Spectrum-Preserving Mapping on $C(X)$ Banach Algebras

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Abstract

In this paper, we showed that $\Delta(C(X))$ can be extended to $\Delta(W)$ where $W$ is a commutative unital Banach algebra including $C(X)$, the space of all continuous functions on $X$. Also it is showed that $\Psi : C(X) \to Y$ is a spectrum-preserving mapping if and only if $\Psi^* (\Delta(Y)) = \Delta(C(X))$.

Keywords: Banach algebra, spectrum, complex homomorphism, spectrum-preserving map

1 Introduction

Let $A$ be a unital Banach algebra. For any element $x \in A$, the spectrum and spectral radius are defined in [5]

$$\sigma_A(x) = \{ \lambda \in \mathbb{C} : (x - \lambda 1) \in A^{-1} \}$$

$$r_A(x) = \text{Sup} \{ ||\lambda|| : \lambda \in \sigma_A(x) \}$$

respectively. Some properties such as $r_A(x) \leq ||x||_A$ and $r_A(x + y) \leq r_A(x) + r_A(y)$ can be found in [2, 6, 7].

Let $A$ and $B$ be Banach algebras and $f$ be a homomorphism from $A$ to $B$. If $\sigma_A(a) = \sigma_B(f(a))$ for each $a \in A$, then $f$ is called a spectrum-preserving mapping [3, 6]. Also $f$ is a spectrum-preserving mapping if $f^{-1}(U_n(B)) =$
$U_n(A)$ where $U_n(A)$ and $U_n(B)$ are the set of unimodular elements of $A$ and $B$, respectively [3].

Let $\Delta(A)$ and $M(A)$ be the set of non-zero complex homomorphisms and the set of all maximal ideals of Banach algebra $A$, respectively. According to [7], there exists a one to one correspondence between the elements of $\Delta(A)$ and $M(A)$ such that $\varphi = \text{Ker}I$ where $\varphi \in \Delta(A)$ and $I \in M(A)$.

Let $\varepsilon > 0$ and $\varphi \in \Delta(A)$. According to Gelfand topology defined on $\Delta(A)$, the neighbourhood of $\varphi$ is the set

$$U_\varepsilon (\varphi ; x_1, x_2, ..., x_n) = \{ \varphi' \in \Delta(A) : |\widehat{x}_i(\varphi) - \widehat{x}_i(\varphi')| < \varepsilon, x_i \in A \} \text{ for } i = 1, 2, ..., n.$$

Let $A$ be a commutative Banach algebra without order. Then $\Delta(A) = \Delta(T) \cup \Delta_0(T)$ where $T$ is a multiplier on $A$. Nevertheless, for any mapping on Banach algebras from $X$ to $Y$, we know that $f^*(\Delta(Y)) \subset \Delta(X)$ by [4].

In this paper, $\Psi : C(X) \rightarrow Y$ will be taken as a continuous homomorphism such that $\Psi(1) = 1$ where $X$ is compact set, $C(X)$ is the set of all continuous functions on $X$ and $Y$ is any commutative unital Banach algebra.

2 Main Results

**Theorem 2.1** Let $A$ and $B$ be to discrete, closed subsets of the space $\Delta(C(X))$. Then, there exist $f, g \in C(X)$ such that $\widehat{f}(A) = 1$, $\widehat{g}(B) = 1$ and $fg = 0$.

**Proof.** For discrete, closed subsets $A$ and $B$, there exists open neighbourhoods $U \in N(A)$ and $V \in N(B)$ such that $U \cap V = \emptyset$. Thus, we have $(\Delta(C(X)) - U) \cap A = \emptyset$ and $(\Delta(C(X)) - V) \cap B = \emptyset$. Using the regularity of the algebra $C(X)$, there exists $f, g \in C(X)$ such that

$$\widehat{f}(\Delta(C(X)) - U) = 0, \widehat{f}(A) = 1$$

and

$$\widehat{g}(\Delta(C(X)) - V) = 0, \widehat{g}(B) = 1.$$

For all $\varphi \in \Delta(C(X))$ only one of $\varphi \in A, \varphi \in B, \varphi \in \Delta(C(X)) - U$ or $\varphi \in \Delta(C(X)) - V$ is true. As Banach algebra $C(X)$ is semisimple and $U \cap V = \emptyset$, we can show the followings:

Let $\varphi \in A$. So, we have $\widehat{f}(\varphi) = 1, \widehat{g}(\varphi) = 0$ and $\widehat{f}g(\varphi) = \widehat{f}(\varphi)\widehat{g}(\varphi) = \varphi(f)\varphi(g) = \varphi(fg) = 0$ which implies $fg = 0$.

If $\varphi \in B$, then $\widehat{f}(\varphi) = 0, \widehat{g}(\varphi) = 1$ and $fg = 0$.

Let $\varphi \in \Delta(C(X)) - U$. Then $\widehat{f}(\varphi) = 0, \widehat{g}(\varphi) = 1$ and $fg = 0$.

Suppose that $\varphi \in \Delta(C(X)) - V$. This implies that $\widehat{f}(\varphi) = 1, \widehat{g}(\varphi) = 0$ and $fg = 0$. ■
**Theorem 2.2** Let \( W \) be a commutative Banach algebra with unity and \( C(X) \subset W \). Then every element of \( \Delta(C(X)) \) can be extended to an element of \( \Delta(W) \). That is, there exist \( g \in \Delta(W) \) such that \( g|_{C(X)} = f \), for all \( f \in \Delta(C(X)) \).

**Proof.** Let \( f \in \Delta(C(X)) \) be arbitrary. There exist \( I \in M(C(X)) \) such that \( I = K \text{erf} \). Denote the minumum ideal containing the ideal \( I \) of algebra \( W \) by \( J_0 \). If \( J_0 = W \), then there exist \( f_1, f_2, \ldots, f_n \in I, g_1, g_2, \ldots, g_n \in W \) such that \( \sum_{i=1}^{n} f_i g_i = 1 \). So, one can take \( \|f_i\| = 1 \) for all \( i = 1, 2, \ldots, n \). Let \( A = \max_{1 \leq i \leq n} |g_i| \) and let us choose a neighbourhood \( U \) of \( f \in \Delta(C(X)) \) according to Gelfand topology such that

\[
U = \left\{ h \in \Delta(C(X)) : \left| \hat{f}_i(h) - \hat{f}_i(f) \right| < \frac{1}{3An}, i = 1, 2, \ldots, n \right\}.
\]

As \( \hat{f}_i(f) = 0 \) for all \( i = 1, 2, \ldots, n \), we have

\[
U = \left\{ h \in \Delta(C(X)) : \left| \hat{f}_i(h) \right| < \frac{1}{3An}, i = 1, 2, \ldots, n \right\}.
\]

Since \( C(X) \) is a regular Banach algebra there exist \( F \in C(X) \) such that

\[
\hat{F}(h) = \begin{cases} 
1, & h = f \\
0, & h \in \Delta(C(X)) - U \\
\leq 1, & \text{otherwise}
\end{cases}
\]

using \( \left| \left( F \hat{f}_i \right)(h) \right| = \left| \hat{F}(h) \hat{f}_i(h) \right| = \left| \hat{f}_i(h) \right| < \frac{1}{3An} \) and \( \sum_{i=1}^{n} (Ff_i) g_i = F \) we get \( \hat{F}(h) = \sum_{i=1}^{n} \left( F \hat{f}_i \right)(h) \hat{g}_i(h) \). Thus, we obtain \( \left| \hat{F}(h) \right| \leq \sum_{i=1}^{n} \left| \hat{f}_i(h) \right| \left| \hat{g}_i(h) \right| < \frac{1}{3} \) which contradicts \( \max_{h \in \Delta(C(X))} \left| \hat{F}(h) \right| = 1 \). So, we should have \( J_0 \neq W \).

In this case, for \( J \in M(W), J_0 \subset J, g \in \Delta(W) \) we have \( J = K \text{erg} \). As a result, for all \( h \in C(X) \) there exist \( \lambda \in \mathbb{C}, k \in I \) such that \( h = \lambda 1 + k \). Using this, we get \( h = \lambda 1 + k = f(h).1 + k \). Hence, we have \( g(h) = f(h) \) which implies \( g|_{C(X)} = f \).

**Corollary 2.3** For the unit embedding \( \varphi : C(X) \to W, \varphi(g) = g \), we have \( \varphi^*(\Delta(W)) = \Delta(C(X)) \).

**Proof.** Using \( \varphi^*(\Delta(W)) \subset \Delta(C(X)) \) and Theorem 2.2, it is straightforward.

**Corollary 2.4** For the map \( \varphi : C(X) \to W, g \to \varphi(g) = g \) all elements of \( \Delta(C(X)) \) can be extended to an element of \( \Delta(W) \).
Proof. Let \( h \in \Delta(C(X)) \) be arbitrary. There exist \( k \in \Delta(W) \) such that \( \varphi^*k = h \). Using this, we have \( h(g) = (\varphi^*k)(g) = k(\varphi(g)) = k(g) \) for all \( g \in C(X) \) which completes the proof.

Theorem 2.5 For all \( h \in C(X) \) if \( r(h) = r(\varphi(h)) \), then \( \Psi^*(\Delta(Y)) = \Delta(C(X)) \).

Proof. Let \( g \in \Delta(C(X)) \). Then, there exist \( I \in M(X) \) such that \( I = \text{Ker}g \). Let \( J_0 \) be the minimal ideal of Banach algebra \( Y \) containing the ideal \( \Psi(I) \). Assume that \( J_0 = Y \). Since \( r(x + y) \leq r(x) + r(y), r(x) \leq \|x\| \) using similar argument in Theorem 1.2 we get \( r(\varphi(F)) = r(\sum_{i=1}^{n} f_i \varphi(g_i F)) \leq \sum_{i=1}^{n} r(f_i)r(\varphi(g_i F)) \leq \sum_{i=1}^{n} \|f_i\| r(\varphi(g_i F)) \leq A\sum_{i=1}^{n} \frac{\varepsilon}{A_n} = \varepsilon \) thus, \( r(\varphi(F)) < \varepsilon < 1 \). On the other hand, as \( r(F) = \text{Sup} \{ \tilde{F}(h) : h \in \Delta(C(X)) \} \) = 1 this contradicts with \( r(\varphi(F)) = r(F) < 1 \). Hence, \( J_0 \neq Y \). There exist \( J \in M(Y) \) such that \( J_0 \subset J \) and \( h \in \Delta(Y) \) such that \( I = \text{Ker}h \). As \( C(X)/I \approx \mathbb{C} \), for all \( k \in C(X) \) there exist \( \lambda \in \mathbb{C}, t \in I \) such that \( k = \lambda t \). As we have \( h(\varphi(t)) = 0 \), we get \( (\varphi^*h)(k) = \lambda + h(\varphi(t)) = \lambda \) so, \( k = (\varphi^*h)(k) + t \). Since \( t \in I = \text{Ker}g \), then \( g(k) = g((\varphi^*h)(k)) + g(t) = (\varphi^*h)(k) \). Thus, as \( g = \varphi^*h \in \varphi^*(\Delta(Y)) \) we obtain \( \varphi^*(\Delta(Y)) = \Delta(C(X)) \).

Theorem 2.6 Let \( \varphi : C(X) \to Y \) be a 1-1 map and \( \overline{\varphi(C(X))} = W \). Then \( W \) is commutative and regular with unity.

Proof. As \( \varphi \) is 1-1 it is obvious that \( W \) has a unity. Let \( w_1, w_2 \in W \) there exist sequences \( (f_n) \) and \( (g_n) \) in \( C(X) \) such that \( \varphi(f_n) \to w_1, \varphi(g_n) \to w_2 \). Since \( \varphi(f_ng_n) = \varphi(f_n)\varphi(g_n) \to w_1w_2 \) and \( \varphi(g_nf_n) = \varphi(g_n)\varphi(f_n) \to w_2w_1 \), we get \( w_1w_2 = w_2w_1 \). So \( W \) is commutative. We have \( \varphi^*(\phi_W(f)) = \langle \varphi^*(\phi_W), f \rangle = \langle \phi_W, \varphi(f) \rangle = \phi_W(\varphi(f)) = 0 \) for all \( \phi_W \in \text{Ker}^c \varphi^* \) and for all non zero \( f \) in \( C(X) \). So we obtain \( f \neq 0 \) and \( \phi_W = 0 \) as \( \varphi \) is 1-1. This shows that the map \( \varphi^* : \Delta(W) \to \Delta(C(X)) \) is 1-1. Let \( K \) be any closed subset of \( \Delta(W) \). For \( f_0 \notin K \) we have \( \varphi^*K \subset \Delta(C(X)) \) and \( \varphi^*f_0 \in \Delta(C(X)) \). Since \( C(X) \) is a regular Banach algebra there exist \( f \in C(X) \) such that \( \varphi(f_0) \neq 0 \). As \( \langle \varphi^*f, K \rangle = 0 \) and \( \langle \varphi^*f_0, f \rangle \neq 0, W \) is regular.

Theorem 2.7 Let \( \varphi : C(X) \to Y \) be 1-1. Then, we have \( \sigma_{C(X)}(f) = \sigma_Y(\varphi(f)) \) for all \( f \in C(X) \).

Proof. Let \( \overline{\varphi(C(X))} = W \). Assume that \( \Delta(C(X)) \notin \varphi^*(\Delta(W)) \). So, there exist \( f_0 \in \Delta(C(X)) \) such that \( f_0 \notin \varphi^*(\Delta(W)) \). \( \{f_0\} \) is closed and by Theorem 1.1 there exist \( f, g \in C(X) \) such that \( \widehat{f}(f_0) = 1, \widehat{g}(\varphi^*(\Delta(W))) = 1 \)
and \( fg = 0 \). So, \( \phi_W(\Psi(g)) \neq 0 \) and \( \Psi(g) \in W^{-1} \) for all \( \phi_W \in \Delta(W) \). On the other hand, \( \Psi(f) = 0 \) as \( fg = 0 \) and \( \Psi(f) \Psi(g) = \Psi(fg) = \Psi(0) = 0 \). Thus we get \( f = 0 \) and this contradicts with \( \hat{f}(f_0) = 1 \). As a result, we obtain \( \Delta(C(X)) = \Psi^*(\Delta(W)) \). For all \( \phi_W \in \Delta(W) \) there exist \( \varphi_{C(X)} \in \Delta(C(X)) \) such that \( \Psi^*\phi_W = \varphi_{C(X)} \) so, for all \( f \in C(X) \) we have \( \Psi^*\phi_W(f) = \varphi_{C(X)}(f) \).

Using \( \varphi_{C(X)}(f) = \Psi^*\phi_W(f) = \phi_W(\Psi(f)) \) we get \( \sigma_Y(\Psi(f)) \) and \( \sigma_{C(X)}(f) \).

**Corollary 2.8** The followings are equivalent:

i. \( \Psi : C(X) \to Y \) is 1-1,

ii. \( \sigma_{C(X)}(f) = \sigma_Y(\Psi(f)) \),

iii. \( r_{C(X)}(f) = r_Y(\Psi(f)) \),

iv. \( \Psi^*(\Delta(Y)) = \Delta(C(X)) \).

**References**


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