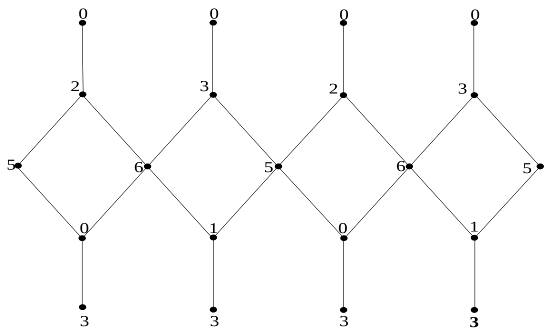

Figure 6

Hence the $L(2,1)$ labeling number $\lambda(G)$ of spiked snake graph $SS(4,n)$) is 6.

5. Conclusion

In this paper the $L(2,1)$ labeling number for the line graph of triangular snake graph and spiked snake graph have been determined.

References

[1] G.J. Chang and D. Kuo, The L(2, 1) labeling problem on graphs, *SIAM Journal on Discrete Math.*, **9** (1996), 309 - 316. http://dx.doi.org/10.1137/s0895480193245339

[2] G.J. Chang and D. Kuo et al., On L (d, 1)-labeling of graphs, *Discrete Math.*, **220** (2000), 57 - 66. http://dx.doi.org/10.1016/s0012-365x(99)00400-8

[3] Z. Fredi and J.H. Kang, L(2,1)-labeling of 3-regular Hamiltonian graphs.

[4] J.A. Gallian, A Dynamic survey of Graph labeling, *The Electronic Journal of Combinatorics*, **16** (2015).

[5] J.R. Griggs and R.K. Yeh, Labeling with a condition of distance two, *SIAM Journal on Discrete Math.*, **5** (1992), 586 - 595. http://dx.doi.org/10.1137/0405048

[6] W.K. Hale, The Frequency Assignment: theory and application, *Proc. IEEE.*, **68** (1980), 1497 - 1514. http://dx.doi.org/10.1109/proc.1980.11899

[7] F. Harary and R.Z. Norman, Some properties of line digraphs, *Rendiconti del Circolo Matematico di Palermo*, **9** (1960), no. 2, 161 - 168. http://dx.doi.org/10.1007/bf02854581

[8] K. Jonas, *Graph Coloring Analogues with a Condition at Distance Two: L(2, 1)-Labeling and List Λ-Labelings*, Ph. D. Thesis, University of South Carolina, (1993).

[9] Gordon and Breach, Dunod, Paris, A. Rosa, On certain valuations of the vertices of a graph, Theory of Graphs, (Intl. Symp. Rome 1966), (1967) 349 - 355.

[10] D. Sakai, Labeling chordal graphs: distance two conditions, *SIAM Journal on Disc. Math.*, **7** (1994), 133 - 140. http://dx.doi.org/10.1137/s0895480191223178

[11] M.A. Whittlesey, J.P. Georges and D.W. Mauro, On the λ number of Q_n and related graphs, *SIAM Journal on Disc. Math.*, **8** (1995), 499 - 506. http://dx.doi.org/10.1137/s0895480192242821

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