International Mathematical Forum, Vol. 11, 2016, no. 12, 547 - 556
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/imf.2016.6443

Transmuted Burr Type XII Distribution: A Generalization of the Burr Type XII Distribution

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Abstract

A new distribution introduced in this paper, called transmuted Burr type XII distribution (TBIIID). Some properties of this distribution will be presented. We obtain the moment generating function and rth moment of the new distribution. We derive representation for the order statistics. The method of maximum likelihood is proposed for estimating the model parameters. We are looking for this generalization may interest wider applications in reliability and lifetime data analysis.

Keywords: Burr XII distribution; Maximum likelihood; Order statistic; moments

1 Introduction

A Burr distribution was raised in 1942 by Irving W. Burr. Rodriguez (1977) presented that the Burr coverage area on a specific plane is working by various well-known and useful distributions, with the normal, log-normal, gamma, logistic and extreme-value type-I distributions. The Burr XII (BXII) distribution, having logistic and Weibull as special sub-models, is a very popular distribution for modeling real time data. The beta Burr XII distribution is defined and investigated by (Patricia et al (2011). Many transmuted distributions are proposed. Aryal and Tsokos (2009) presented a new generalization of Weibull distribution called the transmuted Weibull distribution. Khan and King (2013) are introduced the transmuted modified Weibull distribution. Transmuted Lomax distribution is presented by (Ashour and Eltehiwy 2013a). S.K. Ashour and M.A. Eltehiwy, (2013b) are presented transmuted exponentiated modified Weibull distribution.

Merovci (2013) are introduced Transmuted Rayleigh distribution. Elbatal and M. Elgarhy (2013) Transmuted Quasi Lindley Distribution. The transmutation map method suggested by Shaw and Buckley (2007).

Definition: A random variable X is said to have a transmuted probability distribution with cdf $F(x)$ if

$$F(x) = (1+\lambda)G(x) - \lambda\left[G(x)\right]^2, \quad |\lambda| \le 1, \tag{1}$$

where $G(x)$ is the cdf of the base distribution. If $\lambda = 0$, we have the base distribution of *X*.

$$f(x) = g(x)[1+\lambda-2\lambda G(x)] \tag{2}$$

where $f(x)$ and $g(x)$ are the corresponding probability density functions (pdf) of $F(x)$ and $G(x)$, respectively. We used (1) to obtain a new model as a generalization for the Burr XII distribution, namely; the transmuted Burr XII distribution TBXIID.

Burr distribution is the well-known probability distribution in which the classical properties and the computational methods for the estimation of maximum likelihood estimates to the life time data was proposed. The three parameter Burr XII distributions with the following cumulative distribution function and the probability density function is provided respectively by (D. M. Titterington, et al, 1985).

The probability density function (pdf) of the BURR XII distribution is defined as

$$g(x;k,c,s) = cks^{-c}x^{c-1}\left[1+\left(\frac{x}{s}\right)^c\right]^{-k-1}, \quad ,k>0, c>0, s>0 \tag{3}$$

and its corresponding cumulative distribution function (cdf) is given by

$$G(x;k,c,s) = 1-\left[1+\left(\frac{x}{s}\right)^c\right]^{-k}, \quad ,k>0, c>0, s>0, \tag{4}$$

where k and c are the shape parameters and s is the scale parameter.

The rest of this paper is organized as follows. In Section 2 we demonstrate transmuted BURR XII distribution. Reliability and hazard rate functions are presented in Section 3. In Section 4, the distributions of order statistics are summarized. In Sections 5 and 6, the moment generating function and the quantile function are expressed respectively. The maximum likelihood estimates (MLE) of the distribution parameters are demonstrated in Section 7.

2. Transmuted Burr XII distribution

Using (1) the cdf of a transmuted BURR XII distribution with parameters k , c and s is defined as the following

$$F(x)=(1+\lambda)\left[1-\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k}\right]-\lambda\left[1-\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k}\right]^{2}$$

Simplify we get

$$F(x)=1-\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k}\left[1-\lambda+\lambda\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k}\right] \qquad (5)$$

The pdf of *transmuted BURR XII distribution* can be derived by using (2)

$$f(x)=\left(cks^{-c}x^{c-1}\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k-1}\right)\left\{1+\lambda-2\lambda\left[1-\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k}\right]\right\}$$

$$f(x)=\left(ck\,\frac{1}{x}\left(\frac{x}{s}\right)^{c}\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-(k+1)}\right)\left\{1-\lambda+2\lambda\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k}\right\} \qquad (6)$$

By using negative binomial expansion for the following expression

$$\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k}=\sum_{r=0}^{\infty}(-1)^{r}\binom{k+r-1}{r}\left(\frac{x}{s}\right)^{r}=\sum_{r=0}^{\infty}\binom{-k}{r}\left(\frac{x}{s}\right)^{r}$$

$$\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-(k+1)}=\sum_{m=0}^{\infty}(-1)^{m}\binom{k+m}{m}\left(\frac{x}{s}\right)^{m}=\sum_{m=0}^{\infty}\binom{-k-1}{m}\left(\frac{x}{s}\right)^{m}$$

Then the cdf becomes as follow

$$F(x)=1-\sum_{r=0}^{\infty}\binom{-k}{r}\left(\frac{x}{s}\right)^{r}\left[1-\lambda+2\lambda\sum_{r=0}^{\infty}\binom{-k}{r}\left(\frac{x}{s}\right)^{r}\right], \qquad (7)$$

and the pdf can be formed as

$$f(x)=\left(ck\,\frac{1}{x}y\sum_{r=0}^{\infty}\binom{-k-1}{r}\left(\frac{x}{s}\right)^{r}\right)\left\{1-\lambda+2\lambda\sum_{m=0}^{\infty}\binom{-k}{m}\left(\frac{x}{s}\right)^{m}\right\}.$$

Then the pdf becomes as

$$f(x)=\left(ck\sum_{r=0}^{\infty}(-1)^{r}\binom{k+r-1}{r}\left(\frac{1}{s}\right)^{r+c}(x)^{r+c-1}\right)-\lambda ck\sum_{r=0}^{\infty}(-1)^{r}\binom{k+r-1}{r}\left(\frac{1}{s}\right)^{r+c}(x)^{r+c-1}$$
$$+2\lambda ck\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{r}\binom{k+r-1}{r}\binom{-k}{m}\left(\frac{1}{s}\right)^{r+c+m}(x)^{r+c+m-1} \quad (8)$$

3. Reliability analysis

The reliability function can be called the survival function is the property of the random variable and is related with the failure of some system within a specified time. It is defined as $R_{TBXIID}(x)=1-F_{TBXIID}(x)$. While the Hazard function can be stated as the ratio of the probability density function and reliability function and can be specified in the following form $H_{TBXIID}(x)=\dfrac{f_{TBXIID}(x)}{R_{TBXIID}(x)}$.

The reliability and hazard rate functions of the transmuted Burr XII distribution are defined as the following, respectively.

$$R_{TBXIID}(x)=\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k}\left[1-\lambda+2\lambda\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k}\right] \quad (9)$$

$$H_{TBXIID}(x)=\frac{\left(ck\left(\frac{x}{s}\right)^{c}\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-1}\right)\left\{1+\lambda-2\lambda\left[1-\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k}\right]\right\}}{x\left[1-\lambda+\lambda\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k}\right]} \quad (10)$$

4. Distribution of order statistics from TBIIID

Order statistics are essential in numerous areas of practice and statistical theory. Let $X_1, X_2, ..., X_n$ be a random sample of size n from a pdf $f(x)$ and cdf $F(x)$, and $X_{[1]}, X_{[2]}, ..., X_{[n]}$ be its order statistics. The pdf of the *i*th order statistics is defined as

$$f_{(j)}(x)=\frac{n!}{(j-1)!(n-j)!}f(x)[F(x)]^{j-1}[1-F(x)]^{n-j}, \quad (11)$$

for $i=1,2,...,n$.

Let, $X_{[1]} = \min\{X_1, X_2, ..., X_n\}$, and $X_{[n]} = \max\{X_1, X_2, ..., X_n\}$ with pdfs $f_{(1)}(x)$ and $f_{(n)}(x)$ defined as follow

$$f_{(1)}(x) = nf(x)[1-F(x)]^{n-1} \text{ and } f_{(n)}(x) = nf(x)[F(x)]^{n-1}.$$

As a special case of (11), the pdf of the minimum and maximum order statistics, respectively, are

$$f_{(1)}(x) = ncks^{-c}x^{c-1}\left[1+\left(\frac{x}{s}\right)^c\right]^{-k-1}\left\{1+\lambda-2\lambda\left[1-\left[1+\left(\frac{x}{s}\right)^c\right]^{-k}\right]\right\}\left[\left[1+\left(\frac{x}{s}\right)^c\right]^{-k}\left[1-\lambda+\lambda\left[1+\left(\frac{x}{s}\right)^c\right]^{-k}\right]\right]^{n-1} \quad (12)$$

$$f_{(n)}(x) = n\left\{f(x) = \left(cks^{-c}x^{c-1}\left[1+\left(\frac{x}{s}\right)^c\right]^{-k-1}\right)\left\{1+\lambda-2\lambda\left[1-\left[1+\left(\frac{x}{s}\right)^c\right]^{-k}\right]\right\}\right\}\left[1-\left[1+\left(\frac{x}{s}\right)^c\right]^{-k}\left[1-\lambda+\lambda\left[1+\left(\frac{x}{s}\right)^c\right]^{-k}\right]\right]^{n-1}. \quad (13)$$

5. The moments of the TBXIID

In this section, we derive the expectation of X^i, then the mean and the variance of the TBXIID random variable.

Theorem 5.1: If *X* is a r. v. from TBXIID, then the moment generating function is defined as

$$E(e^{tx}) = (1-\lambda)ck\sum_{r=0}^{\infty}\binom{-k-1}{r}\left(\frac{1}{s}\right)^{r+c}\left(\frac{-1}{t}\right)^{r+c} + 2\lambda ck\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\binom{-k-1}{r}\binom{-k}{m}\left(\frac{1}{s}\right)^{r+c+m}\left(\frac{-1}{t}\right)^{r+c+m} \quad (14)$$

Proof: Assume that *X* is a random variable with TBXIID. Then the MGF of *X* is defined

$$M(x,t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x)dx$$

By using equation (8) of the pdf in the form of negative binomial expansion series then the MGF

$$E(e^{tx}) = \int_0^{\infty} e^{tx}\left\{\left(ck\sum_{r=0}^{\infty}\binom{-k-1}{r}\left(\frac{1}{s}\right)^{r+c}(x)^{r+c-1}\right) - \lambda ck\sum_{r=0}^{\infty}\binom{-k-1}{r}\left(\frac{1}{s}\right)^{r+c}(x)^{r+c-1} + 2\lambda ck\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\binom{-k-1}{r}\binom{-k}{m}\left(\frac{1}{s}\right)^{r+c+m}(x)^{r+c+m-1}\right\}dx$$

$$E(e^{tx}) = \int_0^{\infty}\left\{\left(ck\sum_{r=0}^{\infty}\binom{-k-1}{r}\left(\frac{1}{s}\right)^{r+c}e^{tx}(x)^{r+c-1}\right) - \lambda ck\sum_{r=0}^{\infty}\binom{-k-1}{r}\left(\frac{1}{s}\right)^{r+c}e^{tx}(x)^{r+c-1} + 2\lambda ck\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\binom{-k-1}{r}\binom{-k}{m}\left(\frac{1}{s}\right)^{r+c+m}e^{tx}(x)^{r+c+m-1}\right\}dx$$

$$E(e^{tx})=ck\sum_{r=0}^{\infty}\binom{-k-1}{r}\left(\frac{1}{s}\right)^{r+c}\int_0^{\infty}e^{tx}\left(x\right)^{r+c-1}dx-\lambda ck\sum_{r=0}^{\infty}\binom{-k-1}{r}\left(\frac{1}{s}\right)^{r+c}\int_0^{\infty}e^{tx}\left(x\right)^{r+c-1}dx+2\lambda ck\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\binom{-k-1}{r}\binom{-k}{m}\left(\frac{1}{s}\right)^{r+c+m}\int_0^{\infty}e^{tx}\left(x\right)^{r+c+m-1}dx$$

substitute $tx=-u$ then

$$E(e^{tx})=\left(ck\sum_{r=0}^{\infty}\binom{-k-1}{r}\left(\frac{1}{s}\right)^{r+c}-\lambda ck\sum_{r=0}^{\infty}\binom{-k-1}{r}\left(\frac{1}{s}\right)^{r+c}\right)\int_0^{\infty}e^{-u}\left(\frac{-u}{t}\right)^{r+c-1}\frac{-du}{t}$$
$$+2\lambda ck\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\binom{-k-1}{r}\binom{-k}{m}\left(\frac{1}{s}\right)^{r+c+m}\int_0^{\infty}e^{-u}\left(\frac{-u}{t}\right)^{r+c+m-1}\frac{-du}{t}$$

Then we get

$$E(e^{tx})=ck\sum_{r=0}^{\infty}\binom{-k-1}{r}\left(\frac{1}{s}\right)^{r+c}\left(\frac{-1}{t}\right)^{r+c}-\lambda ck\sum_{r=0}^{\infty}\binom{-k-1}{r}\left(\frac{1}{s}\right)^{r+c}\left(\frac{-1}{t}\right)^{r+c}+2\lambda ck\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\binom{-k-1}{r}\binom{-k}{m}\left(\frac{1}{s}\right)^{r+c+m}\left(\frac{-1}{t}\right)^{r+c+m}$$

The proof is complete.

Theorem 5.2: If X has a TBXIID, then the ith moment is defined as

$$E(x^i)=(1+\lambda)ck\left(s\right)^{i-1}\beta(k-\frac{i+2c-2}{c},\frac{i}{c}-1)-2\lambda(ck\left(s\right)^{i-1}\beta(2k-\frac{i+2c-2}{c},\frac{i}{c}-1)\tag{15}$$

Proof: the rth moment,

$$E(x^i)=\int_0^{\infty}x^i f(x)dx$$

$$E(x^i)=\int_0^{\infty}x^i\left(ck\,\frac{1}{x}\left(\frac{x}{s}\right)^c\left[1+\left(\frac{x}{s}\right)^c\right]^{-(k+1)}\right)\left\{1-\lambda+2\lambda\left[1+\left(\frac{x}{s}\right)^c\right]^{-k}\right\}dx\tag{16}$$

Let $y=\left[1+\left(\frac{x}{s}\right)^c\right]^{-1}$ then $dy=\dfrac{c\left(\frac{x}{s}\right)^c}{x\left(1+\left(\frac{x}{s}\right)^c\right)^2}dx$ and $dx=s\;y^{1-\frac{1}{c}}\left(1-y\right)^{\frac{1}{c}-1}dy$

Substitute in the (16) we have

$$E(x^i)=\int_0^1(1+\lambda)ck\left(s\right)^{i-1}\left(1-y\right)^{\frac{i+c-1}{c}}\left[y\right]^{k+1-\frac{i+c-1}{c}}s\;y^{1-\frac{1}{c}}\left(1-y\right)^{\frac{1}{c}-1}dy$$
$$-2\lambda\int_0^1(ck\left(s\right)^{i-1}\left(1-y\right)^{\frac{i+c-1}{c}}\left[y\right]^{2k+1-\frac{i+c-1}{c}}s\;y^{1-\frac{1}{c}}\left(1-y\right)^{\frac{1}{c}-1}dy$$

With some algebraic manipulation we get

$$E(x^i)=\int_0^1 (1+\lambda)ck\,(s)^i\,(1-y)^{\frac{i+c-1}{c}+\frac{1}{c}-1}[y]^{k+1-\frac{i+c-1}{c}+1-\frac{1}{c}}\,dy-2\lambda\int_0^1 ck\,(s)^i\,(1-y)^{\frac{i+c-1}{c}+\frac{1}{c}-1}[y]^{2k+1-\frac{i+c-1}{c}+1-\frac{1}{c}}\,dy$$

$$E(x^i)=\int_0^1 (1+\lambda)ck\,(s)^i\,(1-y)^{\frac{i}{c}}[y]^{k+1-\frac{i+2c-2}{c}}\,dy-2\lambda\int_0^1 ck\,(s)^i\,(1-y)^{\frac{i}{c}}[y]^{2k+1-\frac{i+2c-2}{c}}\,dy$$

By using beta function which is defined as

$$Beta(\alpha,\beta)=\int_0^1 (1-y)^{\beta-1}\,y^{\alpha-1}\,dy\ .$$

So the integration becomes

$$\int_0^1 (1-y)^{\frac{i}{c}}[y]^{k+1-\frac{i+2c-2}{c}}\,dy=\beta(k+1-\frac{i+2c-2}{c}-1,\frac{i}{c}-1)$$

and

$$\int_0^1 (1-y)^{\frac{i}{c}}[y]^{2k+1-\frac{i+2c-2}{c}}\,dy=\beta(2k+1-\frac{i+2c-2}{c}-1,\frac{i}{c}-1)$$

Then the proof is complete.

Remarks: The first and second moments of the TBXIID random variable are

$$E(x)=(1+\lambda)ck\,\beta(k+1-\frac{2c-1}{c}-1,\frac{1}{c}-1)-2\lambda ck\,\beta(2k+1-\frac{2c-1}{c}-1,\frac{1}{c}-1)$$

and

$$E(x^2)=(1+\lambda)cks\,\beta(k-2,\frac{2}{c}-1)-2\lambda cks\,\beta(2k-2,\frac{2}{c}-1).$$

And the variance can be computed by $\sigma^2=E(X^2)-\left(E(X)\right)^2$

6. The quantile x_q

The quantile x_q of the TBXIID is defined as follows

$$q=F(x)=1-\left[1+\left(\frac{x}{s}\right)^c\right]^{-k}+\lambda\left[1+\left(\frac{x}{s}\right)^c\right]^{-k}-\lambda\left[1+\left(\frac{x}{s}\right)^c\right]^{-2k}$$

Assume that $y=\left[1+\left(\frac{x}{s}\right)^c\right]^{-k}$ then $q=F(x)=1-y+\lambda y-\lambda y^2$ and solve for y we have

$$y=\frac{\sqrt{\lambda^2-4\lambda q+2\lambda+1}+\lambda-1}{2\lambda}$$

then we have

$$x_q = s\left(\frac{1-\left[\dfrac{\sqrt{\lambda^2-4\lambda q+2\lambda+1}+\lambda-1}{2\lambda}\right]}{\dfrac{\sqrt{\lambda^2-4\lambda q+2\lambda+1}+\lambda-1}{2\lambda}}\right)^{\frac{1}{c}} \tag{17}$$

7. Maximum likelihood estimate

The maximum likelihood estimates (MLEs) of the parameters of the TBXIID are derived from complete samples only. Let $X_1, X_2, ..., X_n$ be a random sample of size n from the TBXIID with parameters k, c ,s and λ, let $Lf\ (x, x, ..., x; c, k, s, \lambda) = Lf$ then

$$Lf = \prod_{i=1}^{n}\left(\left(cks^{-c}x^{c-1}\left[1+\left(\frac{x}{s}\right)^c\right]^{-k-1}\right)\left\{1-\lambda+2\lambda\left[1+\left(\frac{x}{s}\right)^c\right]^{-k}\right\}\right) \tag{18}$$

Taking the log-likelihood function for (21) as

$$\ln Lf = \ln\left[(c)^n (k)^n \left(s^{-c}\right)^n \left(x^{c-1}\right)^n \left(\left[1+\left(\frac{x}{s}\right)^c\right]^{-k-1}\right)^n \left(1-\lambda+2\lambda\left[1+\left(\frac{x}{s}\right)^c\right]^{-k}\right)^n\right]$$

$$\ln Lf = n\ln(c)+n\ln(k)-nc\ln(s)+n(c-1)\ln(x)+(-k-1)\sum\ln\left[1+\left(\frac{x}{s}\right)^c\right]+\sum\ln\left(1-\lambda+2\lambda\left[1+\left(\frac{x}{s}\right)^c\right]^{-k}\right) \tag{19}$$

The first-order derivatives of (19) with respect to the four parameters are

$$\frac{\partial \ln Lf}{\partial c} = \frac{n}{c} - n\ln(s) + \sum\ln(x) + \sum\frac{(-k-1)\left(\left(\frac{x}{s}\right)^c\ln\left(\frac{x}{s}\right)\right)}{\left(\frac{x}{s}\right)^c+1} + \sum\frac{2\lambda k\left(\frac{x}{s}\right)^c\ln\left(\frac{x}{s}\right)}{\left(\left(\frac{x}{s}\right)^c+1\right)\left[\lambda\left[\left(\left(\frac{x}{s}\right)^c+1\right)^k-2\right]-\left(\left(\frac{x}{s}\right)^c+1\right)^k\right]} \tag{20}$$

$$\frac{\partial \ln Lf}{\partial k} = \frac{n}{k} - \sum\ln\left[1+\left(\frac{x}{s}\right)^c\right] + \sum\frac{2\lambda\ln\left[1+\left(\frac{x}{s}\right)^c\right]}{\lambda\left\{\left[1+\left(\frac{x}{s}\right)^c\right]^k-2\right\}-\left[1+\left(\frac{x}{s}\right)^c\right]^k} \tag{21}$$

$$\frac{\partial \ln Lf}{\partial s}=\sum \frac{2\lambda kc\left(\frac{x}{s}\right)^{c}}{s\left\{(\lambda-1)\left[1+\left(\frac{x}{s}\right)^{c}\right]^{k}-2\lambda\right\}\left[1+\left(\frac{x}{s}\right)^{c}\right]}-\sum \frac{c(-k-1)\left(\frac{x}{s}\right)^{c}}{s\left[1+\left(\frac{x}{s}\right)^{c}\right]}-\frac{nc}{s} \quad (22)$$

$$\frac{\partial \ln Lg}{\partial \lambda}=\sum \frac{\left(2\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k}-\lambda\right)}{1-\lambda+2\lambda\left[1+\left(\frac{x}{s}\right)^{c}\right]^{-k}} \quad (23)$$

Setting these expressions to zero and solving them simultaneously produces the maximum-likelihood estimates of the four parameters.

References

[1] D. M. Titterington, A. F. M. Smith and U. E. Makov, *Statistical Analysis of Finite Mixture Distributions*, John Wiley & Sons, New York, 1985.

[2] I. Elbatal and M. Elgarhy, Transmuted Quasi Lindley Distribution: A Generalization of the Quasi Lindley Distribution, *International Journal of Pure and Applied Sciences and Technology*, **18** (2013), no. 2, 59-70.

[3] F. Merovci, Transmuted Rayleigh distribution, *Austrian Journal of Statistics*, **42** (2013), no. 1, 21-31.

[4] G.R. Aryal and C.P. Tsokos, On the transmuted extreme value distribution with applications, *Nonlinear Analysis: Theory, Methods and Applications*, **71** (2009), 1401-1407. http://dx.doi.org/10.1016/j.na.2009.01.168

[5] I. W. Burr, Cumulative frequency functions, *Annals of Mathematical Statistics*, **13** (1942), 215-232. http://dx.doi.org/10.1214/aoms/1177731607

[6] M. S. Khan and R. King, Transmuted modified Weibull distribution: A generalization of the modified Weibull probability distribution, *European Journal of Pure and Applied Mathematics*, **6** (2013), 66-88.

[7] Patricia F. Paranaiba, Edwin M. M. Orteqa, Gauss M. Coredeiro and Rodrigo R. Pescim, The beta Burr XII distribution with application to lifetime data, *Computational Statistics & Data Analysis*, **55** (2011), 1118-1136. http://dx.doi.org/10.1016/j.csda.2010.09.009

[8] R. N. Rodriguez, A guide to the Burr Type XII distributions, *Biometrika*, **64** (1977), 129-134. http://dx.doi.org/10.1093/biomet/64.1.129

[9] S.K. Ashour and and M.A. Eltehiwy, Transmuted exponentiated Lomax distribution, *Australian Journal of Basic and Applied Sciences*, **7** (2013a), no. 7, 658-667.

[10] S.K. Ashour and M.A. Eltehiwy, Transmuted exponentiated modified Weibull distribution, *International Journal of Basic and Applied Sciences*, **2** (2013b), no. 3, 258-269. http://dx.doi.org/10.14419/ijbas.v2i3.1074

[11] W. Shaw and I. Buckley, The Alchemy of Probability Distributions: Beyond Gram-Charlier expansions, and a skew-Kurtotic-normal distribution from a rank transmutation map. Research report, (2007).

Received: April 24, 2016; Published: June 10, 2016