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Spectrum-Preserving Mapping on $C\,(X)$ Banach Algebras

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Abstract

In this paper, we showed that $\Delta(C(X))$ can be extended to $\Delta(W)$ where W is a commutative unital Banach algebra including $C(X)$, the space of all continuous functions on X. Also it is showed that $\Psi : C(X) \to Y$ is a spectrum- preserving mapping if and only if $\Psi^* (\Delta (Y)) = \Delta (C(X))$.

Keywords: Banach algebra, spectrum, complex homomorphism, spectrum-preserving map

1 Introduction

Let A be a unital Banach algebra. For any element $x \in A$, the spectrum and spectral radius are defined in [5]

$$\sigma_A(x) = \left\{\lambda \in \mathbb{C} : (x - \lambda 1) \in A^{-1}\right\}$$

$$r_A(x) = Sup\left\{|\lambda| : \lambda \in \sigma_A(x)\right\}$$

respectively. Some properties such as $r_A(x) \leq \|x\|_A$ and $r_A(x+y) \leq r_A(x) + r_A(y)$ can be found in [2, 6, 7].

Let A and B be Banach algebras and f be a homomorphism from A to B. If $\sigma_A(a) = \sigma_B(f(a))$ for each $a \in A$, then f is called a spectrum-preserving mapping [3, 6]. Also f is a spectrum-preserving mapping if $f^{-1}\left(U_n(B)\right) =$

$U_n(A)$ where $U_n(A)$ and $U_n(B)$ are the set of unimodular elements of A and B, respectively [3].

Let $\Delta(A)$ and $M(A)$ be the set of non-zero complex homomorphisms and the set of all maximal ideals of Banach algebra A, respectively. According to [7], there exists a one to-one correspondence between the elements of $\Delta(A)$ and $M(A)$ such that $\varphi = KerI$ where $\varphi \in \Delta(A)$ and $I \in M(A)$.

Let $\varepsilon > 0$ and $\varphi \in \Delta(A)$. According to Gelfand topology defined on $\Delta(A)$, the neighbourhood of φ is the set

$U_\varepsilon(\varphi; x_1, x_2, ..., x_n) = \{\varphi' \in \Delta(A) : |\widehat{x_i}(\varphi) - \widehat{x_i}(\varphi')| < \varepsilon, x_i \in A\}$ for $i = 1, 2, ..., n$.

Let A be a commutative Banach algebra without order. Then $\Delta(A) = \Delta(\widehat{T}) \cup \Delta_0(\widehat{T})$ where T is a multiplier on A. Nevertheless, for any mapping on Banach algebras from X to Y, we know that $f^*\Delta(Y) \subset \Delta(X)$ by [4].

In this paper, $\Psi : C(X) \to Y$ will be taken as a continuous homomorphism such that $\Psi(1) = 1$ where X is compact set, $C(X)$ is the set of all continuous functions on X and Y is any commutative unital Banach algebra.

2 Main Results

Theorem 2.1 *Let A and B be to discrete, closed subsets of the space $\Delta(C(X))$. Then, there exist $f, g \in C(X)$ such that $\widehat{f}(A) = 1, \hat{g}(B) = 1$ and $fg = 0$.*

Proof. For discrete, closed subsets A and B, there exists open neighbourhoods $U \in N(A)$ and $V \in N(B)$ such that $U \cap V = \varnothing$. Thus, we have $(\Delta(C(X)) - U) \cap A = \varnothing$ and $(\Delta(C(X)) - V) \cap B = \varnothing$. Using the regularity of the algebra $C(X)$, there exists $f, g \in C(X)$ such that

$$\widehat{f}(\Delta(C(X)) - U) = 0, \widehat{f}(A) = 1$$

and

$$\hat{g}(\Delta(C(X)) - V) = 0, \hat{g}(B) = 1.$$

For all $\varphi \in \Delta(C(X))$ only one of $\varphi \in A, \varphi \in B, \varphi \in \Delta(C(X)) - U$ or $\varphi \in \Delta(C(X)) - V$ is true. As Banach algebra $C(X)$ is semisimple and $U \cap V = \varnothing$, we can show the followings:
Let $\varphi \in A$. So, we have $\widehat{f}(\varphi) = 1, \hat{g}(\varphi) = 0$ and $\widehat{f.g}(\varphi) = \widehat{f}(\varphi)\hat{g}(\varphi) = \varphi(f)\varphi(g) = \varphi(fg) = 0$ which implies $fg = 0$.
If $\varphi \in B$, then $\widehat{f}(\varphi) = 0, \hat{g}(\varphi) = 1$ and $fg = 0$.
Let $\varphi \in \Delta(C(X)) - U$. Then $\widehat{f}(\varphi) = 0, \hat{g}(\varphi) = 1$ and $fg = 0$.
Suppose that $\varphi \in \Delta(C(X)) - V$. This implies that $\widehat{f}(\varphi) = 1, \hat{g}(\varphi) = 0$ and $fg = 0$. ■

Theorem 2.2 *Let W be a commutative Banach algebra with unity and $C(X) \subset W$. Then every element of $\Delta(C(X))$ can be extended to an element of $\Delta(W)$. That is, there exist $g \in \Delta(W)$ such that $g|_{C(X)} = f$, for all $f \in \Delta(C(X))$.*

Proof. Let $f \in \Delta(C(X))$ be arbitrary. There exist $I \in M(C(X))$ such that

$I = K\text{erf}$. Denote the minumum ideal containing the ideal I of algebra W by J_0. If $J_0 = W$, then there exist $f_1, f_2, ..., f_n \in I, g_1, g_2, ..., g_n \in W$ such that $\sum_{i=1}^{n} f_i g_i = 1$. So, one can take $\|f_i\| = 1$ for all $i = 1, 2, ..., n$. Let $A = \max\limits_{1 \le i \le n} |g_i|$ and let us choose a neighbourhood U of $f \in \Delta(C(X))$ according to Gelfand topology such that

$$U = \left\{ h \in \Delta(C(X)) : \left| \widehat{f_i}(h) - \widehat{f_i}(f) \right| < \frac{1}{3An}, i = 1, 2, ..., n \right\}.$$

As $\widehat{f_i}(f) = 0$ for all $i = 1, 2, ..., n$, we have

$$U = \left\{ h \in \Delta(C(X)) : \left| \widehat{f_i}(h) \right| < \frac{1}{3An}, i = 1, 2, ..., n \right\}.$$

Since $C(X)$ is a regular Banach algebra there exist $F \in C(X)$ such that

$$\widehat{F}(h) = \begin{cases} 1, & h = f \\ 0, & h \in \Delta(C(X)) - U \\ \le 1, & \text{otherwise} \end{cases}$$

using $\left| \left(\widehat{Ff_i} \right)(h) \right| = \left| \widehat{F}(h) \widehat{f_i}(h) \right| = \left| \widehat{F}(h) \right| = \left| \widehat{f_i}(h) \right| < \frac{1}{3An}$ and $\sum_{i=1}^{n} (Ff_i)\, g_i = F$ we get $\widehat{F}(h) = \sum_{i=1}^{n} \left(\widehat{Ff_i} \right)(h) \widehat{g_i}(h)$. Thus,we obtain $\left| \widehat{F}(h) \right| \le \sum_{i=1}^{n} \left| \widehat{Ff_i}(h) \right| |\widehat{g_i}(h)| < \frac{1}{3}$ which contradict, $\max\limits_{h \in \Delta(C(X))} \left| \widehat{F}(h) \right| = 1$. So, we should have $J_0 \neq W$.
In this case, for $J \in M(W), J_0 \subset J, g \in \Delta(W)$ we have $J = Kerg$. As a result, for all $h \in C(X)$ there exist $\lambda \in \mathbb{C}, k \in I$ such that $h = \lambda 1 + k$. using this, we get $h = \lambda 1 + k = f(h).1 + k$. Hence, we have $g(h) = f(h)$ which implies $g|_{C(X)} = f$. ■

Corollary 2.3 *For the unit embedding $\varphi : C(X) \to W, \varphi(g) = g$, we have $\varphi^* (\Delta(W)) = \Delta(C(X))$.*

Proof. Using $\varphi^* (\Delta(W)) \subset \Delta(C(X))$ and Theorem 2.2, it is straightforward. ■

Corollary 2.4 *For the map $\varphi : C(X) \to W, g \to \varphi(g) = g$ all elements of $\Delta(C(X))$ can be extended to an element of $\Delta(W)$.*

Proof. Let $h \in \Delta(C(X))$ be arbitrary. There exist $k \in \Delta(W)$ such that $\varphi^* k = h$. Using this, we have $h(g) = (\varphi^* k)(g) = k(\varphi(g)) = k(g)$ for all $g \in C(X)$ which completes the proof. ■

Theorem 2.5 *For all* $h \in C(X)$ *if* $r(h) = r(\Psi(h))$*, then* $\Psi^*(\Delta(Y)) = \Delta(C(X))$.

Proof. Let $g \in \Delta(C(X))$. Then, there exist $I \in M(X)$ such that $I = Kerg$. Let J_0 be the minimal ideal of Banach algebra Y containing the ideal $\Psi(I)$. Assume that $J_0 = Y$. Since $r(x+y) \leq r(x) + r(y), r(x) \leq \|x\|$ using smilar argument in Theorem 1.2 we get $r(\Psi(F)) = r\left(\sum_{i=1}^n f_i \Psi(g_i F)\right) \leq \sum_{i=1}^n r(f_i) r(\Psi(g_i F)) \leq \sum_{i=1}^n \|f_i\| r(\Psi(g_i F)) \leq \max\|f_i\| \sum_{i=1}^n r(\Psi(g_i F)) < A \sum_{i=1}^n \frac{\varepsilon}{An} = \varepsilon$ thus, $r(\Psi(F)) < \varepsilon < 1$. On the other hand, as $r(F) = Sup\left\{\widehat{F}(h) : h \in \Delta(C(X))\right\} = 1$ this contradicts with $r(\Psi(F)) = r(F) < 1$. Hence, $J_0 \neq Y$. There exist $J \in M(Y)$ such that $J_0 \subset J$ and $h \in \Delta(Y)$ such that $I = Kerh$. As $C(X) \diagup I \approx \mathbb{C}$, for all $k \in C(X)$ there exist $\lambda \in \mathbb{C}, t \in I$ such that $k = \lambda 1 + t$. As we have $h(\Psi(t)) = 0$, we get $(\Psi^* h)(k) = \lambda + h(\Psi(t)) = \lambda$ so, $k = (\Psi^* h)(k) + t$ Since $t \in I = Kerg$, then $g(k) = g((\Psi^* h)(k)) + g(t) = (\Psi^* h)(k)$. Thus, as $g = \Psi^* h \in \Psi^*(\Delta(Y))$ we optain $\Psi^*(\Delta(Y)) = \Delta(C(X))$. ■

Theorem 2.6 *Let* $\Psi : C(X) \to Y$ *be a 1-1 map and* $\overline{\Psi(C(X))} = W$*. Then* W *is commutative and regular with unity.*

Proof. As Ψ is 1-1 it is obvious that W has a unity. Let $w_1, w_2 \in W$ there exist sequences (f_n) and (g_n) in $C(X)$ such that $\Psi(f_n) \to w_1, \Psi(g_n) \to w_2$. Since $\Psi(f_n g_n) = \Psi(f_n)\Psi(g_n) \to w_1 w_2$ and $\Psi(g_n f_n) = \Psi(g_n)\Psi(f_n) \to w_2 w_1$, we get $w_1 w_2 = w_2 w_1$. So W is commutative. We have $\Psi^*(\phi_W(f)) = \langle \Psi^* \phi_W, f \rangle = \langle \phi_W, \Psi(f) \rangle = \phi_W(\Psi(f)) = 0$ for all $\phi_W \in Ker\Psi^*$ and for all non zero f in $C(X)$. So we obtain $f \neq 0$ and $\phi_W = 0$ as Ψ is 1-1. This showns that the map $\Psi^* : \Delta(W) \to \Delta(C(X))$ is 1-1. Let K be any closed subset of $\Delta(W)$. For $f_0 \notin K$ we have $\Psi^* K \subset \Delta(C(X))$ and $\Psi^* f_0 \in \Delta(C(X))$. Since $C(X)$ is a regular Banach algebra there exist $f \in C(X)$ such that $\widehat{f}(\Psi^* f_0) \neq 0$. As $\left\langle \Psi\widehat{f}, K \right\rangle \equiv 0$ and $\left\langle \Psi\widehat{f}, f_0 \right\rangle \neq 0, W$ is regular. ■

Theorem 2.7 *Let* $\Psi : C(X) \to Y$ *be 1-1. Then, we have* $\sigma_{C(X)}(f) = \sigma_Y(\Psi(f))$ *for all* $f \in C(X)$.

Proof. Let $\overline{\Psi(C(X))} = W$. Assume that $\Delta(C(X)) \nsubseteq \Psi^*(\Delta(W))$. So, there exist $f_0 \in \Delta(C(X))$ such that $f_0 \notin \Psi^*(\Delta(W))$. $\{f_0\}$ is closed and by Theorem 1.1 there exist $f\ , g \in C(X)$ such that $\widehat{f}(f_0) = 1, \widehat{g}(\Psi^* \Delta(W)) \equiv 1$

and $fg = 0$. So, $\phi_W(\Psi(g)) \neq 0$ and $\Psi(g) \in W^{-1}$ for all $\phi_W \in \Delta(W)$. On the other hand, $\Psi(f) = 0$ as $fg = 0$ and $\Psi(f)\Psi(g) = \Psi(fg) = \Psi(0) = 0$. Thus we get $f = 0$ and this contradicts with $\widehat{f}(f_0) = 1$. As a result, we obtain $\Delta(C(X)) = \Psi^*(\Delta(W))$. for all $\phi_W \in \Delta(W)$ there exist $\varphi_{C(X)} \in \Delta(C(X))$ such that $\Psi^*\phi_W = \varphi_{C(X)}$ so, for all $f \in C(X)$ we have $\Psi^*\phi_W(f) = \varphi_{C(X)}(f)$. Using $\phi_{C(X)}(f) = \Psi^*\phi_W(f) = \phi_W(\Psi(f))$ we get $\sigma_Y(\Psi(f))$ and $\sigma_{C(X)}(f)$. ■

Corollary 2.8 *The followings are equivalent:*
i. $\Psi : C(X) \to Y$ *is 1-1,*
ii. $\sigma_{C(X)}(f) = \sigma_Y(\Psi(f))$,
iii. $r_{C(X)}(f) = r_Y(\Psi(f))$,
iv. $\Psi^*(\Delta(Y)) = \Delta(C(X))$.

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