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OU-Operators and Identities between Holomorphic Vector Fields on $\Im(z_1) > |z_2|^2$

Hélène Airault

Lamfa, UMR 7352 CNRS, 33, rue Saint-Leu, 80039 Amiens, France

Abstract

Consider the group $G = Aut_{Hol}\mathcal{D}$ of holomorphic automorphisms of the domain $\mathcal{D} = \{\Im(z_1) > |z_2|^2\}$. We give the expression of the Kähler Laplacian Δ^K in $\mathcal{D}$ in terms of the holomorphic vector fields in the Lie algebra $\mathcal{G}$ of G. We show how some identities between the holomorphic vector fields imply the invariance of the Kähler Laplacian with respect to the volume measure. On $\mathcal{D}$, we define operators of Ornstein-Uhlenbeck type and we calculate their invariant measure.

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Introduction

Let

$$u(z_1, z_2) = \frac{z_1 - \overline{z_1}}{2i} - z_2\overline{z_2} \tag{0.1}$$

On the Siegel domain

$$\mathcal{D} = \{(z_1, z_2) \in \mathbf{C}^2 \ |u(z_1, z_2) > 0\} \tag{0.2}$$

we consider the Bergman metric,

$$ds^2 = -\sum_{k,j} \frac{\partial^2}{\partial z_k \partial \overline{z_j}} \log u(z_1, z_2)\, dz_k d\overline{z_j} \tag{0.3}$$

$$ds^2 = \frac{1}{4u^2(z_1,z_2)}(dz_1 - 2i\overline{z_2}dz_2)(d\overline{z_1} + 2iz_2 d\overline{z_2}) + \frac{1}{u(z_1,z_2)}dz_2 d\overline{z_2}$$

We denote dv the volume measure on $\mathcal{D}$,

$$dv = u(z_1,z_2)^{-3}\, dz_1 d\overline{z_1} dz_2 d\overline{z_2} \tag{0.4}$$

The differential two-form

$$\Omega = i \sum_{j,k} \frac{\partial^2 \log u(z_1,z_2)}{\partial z_j \partial \overline{z_k}} dz_j \wedge d\overline{z_k} \tag{0.5}$$

is exact ($d\Omega = 0$). Let S^1 be the unit circle parametrized by $e^{i\theta}$. On $\mathcal{D} \times S^1$, we define the $(1,0)$ differential forms

$$\sigma_1 = \frac{1}{2iu}(dz_1 - 2i\overline{z_2}dz_2) \quad \text{and} \quad \sigma_2 = e^{-i\theta}\frac{dz_2}{\sqrt{u}} \tag{0.6}$$

where $u = \Im z_1 - z_2\overline{z_2}$. It is classical that $\Omega \wedge \Omega = dv$ where dv is the differential form associated to (0.4). Also $\sigma_1 = \partial \log(u)$ and $\overline{\partial}\sigma_1 = \sigma_1 \wedge \overline{\sigma_1} + \sigma_2 \wedge \overline{\sigma_2} = -i\,\Omega$. The Kähler Laplacian Δ^K on $\mathcal{D}$ with the Bergman metric (0.3) is given by

$$\Delta^K = u(z_1,z_2) \times [2i(z_1 - \overline{z_1})\frac{\partial^2}{\partial z_1 \partial \overline{z_1}} - 2i\overline{z_2}\frac{\partial^2}{\partial z_1 \partial \overline{z_2}} + 2i\, z_2 \frac{\partial^2}{\partial z_2 \partial \overline{z_1}} - \frac{\partial^2}{\partial z_2 \partial \overline{z_2}}] \tag{0.7}$$

See for example [7], p.446 and see [4], [18]. We have

$$\Delta^K = -4\,u^2 \Delta_1^K + u\,\frac{1}{2i}(z_1 - \overline{z_1} - 2iz_2\overline{z_2})\Delta_2^K \quad \text{with} \quad \Delta_1^K = \frac{\partial^2}{\partial z_1 \partial \overline{z_1}} \tag{0.8}$$

$$\begin{aligned} \Delta_2^K &= -4z_2\overline{z_2}\frac{\partial^2}{\partial z_1 \partial \overline{z_1}} - 2i\overline{z_2}\frac{\partial^2}{\partial z_1 \partial \overline{z_2}} + 2iz_2\frac{\partial^2}{\partial z_2 \partial \overline{z_1}} - \frac{\partial^2}{\partial z_2 \partial \overline{z_2}} \\ &= -\frac{1}{4}(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y_2^2}) - y_2(\frac{\partial^2}{\partial x_1 \partial x_2} + \frac{\partial^2}{\partial y_1 \partial y_2}) \\ &\quad + x_2(\frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial y_1 \partial x_2}) - (x_2^2 + y_2^2)(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}) \end{aligned} \tag{0.9}$$

Let $\epsilon = 1$ or -1 and consider the non holomorphic vector fields,

$$X_1 = u(z_1,z_2)\frac{\partial}{\partial z_1} \quad \text{and} \quad X_2 = \sqrt{u(z_1,z_2)}(2i\,\epsilon\,\overline{z_2}\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}) \tag{0.10}$$

The vector fields $2X_1$ and X_2 are orthonormal with respect to the metric (0.3): Let $Y_1 = 2X_1$ and $Y_2 = X_2$ where X_1, X_2 as in (0.10), then $\sigma_j(Y_k) = 0$ if $j \neq k$, $\sigma_1(Y_1) = -i$ and $\sigma_2(Y_2) = 1$. It holds $X_2 u = 0$. It is classical that

$$\Delta^K = -4X_1\overline{X_1} - X_2\overline{X_2} \tag{0.11}$$

Consider a Kähler domain and the group G of holomorphic automorphisms of this domain. Tne Lie algebra $\mathcal{G}$ of G is constituted of *holomorphic* vector fields. In [3], for the Siegel disk of complex symmetric matrices and in [1], for the n-dimensional complex ball, the Kähler Laplacian has been expressed in terms of a basis of $\mathcal{G}$. In the following, we extend to $\mathcal{D}$, the results obtained in [3]-[1]. By direct calculation, we write Δ^K in terms of the holomorphic vector fields of a basis in the algebra $\mathcal{G}$. This expression of Δ^K differs from those in [10], [6], p. 49 or [15], Chap. 4 for example and also from (0.8)-(0.11). The domain $\mathcal{D}$ is bi-rationally equivalent to the hyperball, i.e. the unit ball in $\mathbf{C^2}$, sse [14] or [15], Chap. 2. Identities similar to those in subsection 1.3 have been explicated in [1] for the unit ball in $\mathbf{C}^n$ but their expressions in the case of the domain (0.2) is new. Since the change of variables from $\mathcal{D}$ to the unit ball of $\mathbf{C^2}$ is not trivial at all, it is interesting to consider the problems for the domain $\mathcal{D}$. In the present work, we are interested by expressions of the Laplacian Δ^K with vector fields in $\mathcal{G}$ and by operators (OU-operators) which are adjoint of Δ^K with respect to the measure $d\mu = u^{3c}\, dv$ as explained in [1] for the unit ball of $\mathbf{C}^n$. Such investigations have not been carried out in [15] or [17]. The following is an attempt to construct OU-operators. It gives one more example for some of the problems raised in [3].

In Part I, let $G = Aut_{Hol}(\mathcal{D})$ be the group of holomorphic automorphisms of $\mathcal{D}$, see [13]-[15] Chap 2, for more details, we give a calculation of its Lie algebra $\mathcal{G}$. Then we state the holomorphic identities (1.28)-(1.29)-(1.30)-(1.31) between vector fields in $\mathcal{G}$. These are new.

In Part II, we verify (0.11). By analogy with [1], we relate Δ^K to $\mathcal{G}$ by proving (2.3). We define a complex Laplacian $\Delta^K_{\mathbf{C}}$ such that $\Delta^K = \Re\,(\Delta^K_{\mathbf{C}})$ and we show how the invariance of dv for this complex operator ($\int \Delta^K_{\mathbf{C}} F\, dv = 0$) is a consequence of the (1.28)-(1.29). In Remark 2.7, we deduce from (2.6) that the Kohn Laplacian on the boundary of $\mathcal{D}$ can be expressed with Lie algebra of the subgroup of affine holomorphic automorphisms of $\mathcal{D}$.

In Part III, let $\mathcal{D} \subset \mathbf{C}^n$ be a complex domain. We extend the Lie algebra of $Aut_{Hol}\mathcal{D}$: For a holomorphic vector field V, we define the inner contraction $\iota(V)$, see (3.1) and the operator $\rho(V) = V + c\ \iota(V)$ where c is a constant, see (3.4). This allows us to obtain on $\mathcal{D}$, Ornstein-Uhlenbeck type operators that we call OU-operators. We find these operators and their invariant measures for the domain (0.2).

1 The group $G = Aut_{Hol}\mathcal{D}$, its Lie algebra, identities between holomorphic vector fields.

Subsections 1.1 and 1.2 are known facts, see [14]- [13] for 1.1 and see [16], p. 215 - [8]- [9]- [11] for 1.2. We need them for our further investigations. The identities in subsection 1.3 are new.

The domain $\mathcal{D}$ is mapped to the hyperball $\mathcal{B}:\ 1-w_1\overline{w_1}-w_2\overline{w_2}>0$ with

$$w_1=\frac{z_1-i}{z_1+i},\qquad w_2=\frac{2\,z_2}{z_1+i} \tag{1.1}$$

since (1.1) implies $1-w_1\overline{w_1}-w_2\overline{w_2}=\dfrac{4}{|z_1+i|^2}(\dfrac{z_1-\overline{z_1}}{2i}-z_2\overline{z_2})$. The equivalent domain $z_1+\overline{z_1}>|z_2|^2$ was introduced in [14] to study the hypersphere $w_1\overline{w_1}+w_2\overline{w_2}=1$. Let

$$w_1=\frac{z_1-1}{z_1+1},\ w_2=\frac{\sqrt{2}\,z_2}{z_1+1}\ \text{ then } 1-w_1\overline{w_1}-w_2\overline{w_2}=2\,\frac{z_1+\overline{z_1}-z_2\overline{z_2}}{(z_1+1)(\overline{z_1}+1)} \tag{1.2}$$

The holomorphic transformation (1.2) leaves $\mathcal{D}$ invariant since

$$\frac{w_1-\overline{w_1}}{2i}-w_2\overline{w_2}=\frac{2}{|z_1+1|^2}(\frac{z_1-\overline{z_1}}{2i}-z_2\overline{z_2})$$

1.1 The group of holomorphic automorphisms of $\mathcal{D}$

The group G is generated by $\Psi_{t,\xi}$, $(t,\xi)\in\mathbf{R}\times\mathbf{C}$, the dilations $\mathcal{H}_c$, $c\in\mathbf{C}$ and the involution $\mathcal{I}$,

$$\Psi_{t,\xi}\begin{pmatrix}z_1\\ z_2\end{pmatrix}=\begin{pmatrix}z_1+2i\overline{\xi}z_2+\ t+i\,\xi\overline{\xi}\\ z_2+\xi\end{pmatrix}=\begin{pmatrix}1 & 2i\overline{\xi}\\ 0 & 1\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix}+\begin{pmatrix}t+i\xi\overline{\xi}\\ \xi\end{pmatrix} \tag{1.3}$$

$$\mathcal{H}_c:(z_1,z_2)\to(\,c\overline{c}z_1,cz_2)\qquad\text{and}\quad \mathcal{I}(z_1,z_2)=\,(-\frac{1}{z_1},\ -i\frac{z_2}{z_1}) \tag{1.4}$$

The subgroup of affine holomorphic transformations of $\mathcal{D}$ is generated by $\Psi_{t,\xi}$, $\mathcal{H}_c$. We have $\mathcal{I}(i,0)=(i,0)$ and $(i,0)$ is the only point fixed under $\mathcal{I}$. The subgroup $G^+=\{\Psi_{t,\xi},\quad t\in\mathbf{R},\,\xi\in\mathbf{C}\}$ of $Aut_{Hol}(\mathcal{D})$ is isomorphic to $\mathcal{D}_0=\{\Im(z_1)=|z_2|^2\}=\quad\{(x_1+iz_2\overline{z_2}\,,\,z_2)\}$ since $\Psi_{t,\xi}$ is determined by $\Psi_{t,\xi}(0,0)=(t+i\xi\overline{\xi},\,\xi)\in\mathcal{D}_0$. The composition $\Psi_{t_1,\xi_1}\circ\Psi_{t_2,\xi_2}=\Psi_{\tau,\xi_1+\xi_2}$ induces the Heisenberg group law on $\mathcal{D}_0$, $(t_1,\xi_1)*(t_2,\xi_2)=(\tau,\ \xi_1+\xi_2)$ where τ is given by $\tau=t_1+t_2+i(\overline{\xi}_1\,\xi_2\,-\,\xi_1\overline{\xi_2})$.

Let $(u_1,u_2)=(\mathcal{I}o\Psi_{t,\xi}o\mathcal{I})(z_1,z_2)$. Then

$$(u_1,u_2)=(\frac{z_1}{Q}\,,\,\frac{i\xi z_1+z_2}{Q})\quad\text{with}\quad Q=Q(t,\xi)=1-2\overline{\xi}z_2-(t+i\xi\overline{\xi})z_1 \tag{1.5}$$

and $u_1-\overline{u_1}-2iu_2\overline{u_2}=\frac{1}{|Q|^2}\,(z_1-\overline{z_1}-\,2\,i\,z_2\overline{z_2})$. The Jacobian J of the map $\mathcal{I}o\,\Psi_{t,\xi}\,o\mathcal{I}$ and its determinant are

$$J=\frac{1}{Q^2}\times\begin{pmatrix}1-2\overline{\xi}z_2 & 2\overline{\xi}\,z_1\\ i\,\xi\,+\,z_2(t-i\xi\overline{\xi}) & 1\,-\,z_1(t-i\xi\overline{\xi})\end{pmatrix}\ \text{and}\ \det J=\frac{1}{Q^3}$$

Let $Q(t,\xi)$ as in (1.5). For a holomorphic function $f(z_1,z_2)$ and $g=(t,\xi,)$, we define

$$T_g^p f(z_1,z_2) = Q(t,\xi)^p f\left(\frac{z_1}{Q}\ ,\ \frac{z_2+i\xi\, z_1}{Q}\right) \tag{1.6}$$

then

$$\int |T_g^p f(z_1,z_2)|^2\, u^{-p}\, dv = \int |f(z_1,z_2)|^2\, u^{-p}\, dv \tag{1.7}$$

$(T_g^p,\, u^{-p}\, dv)$ is a holomorphic unitary representation for the subgroup of transformations (1.5).

Proof of (1.7). Denote $T_g^p f(z_1,z_2) = Q(t,\xi)^p f(\,k_g(Z)\,)$ with $Z=(z_1,z_2)$. The Jacobien determinant of $k_g(Z)$ is equal to J. Let $R(Z,\overline{Z})$ such that

$$\int |Q(Z)|^{2p} |f(k_g(Z)\,)|^2 R(Z,\overline{Z}) dx_1 dy_1 dx_2 dy_2 = \int |f(Z)|^2 R(Z,\overline{Z}) dx_1 dy_1 dx_2 dy_2$$

We have $\int |Q(Z)|^{2p} |f(k_g(Z)\,)|^2 R(Z,\overline{Z}) dx_1 dy_1 dx_2 dy_2$

$$= \int |Q(Z)|^{2p+6} |f(k_g(Z))\,|^2 R(Z,\overline{Z}) |J|^2 dx_1 dy_1 dx_2 dy_2$$

$$= \int |Q(k_g^{-1}(Z))|^{2p+6} |f(Z)|^2 R(k_g^{-1}(Z),\, \overline{k_g^{-1}(Z)}) dx_1 dy_1 dx_2 dy_2$$

Then (1.7) is a consequence of $|Q(Z)|^{2p+6}\, R(Z,\overline{Z}) = R(k_g(Z), \overline{k_g(Z)})$. •

1.1.1 Some interesting holomorphic transformations of $\mathcal{D}$.

Holomorphic involutions of $\mathcal{D}$ are $(z_1,z_2)\to(z_1,-z_2)$ and

$$\mathcal{I}_\beta(z_1,z_2) = \left(-\frac{\beta^2}{z_1}\ ,\ i\,\beta\,\frac{z_2}{z_1}\right) = \mathcal{H}_{-\beta}\circ\mathcal{I} \qquad \text{for} \quad \beta\in\mathbf{R} \tag{1.8}$$

More generally, for $\beta\in\mathbf{R}$, $\theta\in\mathbf{R}$, let

$$\mathcal{I}_{\theta,\beta}(z_1,z_2) = \left(-\frac{\beta^2}{z_1}\ ,\ i\, e^{i\theta}\beta\,\frac{z_2}{z_1}\right) \quad \text{then} \quad \mathcal{I}_{\theta_1,\beta_1}\circ\mathcal{I}_{\theta_2,\beta_2} = \mathcal{H}_{e^{i(\theta_1+\theta_2)}\,(\beta_1/\beta_2)}$$

Consider (1.2), $\mathcal{T}(z_1,\, z_2) = \left(\frac{z_1-1}{z_1+1}\, ,\ \frac{\sqrt{2}\, z_2}{z_1+1}\right)$

$$= [\mathcal{H}_{i\sqrt{2}}\circ\Psi_{1/2\,,0}\circ\mathcal{I}\circ\Psi_{1,0}](z_1,z_2) = [\Psi_{1,0}\circ\mathcal{I}\circ\Psi_{(1/2),0}\circ\mathcal{H}_{(i/\sqrt{2})}](z_1,z_2) \tag{1.9}$$

We have $\mathcal{T}\circ\mathcal{T} = \mathcal{H}_i\circ\mathcal{I}$ and $\mathcal{T}^4(z_1,z_2) = (z_1,\,-z_2)$ with $\mathcal{T}^4 = \mathcal{T}\circ\mathcal{T}\circ\mathcal{T}\circ\mathcal{T}$.

Let $\mathcal{T}_\theta(z_1,\, z_2) = \left(\frac{\cos\theta\, z_1 - \sin\theta}{\sin\theta\, z_1+\cos\theta},\ \frac{z_2}{\sin\theta\, z_1+\cos\theta}\right)$

$$= [\,\Psi_{-\tan\theta,0}\circ\mathcal{H}_{1/\cos\theta}\circ\mathcal{I}\circ\Psi_{-\tan\theta,0}\circ\mathcal{I}](z_1,z_2) \tag{1.10}$$

If $\theta=\pi/4$, (1.10) gives (1.9). We have $\mathcal{T}_{\theta_1+\theta_2} = \mathcal{T}_{\theta_1}\circ\mathcal{T}_{\theta_2}$.

Let $\mathcal{S}(z_1,\, z_2) = \left(\frac{z_1-1}{z_1+1},\ \frac{\sqrt{2}\, e^{-i(\pi/4)} z_2}{z_1+1}\right)$, then $\mathcal{S}\circ\mathcal{S} = \mathcal{I}$ and

$$\mathcal{S}^{-1}(u_1,\, u_2) = \left(\frac{1+u_1}{1-u_1},\ \frac{\sqrt{2}\, e^{i(\pi/4)} u_2}{1-u_1}\right) \tag{1.11}$$

1.1.2 The non holomorphic vector fields (0.10)

Let X_1 and X_2 as in (0.10).

$$X_1\overline{X_1} = u^2\Delta_1^K + \frac{u}{2\,i}\frac{\partial}{\partial \overline{z_1}} \tag{1.12}$$

$$X_2\overline{X_2} = u\,(\,2i\epsilon\,\overline{z_2}\,\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2})\,(\,-2iz_2\,\frac{\partial}{\partial \overline{z_1}} + \frac{\partial}{\partial \overline{z_2}})$$

$$= -\,u\Delta_2^K - 2i\,u\,\frac{\partial}{\partial \overline{z_1}} \tag{1.13}$$

Adding (1.12) and (1.13) and comparing with (0.8), we obtain (0.11).

Let F be a holomorphic function on $\mathcal{D}$. We have

$$(X_kF)(z_1,z_2) = \frac{d}{dz'_{k\,|z'_1=i,\,z'_2=0}} F([\Psi_{\Re z_1,z_2}\circ\mathcal{H}_{\sqrt{u}}](z'_1,z'_2))\,, \quad k=1,\ 2 \tag{1.14}$$

We denote $u = \Im z_1 - z_2\overline{z_2}$ and $\Re z_1 = (1/2)(z_1+\overline{z_1})$. On $\mathcal{D}\times S^1$, we define the group law

$$(\mathcal{Z}_1,\,\mathcal{Z}_2,\Theta) = (z_1,z_2,\theta)*(z'_1,z'_2,\theta') \tag{1.15}$$

where $\Theta = \theta+\theta'$ and $(\mathcal{Z}_1,\mathcal{Z}_2) = (\Psi_{\Re z_1,\,z_2}\circ\mathcal{H}_{e^{i\theta}\,\sqrt{u}})(z'_1,z'_2)$. The neutral element is $(i,0,0)$. In (1.15), we have

$$\mathcal{Z}_1 = z_1 + uz'_1 + 2i\,e^{i\theta}\sqrt{u}\,\overline{z_2}\,z'_2 - i\,u$$

$$\mathcal{Z}_2 = z_2 + e^{i\theta}\sqrt{u}z'_2 \tag{1.16}$$

With (0.11), we verify that the Laplacian Δ^K is left invariant with respect to this group operation. Moreover in (1.16), we have

$$\Im Z_1 - Z_2\overline{Z_2} = u\,u' = U \quad \text{where} \quad u' = \Im z'_1 - z'_2\overline{z'_2}$$

$$\Re Z_1 = \Re z_1 + u\Re z'_1 + i\sqrt{u}(e^{i\theta}\overline{z_2}z'_2 - e^{-i\theta}z_2\overline{z'_2}) \tag{1.17}$$

1.2 The Lie algebra $\mathcal{G}$ of $Aut_{Hol}(\mathcal{D})$

Let ϕ_ϵ be a parametrized curve in $Aut_{Hol}(\mathcal{D})$ such that $\phi_0 = Identity$ and let F be a holomorphic function. We consider $VF = \frac{d}{d\epsilon}_{\epsilon=0}(F\circ\phi_\epsilon)$. In the following, $t\in\mathbf{R}$ and $\xi = \alpha+i\beta\in\mathbf{C}$. The real vector space $\mathcal{G}_{-1}$ is generated by

$$L_{-1} = \frac{\partial}{\partial z_1} \tag{1.18}$$

where $(L_{-1}F)(z_1,z_2) = \frac{\partial}{\partial t}_{|t=0,\xi=0} F(\Psi_{t,\xi}(z_1,z_2)) = \frac{\partial}{\partial z_1}F$. On the other hand,

$$V_1F = \frac{\partial}{\partial \alpha}_{|t=0,\xi=0} F(\Psi_{t,\xi}(z_1,z_2)) = (2\,i\,z_2\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2})F$$

$$V_iF = \frac{\partial}{\partial \beta}_{|t=0,\xi=0} F(\Psi_{t,\xi}(z_1,z_2)) = (2\,z_2\frac{\partial}{\partial z_1} + i\,\frac{\partial}{\partial z_2})F$$

If $\gamma = a+ib$, we denote $V_\gamma = a\frac{\partial}{\partial\alpha} + b\frac{\partial}{\partial\beta}$. The two dimensional real vector space $\mathcal{G}_{-1/2}$ is the set of vectors

$$V_\gamma \,=\, 2i\,z_2\,\overline{\gamma}\frac{\partial}{\partial z_1} \,+\, \gamma\,\frac{\partial}{\partial z_2} \,=\, L_{-(1/2)} \tag{1.19}$$

It holds

$$W_iF = \frac{\partial}{\partial \alpha}_{|t=0,\xi=0} F(\mathcal{I}\circ\Psi_{t,\xi}\circ\mathcal{I}(z_1,z_2)) \,= (2\,z_1z_2\frac{\partial}{\partial z_1} + \,i\,z_1\frac{\partial}{\partial z_2} \,+\, 2\,z_2^2\frac{\partial}{\partial z_2})F$$

$$W_{-1}F = \frac{\partial}{\partial \beta}_{|t=0,\xi=0} F(\mathcal{I}\circ\Psi_{t,\xi}\circ\mathcal{I}(z_1,z_2)) \,= (\,-\,2\,i\,z_1z_2\frac{\partial}{\partial z_1} - z_1\frac{\partial}{\partial z_2} - 2\,i\,z_2^2\frac{\partial}{\partial z_2})F$$

We denote $\mathcal{G}_{1/2}$, the 2-dimensional real vector space constituted of vectors

$$W_\delta \,=\, 2i\,\overline{\delta}\,z_2^2\,\frac{\partial}{\partial z_2} \,+\, 2i\,\overline{\delta}\,z_1z_2\,\frac{\partial}{\partial z_1} \,+\, \delta\,z_1\frac{\partial}{\partial z_2} \,=\, L_{1/2} \tag{1.20}$$

where $\delta\in\mathbf{C}$ is a constant. The real vector space $\mathcal{G}_1$ is generated by

$$L_1 \,=\, z_1z_2\frac{\partial}{\partial z_2} \,+\, z_1^2\frac{\partial}{\partial z_1} \tag{1.21}$$

where $L_1F = \frac{\partial}{\partial t}_{|t=0,\xi=0} F(\mathcal{I}\circ\Psi_{t,\xi}\circ\mathcal{I}(z_1,z_2))$. Let $H_\alpha\,F = \frac{\partial}{\partial\alpha}_{|\lambda=1} F(\mathcal{H}_\lambda(z_1,z_2))$ and $H_\beta F = \frac{\partial}{\partial\beta}_{|\beta=0} F((z_1, e^{i\beta}z_2))$ We denote $\mathcal{G}_0$, the real vector space generated by the two vector fields

$$H_\alpha = 2z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} \quad \textit{and} \quad H_\beta = iz_2\frac{\partial}{\partial z_2} \tag{1.22}$$

Equivalently $\mathcal{G}_0$ is the set of vectors $L_0(\gamma) = (\gamma+\overline{\gamma})z_1\frac{\partial}{\partial z_1} + \gamma\,z_2\frac{\partial}{\partial z_2}$ with $\gamma\in\mathbf{C}$. The Lie algebra $\mathcal{G}$ of the group G is the direct sum of real vector spaces

$$\mathcal{G} = \mathcal{G}_{-1}\oplus\mathcal{G}_{-1/2}\oplus\mathcal{G}_0\oplus\mathcal{G}_{1/2}\oplus\mathcal{G}_1 \tag{1.23}$$

We denote $\mathcal{G}_{aff} = \mathcal{G}_{-1}\oplus\mathcal{G}_{-1/2}\oplus\mathcal{G}_0$, the Lie algebra of $Affine_{Hol}(\mathcal{D})$. The Lie algebra structure on the real vector space $\mathcal{G}$ is given by the Lie brackets

$$[V_\delta, V_\gamma] = 2i(\delta\overline{\gamma} - \gamma\overline{\delta})\,L_{-1}, \quad [V_\delta, L_{-1}] = 0, \quad [L_{-1}, L_1] = H_\alpha$$

$$[V_\gamma, H_\beta] = V_{i\gamma}, \quad [L_{-1}, H_\beta] = 0$$
$$[L_{-1}, H_\alpha] = 2L_{-1}, \quad [W_\delta, H_\alpha] = -W_\delta, \quad [L_{-1}, W_\delta] = V_\delta$$
$$[L_1, H_\alpha] = -2L_1, \quad [L_1, H_\beta] = 0\,, \quad [H_\alpha, H_\beta] = 0 \quad [W_\delta, L_1] = 0$$
$$[W_\delta, W_\gamma] = 2i(\delta\overline{\gamma} - \gamma\overline{\delta})\, L_1, \quad [W_\delta, V_\gamma] = i(\delta\overline{\gamma} - \gamma\overline{\delta})H_\alpha - 3\,(\delta\overline{\gamma} + \gamma\overline{\delta})H_\beta \quad (1.24)$$

We have $[\mathcal{G}_{-(1/2)}, \mathcal{G}_{-(1/2)}] \subset \mathcal{G}_{-1}$, $[\mathcal{G}_{-(1/2)}, \mathcal{G}_{-1}] = 0$, $[\mathcal{G}_{-(1/2)}, \mathcal{G}_0] = \mathcal{G}_{-(1/2)}$ and $[\mathcal{G}_j\,, \mathcal{G}_k] \subset \mathcal{G}_{j+k}$. We assign weights to the variables z_1 and z_2 by saying that the constants have weight zero, z_1 has weight 1 and z_2 has weight $1/2$. With this convention, $\frac{\partial}{\partial z_1}$ has weight -1 and $\frac{\partial}{\partial z_2}$ has weight $-(1/2)$. The subspace $\mathcal{G}_k$ is constituted of vector fields of weight k. The brachets in (1.24) show that

$$\mathcal{G}_{-1} \oplus \mathcal{G}_{-1/2} \quad \text{and} \quad \mathcal{A}_0 = \mathcal{G}_{-1} \oplus \mathcal{G}_{-1/2} \oplus \mathbf{R}.H_\beta \qquad (1.25)$$

are subalgebras of $\mathcal{G}$. The direct sum $\mathcal{A}_0$ is the Lie algebra of the group A_0 of holomorphic affine automorphisms with Jacobian determinant equal to one.

1.2.1 The involution $\mathcal{I}^*$ on the Lie algebra $\mathcal{G}$.

The involution $\mathcal{I}$ induces an involution $\mathcal{I}^*$ on $\mathcal{G}$, see for example [5],

$$\mathcal{I}^*(\mathcal{G}_k) = \mathcal{G}_{-k} \quad for \quad k \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\} \qquad (1.26)$$

Let X be a vector field on $\mathcal{D}$, we define the vector field X^* as

$$(X^*F)(z_1, z_2) = X(F \circ \mathcal{I})(\mathcal{I}^{-1}(z_1, z_2)) \qquad (1.27)$$

where $F : \mathcal{D} \to \mathbf{C}$ is differentiable and $\mathcal{I} = \mathcal{I}^{-1}$ is the involution. It holds $X(F \circ \mathcal{I}) = (X^*(F)) \circ \mathcal{I}$. The map $\mathcal{I}^*$ defined by $\mathcal{I}^*(X) = X^*$, is an involution on $\mathcal{G}$. We have $\mathcal{I}^*(L_{-1}) = L_1$, $\mathcal{I}^*(L_1) = L_{-1}$, $\mathcal{I}^*(V_\gamma) = W_{i\gamma}$, $\mathcal{I}^*(W_\delta) = V_{-i\delta}$ and $\mathcal{I}^*(H_\alpha) = -H_\alpha$, $\mathcal{I}^*(H_\beta) = H_\beta$, $\mathcal{I}^*(L_0(\gamma)) = -L_0(\overline{\gamma})$.

1.2.2 The map $\mathcal{S}$ such that $\mathcal{S} \circ \mathcal{S} = \mathcal{I}$.

Let $\mathcal{S}(z_1, z_2) = (\frac{z_1 - 1}{z_1 + 1}, \frac{\sqrt{2}\, e^{-i(\pi/4)} z_2}{z_1 + 1})$ as in (1.11). The inverse map $\mathcal{S}^{-1}$ is given by $\mathcal{S}^{-1} = \mathcal{S} \circ \mathcal{I} = \mathcal{I} \circ \mathcal{S}$. For a vector field X in $\mathcal{G}$ and a differentiable function F, we put

$$\mathcal{S}^*(X)F = [X(F \circ \mathcal{S})](\mathcal{S}^{-1})$$

We have $\mathcal{S}^*(\mathcal{G}) \subset \mathcal{G}$ and $\mathcal{S}^* \circ \mathcal{S}^* = \mathcal{I}^*$, $\mathcal{S}^*(H_\alpha) = L_{-1} - L_1$, $\mathcal{S}^*(H_\beta) = H_\beta$

$$\mathcal{S}^*(L_{-1}) = \frac{1}{2}(L_1 + L_{-1} - H_\alpha), \qquad \mathcal{S}^*(L_1) = \frac{1}{2}(L_1 + L_{-1} + H_\alpha)$$

$$\mathcal{S}^*(V_\gamma) = \frac{1}{2}(V_\delta - W_\delta) \quad \text{and} \quad \mathcal{S}^*(W_\gamma) = \frac{1}{2}(V_\delta + W_\delta) \quad , \qquad \delta = \sqrt{2}e^{-i(\pi/4)}\gamma$$

1.3 Identities between holomorphic vector fields

By direct calculation, we verify the holomorphic identities where the right hand side is a first order *holomorphic* operator,

$$(i) \qquad 4L_{-1}\,L_1 \,-\, H_\alpha^2 \,-\, H_\beta^2 \,=\, 4(z_1\frac{\partial}{\partial z_1} \,+\, z_2\frac{\partial}{\partial z_2})$$

$$(ii) \qquad 4\,L_1\,L_{-1} \,-\, H_\alpha^2 \,-\, H_\beta^2 \,=\, -\,4\,z_1\frac{\partial}{\partial z_1} \tag{1.28}$$

Let $\gamma \in \mathbf{R}$, $\delta \in \mathbf{R}$ and $\epsilon \in \mathbf{R}$, then

$$(i) \qquad \frac{1}{\gamma\delta}[V_\gamma W_{-\epsilon\, i\, \delta} + V_{\epsilon\, i\, \gamma} W_\delta\,] \,-\, 4\epsilon\, H_\beta^2 = \,-\,4\,\epsilon\, z_1\frac{\partial}{\partial z_1}$$

$$(ii) \qquad \frac{1}{\gamma\delta}[W_{-\epsilon i\delta} V_\gamma \,+\, W_\delta\, V_{\epsilon\, i\gamma}] \,-\, 4\,\epsilon\, H_\beta^2 \,=\, 4\,\epsilon\,(z_1\frac{\partial}{\partial z_1} \,+\, z_2\frac{\partial}{\partial z_2}) \tag{1.29}$$

For γ, $\delta \in \mathbf{C}$,

$$(i) \quad V_\delta\, V_\gamma + V_{i\,\delta}\, V_{i\,\gamma} \,-\, 4(\overline{\gamma}\delta + \overline{\delta}\gamma)L_{-1}H_\beta = \,4\,i\,\delta\overline{\gamma}L_{-1}$$

$$(ii) \quad W_\delta W_\gamma + W_{i\delta} W_{i\gamma} - 4(\overline{\gamma}\delta + \overline{\delta}\gamma)L_1 H_\beta = 4i\delta\overline{\gamma}L_1$$

$$(iii) \quad W_\delta\, V_\gamma + W_{i\gamma} V_{i\delta} = 2(\overline{\delta}\gamma + \overline{\gamma}\delta)[H_\alpha H_\beta \,+\, z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}]$$

$$(iv) \quad W_\delta V_\gamma \,-\, W_\gamma V_\delta \,= 2i(\overline{\delta}\gamma - \overline{\gamma}\delta)\, z_2^2\frac{\partial^2}{\partial z_2^2} \,+\, 2i(\overline{\delta}\gamma - \overline{\gamma}\delta)\, z_1\frac{\partial}{\partial z_1} \tag{1.30}$$

In (1.29)-(1.30), we pass from (i) to (ii) with the involution $\mathcal{I}^*$, see (1.26). If $\overline{\delta}\gamma - \overline{\gamma}\delta = 0$, then $W_\delta V_\gamma \,-\, W_\gamma V_\delta \,=\, 0$. Moreover,

$$4[L_1,\, L_{-1}] + [V_1, W_i] + [V_{-i}, W_1] =\, 0 \tag{1.31}$$

since $4[L_1,\, L_{-1}] = -4H_\alpha$, $[V_1,\, W_i] \,= 2H_\alpha$, $[V_{-i},\, W_1] \,= 2H_\alpha$.

2 The Kähler Laplacian Δ^K on $\mathcal{D}$

2.1 Δ^K calculated with the Bergman metric.

Let $H(z_1, z_2) = \log u(z_1, z_2)$. We have

$$\frac{\partial^2}{\partial z_1 \partial \overline{z_1}} H = (z_1 - \overline{z_1} - 2iz_2\overline{z_2})^{-2}, \qquad \frac{\partial^2}{\partial z_1 \partial \overline{z_2}} H =\, 2iz_2\,\times\,(z_1 - \overline{z_1} - 2iz_2\overline{z_2})^{-2}$$

$$\frac{\partial^2}{\partial z_2 \partial \overline{z_2}} H =\, -\,2i(z_1 - \overline{z_1})\,\times\,(z_1 - \overline{z_1} - 2iz_2\overline{z_2})^{-2}$$

We find for the Bergman metric (0.3),

$$ds^2 = \frac{1}{4u^2}[dz_1 d\overline{z_1} + 2iz_2 dz_1 d\overline{z_2} - 2i\overline{z_2} dz_2 d\overline{z_1} + 4(\frac{z_1 - \overline{z_1}}{2i}) dz_2 d\overline{z_2}]$$
$$= \frac{1}{4u^2}[dz_1 d\overline{z_1} + 2idz_2(\overline{z_1} d\overline{z_2} - \overline{z_2} d\overline{z_1}) + (z_1 dz_2 - z_2 dz_1)(\overline{2i\, dz_2})]$$

The inverse of $P = (2iu)^{-2} \times \begin{pmatrix} 1 & 2iz_2 \\ -2i\overline{z_2} & -2i\,(z_1 - \overline{z_1}) \end{pmatrix}$ is

$$P^{-1} = (2iu) \times \begin{pmatrix} z_1 - \overline{z_1} & z_2 \\ -\overline{z_2} & -\frac{1}{2i} \end{pmatrix} = (2iu)^3 \times \begin{pmatrix} \frac{i}{2}\frac{\partial}{\partial y_1}\frac{1}{u} & -\frac{1}{4}\frac{\partial}{\partial \overline{z_2}}\frac{1}{u} \\ \frac{1}{4}\frac{\partial}{\partial z_2}\frac{1}{u} & -\frac{i}{8u^2} \end{pmatrix}$$

The determinant of the matrix P^{-1} is $(1/4)(u(z_1, z_2))^{-3}$. We find (0.7) since the Kähler Laplacian is $\Delta^K = \sum_{jk} m_{jk} \frac{\partial^2}{\partial z_j \partial \overline{z_k}}$ where the matrix $(m_{jk}) = \overline{P^{-1}}$.

Lemma 2.1 *We have $\int \Delta^K F dv = 0$. The volume measure is invariant for Δ^K as well as for Δ_1^K and Δ_2^K.*

Proof. We integrate by parts or we verify that $\sum_j \frac{\partial}{\partial z_j} m_{jk} (\Im(z_1) - z_2\overline{z_2})^{-3} = 0$ for $k = 1, 2$, as in [3], Theor. 10.2. •

In (2.14)-(2.16) of [3], the vector field

$$\mathcal{V}^K = \sum_{jk} m_{jk} [\frac{\partial}{\partial z_j} \log u] \frac{\partial}{\partial \overline{z_k}} \tag{2.1}$$

is associated to $\Delta^K = \sum_{jk} m_{jk} \frac{\partial^2}{\partial z_j \partial \overline{z_k}}$. The operator $\Delta^K - c\mathcal{V}^K$ has the measure $\exp(-c \log u) dv$ as invariant measure. When m_{jk} is given by (0.7), this gives

$$\mathcal{V}^K = u[\, 2i(z_1 - \overline{z_1})(\frac{\partial}{\partial z_1} \log u) \frac{\partial}{\partial \overline{z_1}} - 2i\overline{z_2}(\frac{\partial}{\partial z_1} \log u) \frac{\partial}{\partial \overline{z_2}} +$$
$$2iz_2(\frac{\partial}{\partial z_2} \log u) \frac{\partial}{\partial \overline{z_1}} - (\frac{\partial}{\partial z_2} \log u) \frac{\partial}{\partial \overline{z_2}}]$$

We find

$$\mathcal{V}^K = 2i\, u\, \frac{\partial}{\partial \overline{z_1}} \quad \text{and} \quad \mathcal{V}^K + \overline{\mathcal{V}^K} = -2\, u\, \frac{\partial}{\partial y_1} \tag{2.2}$$

2.2 Δ^K in terms of the holomorphic vector fields in $\mathcal{G}$.

Theorem 2.2

$$4\Delta^K = B + \overline{B} \quad \text{with} \quad B = B_1 + \overline{B_2}$$

$$B_1 = 4L_1 \overline{L_{-1}} - H_\alpha \overline{H_\alpha} - H_\beta \overline{H_\beta}$$

$$B_2 = \frac{1}{\gamma\delta} W_{i\delta}\overline{V_\gamma} + \frac{1}{\gamma'\delta'} W_{\delta'}\,\overline{V_{-i\gamma'}} + 4H_\beta\overline{H_\beta} \tag{2.3}$$

where γ, γ', δ, δ' are real, non zero constants. We have

$$\begin{aligned} 2\Delta^K &= \Re(B_1 + \overline{B_2}) \\ &= \Re(B_1 + B_2) \end{aligned} \tag{2.4}$$

Proof. We have

$$B_1 = 4(z_1z_2\frac{\partial}{\partial z_2} + z_1^2\frac{\partial}{\partial z_1})\frac{\partial}{\partial \overline{z_1}} - (2z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2})(2\overline{z_1}\frac{\partial}{\partial \overline{z_1}} + \overline{z_2}\frac{\partial}{\partial \overline{z_2}}) - z_2\overline{z_2}\frac{\partial^2}{\partial z_2\partial\overline{z_2}}$$

and $\overline{B_2} = V_1\overline{W_i} + V_{-i}\overline{W_1} + 4H_\beta\overline{H_\beta} =$

$$8iz_2\overline{z_2}^2\frac{\partial^2}{\partial z_1\partial\overline{z_2}} + 8iz_2\overline{z_2}\overline{z_1}\frac{\partial^2}{\partial z_1\partial\overline{z_1}} + 4(\frac{\overline{z_1}}{2i} + z_2\overline{z_2})\frac{\partial^2}{\partial z_2\partial\overline{z_2}}$$

Then $B = B_1 + \overline{B_2}$

$$= 4u(z_1,z_2) \times (2i)\, z_1\frac{\partial^2}{\partial z_1\partial\overline{z_1}} + 8iz_2\overline{z_2}(z_1 + \overline{z_1})\frac{\partial^2}{\partial z_1\partial\overline{z_1}} + (4z_1z_2 - 2\overline{z_1}z_2)\frac{\partial^2}{\partial z_2\partial\overline{z_1}}$$
$$+ (8iz_2\overline{z_2}^2 - 2z_1\overline{z_2})\frac{\partial^2}{\partial z_1\partial\overline{z_2}} + 4(\frac{\overline{z_1}}{2i} + \frac{1}{2}z_2\overline{z_2})\frac{\partial^2}{\partial z_2\partial\overline{z_2}}$$

By identification with (0.7), we find $B + \overline{B} = 4\Delta^K$ •

Theorem 2.3

$$2i\Im B = 4(L_1\overline{L_{-1}} - L_{-1}\overline{L_1}) + (V_1\overline{W_i} - W_i\overline{V_1}) + (V_{-i}\overline{W_1} - W_1\overline{V_{-i}}) \tag{2.5}$$

Δ_2^K *in (0.9) satisfies*

$$\Delta_2^K = \frac{1}{2}[-V_1\overline{V_1} - V_i\overline{V_i} + 4L_{-1}\overline{H_\beta} + 4H_\beta\overline{L_{-1}}] \tag{2.6}$$

Proof.

$$\frac{1}{2i}(B - \overline{B}) = 4(\Im z_1 + z_2\overline{z_2})(z_1 + \overline{z_1})\frac{\partial^2}{\partial z_1\partial\overline{z_1}} + 4\,z_2\,(\frac{z_1}{2i} + z_2\overline{z_2})\frac{\partial^2}{\partial z_2\partial\overline{z_1}}$$
$$+ 4\,\overline{z_2}\,(-\frac{\overline{z_1}}{2i} + z_2\overline{z_2})\frac{\partial^2}{\partial z_1\partial\overline{z_2}} - (z_1 + \overline{z_1})\frac{\partial^2}{\partial z_2\partial\overline{z_2}}$$

We verify (2.5) since $\Im B = C_1 + C_2$ with

$$\begin{aligned} C_1 &= 4\,(\Im\, z_1)(z_1 + \overline{z_1})\frac{\partial^2}{\partial z_1\partial\overline{z_1}} + \frac{4z_1z_2}{2i}\frac{\partial^2}{\partial z_2\partial\overline{z_1}} - \frac{4\overline{z_1}\overline{z_2}}{2i}\frac{\partial^2}{\partial z_1\partial\overline{z_2}} \\ &= -2i(L_1\overline{L_{-1}} - \overline{L_1}L_{-1}) \end{aligned}$$

2.3 It results from (1.28)-(1.29), that the volume dv is invariant for the complex operator $B_1 + \overline{B_2}$

It is well known that the volume measure (0.4) is invariant with respect to Δ^K, see for example [7]. In this subsection, we prove this fact by using the identities (1.28)-(1.29). For $z = (z_1, z_2) \in \mathcal{D}$, we consider the measure $d\mu = k(z) dz_1 d\overline{z_1} dz_2 d\overline{z_2}$ where k is a differentiable real-valued function. Let $V = a_1(z)\frac{\partial}{\partial z_1} + a_2(z)\frac{\partial}{\partial z_2}$ be a holomorphic vector field, ($a_j(z)$, $j = 1, 2$ are holomorpic functions). We define

$$\iota(V)(z) = \frac{\partial}{\partial z_1}[a_1(z)] \; + \; \frac{\partial}{\partial z_2}[a_2(z)] \tag{2.7}$$

By integration by parts, if F is a differentiable function

$$\int VF \; d\mu \; = \; - \int F(z) \times [\iota(V)(z) + \; V(\log k)(z)] \, d\mu \tag{2.8}$$

For the vector fields (1.18)-(1.19)-(1.20)-(1.21)-(1.22), we have

$$\iota(L_{-1})(z) = \iota(V_\gamma)(z) = 0 \, , \quad \iota(H_\alpha)(z) = 3 \, , \quad \iota(H_\beta)(z) = \; i$$
$$\iota(W_\delta)(z) = 6 \, i \, \overline{\delta} \, z_2 \, , \quad \iota(L_1)(z) = 3 z_1 \tag{2.9}$$

It is not difficult to verify the two following lemmas.

Lemma 2.4 *We have*

$$\Re[V \, (k)] = 0 \quad \textit{for} \quad V = L_{-1}, \; V = V_\gamma, \; V = H_\beta \tag{2.10}$$

if and only if

$$k(z) = \phi(\frac{z_1 - \overline{z_1}}{2i} - z_2 \overline{z_2}) \tag{2.11}$$

where $\phi : \mathbf{R} \to \mathbf{R}^+$ is a differentiable function. If (2.11) is verified and V is one of the vector fields in (2.10), then

$$\int (V + \overline{V}) F \, d\mu = 0 \qquad \textit{for} \quad d\mu = k(z) dz_1 d\overline{z_1} dz_2 d\overline{z_2} \tag{2.12}$$

Lemma 2.5 *Assume that $d\mu = \phi(\frac{z_1 - \overline{z_1}}{2i} - z_2\overline{z_2}) dz_1 d\overline{z_1} dz_2 d\overline{z_2}$. Then*

$$\int (V + \overline{V}) F \, d\mu = 0 \qquad \textit{for} \quad V = H_\alpha \tag{2.13}$$

if and only if $\phi(u) = u^{-3}$. Moreover, in that case, we have

$$\int (V + \overline{V}) F \, d\mu = 0 \qquad \textit{for any} \quad V \in \mathcal{G} \tag{2.14}$$

Theorem 2.6 *Let $B = B_1 + \overline{B_2}$ as in (2.3), then B has dv for invariant measure. In particular Δ^K and $\Im B$ have dv as invariant measure.*

Proof. Let

$$B_1 = 4L_1\overline{L_{-1}} - H_\alpha\overline{H_\alpha} - H_\beta\overline{H_\beta}$$

$$\overline{B_2} = V_1\overline{W_i} + V_{-i}\overline{W_1} + 4H_\beta\overline{H_\beta}$$

then with (1.28)-(ii),

$$4(L_1 + \overline{L_1})\overline{L_{-1}} - (H_\alpha + \overline{H_\alpha})\overline{H_\alpha} - (H_\beta + \overline{H_\beta})\overline{H_\beta} = B_1 - 4\overline{z_1}\frac{\partial}{\partial\overline{z_1}} \tag{2.15}$$

and with (1.29)-(i),

$$(V_1 + \overline{V_1})\overline{W_i} + (V_{-i} + \overline{V_{-i}})\overline{W_1} + 4(H_\beta + \overline{H_\beta})\overline{H_\beta} = \overline{B_2} + 4\overline{z_1}\frac{\partial}{\partial\overline{z_1}} \tag{2.16}$$

Adding and using (2.14), we obtain $\int(BF)\,dv = 0$. •

Remark 2.7 *By changing variables, on the boundary $\mathcal{D}_0$, the operator Δ_2^K in (0.9) is identical to the Kohn Laplacian, see [12]. We have proved that Δ_2^K satisfies (2.6).*

3 Divergence of vector fields in finite dimension. Invariant measures for OU-operators

The first two subsections 3.1-3.2 are valid for a domain $\mathcal{D}$ in $\mathbf{C}^n$. Of course, they do not extend to infinite dimensional domains. In subsection 3.3, with 3.1-3.2, we obtain OU-operators on (0.2).

3.1 Extension of the Lie algebra $\mathcal{G}$

Let $\mathcal{D} \subset \mathbf{C}^n$. For a holomorphic vector field $V = \sum_j a_j(z)\frac{\partial}{\partial z_j}$, we define the inner contraction, see [2]

$$\iota(V)(z) = \sum_j \frac{\partial}{\partial z_j} a_j(z) \tag{3.1}$$

For constants c_1, c_2, we have $\iota(c_1V_1 + c_2V_2) = c_1\iota(V_1) + c_2\iota(V_2)$.

Lemma 3.1 *For (3.1), it holds*

$$V(\iota(W)) - W(\iota(V)) = \iota([V, W]) \tag{3.2}$$

Proof. To prove (3.2), let $V = \sum_j a_j^V(z)\dfrac{\partial}{\partial z_j}$ and $W = \sum_k a_k^W(z)\dfrac{\partial}{\partial z_k}$. We find

$$V(\iota(W)) - W(\iota(V)) = \sum_{j,k} a_j^V(z)\frac{\partial^2}{\partial z_j \partial z_k} a_k^W(z) \; - \; \sum_{j,k} a_j^W(z)\frac{\partial^2}{\partial z_j \partial z_k} a_k^V(z) \quad (3.3)$$

On the other hand

$$[V,W] = \sum_{j,k} a_j^V(z)\frac{\partial}{\partial z_j}(a_k^W(z))\frac{\partial}{\partial z_k} \; - \; \sum_{j,k} a_j^W(z)\frac{\partial}{\partial z_j}(a_k^V(z))\frac{\partial}{\partial z_k}$$

This permits to identify the two sides of (3.2). •

Definition 3.2 *Consider the operator*

$$\rho(V) = \; V + \; c \; \iota(V) \quad \text{where } c \text{ is a real constant} \quad (3.4)$$

For a function F differentiable on $\mathcal{D}$, $\rho(V)F = \; VF + \; c \; \iota(V) \times F$.

Theorem 3.3 *The identity (3.2) with (3.4) imply*

$$[\rho(V), \rho(W)] = \; \rho([V,W]) \quad (3.5)$$

Let $V = \sum_p b_p(z) V_p$, then

$$\iota(V) = \sum_p V_p(b_p) \; + \; b_p \; \iota(V_p) \quad (3.6)$$

In particular, if $\iota(V_p) = 0$, $\forall p$, we have $\rho(V) = V + \; c \; \sum_p V_p(b_p)$.

Proof. Let $V_p = \sum_k c_{kp}(z)\frac{\partial}{\partial z_k}$, then $\iota(V) = \sum_{p,k} \frac{\partial}{\partial z_k}(b_p \, c_{kp})$. This proves (3.6) •

Remark 3.4 *Let $\iota(V)$ be a function defined on $\mathcal{D}$ and depending on the vector field V. We define the operator $\rho(V) = \; V + \; c \; \iota(V)$ as in (3.4). Then for two vevtor fields V and W, the condition $[\rho(V), \rho(W)] = \; \rho([V,W])$ is valid if and only if $V(\iota(W)) - W(\iota(V)) = \iota([V,W])$.*

Remark 3.5 *Let ϕ be a holomorphic function on $\mathcal{D}$. For holomorphic $V = \sum_j a_j(z)\frac{\partial}{\partial z_j}$, we define*

$$\iota_\phi(V)(z) = V(\phi)(z) \; + \sum_j \frac{\partial}{\partial z_j} a_j(z) \quad (3.7)$$

We have $V(\iota_\phi(W)) - W(\iota_\phi(V)) = \iota_\phi([V,W])$. Assume that $V = \sum_p b_p(z) V_p$, then

$$\iota_\phi(V) = \sum_p V_p(b_p) \; + \; b_p \; \iota_\phi(V_p) \quad (3.8)$$

Compare (3.8) with (3.6).

3.2 Unitarity condition and invariant measures

We proceed as in [3]. Consider a second order differential operator

$$\Delta = \sum_{j_1 \in J_1,\, j_2 \in J_2} V_{j_1} \overline{V_{j_2}} \quad \text{such that} \quad \sum_{j_1 \in J_1,\, j_2 \in J_2} V_{j_1} V_{j_2} = \mathcal{V} \tag{3.9}$$

where V_{j_1}, V_{j_2}, $\mathcal{V}$ are holomorphic vector fields. Then

$$\int [(V_{j_1} + \overline{V_{j_1}})F] d\mu = 0 \quad \forall\, j_1 \in J_1 \qquad \text{implies} \quad \int (\Delta + \overline{\mathcal{V}}) F d\mu = 0 \tag{3.10}$$

On the other hand, from (3.9),

$$\overline{\Delta} = \sum_{j_1 \in J_1,\, j_2 \in J_2} V_{j_2} \overline{V_{j_1}} \quad \text{with} \quad \mathcal{W} = \sum_{j_1 \in J_1,\, j_2 \in J_2} V_{j_2} V_{j_1} \tag{3.11}$$

where $\mathcal{W} = \mathcal{V} + \sum_{j_1 \in J_1,\, j_2 \in J_2} [V_{j_2}\, ,\, V_{j_1}]$ is also holomorphic. We have

$$\int [(V_{j_2} + \overline{V_{j_2}})F] d\mu = 0 \quad \forall\, j_2 \in J_2 \qquad \text{implies} \quad \int (\overline{\Delta} + \overline{\mathcal{W}}) F d\mu = 0 \tag{3.12}$$

If (3.10)-(3.11)-(3.12) are satisfied, the vector field $\overline{\mathcal{W}} - \mathcal{V}$ is divergence free for $d\mu$,

$$\int (\overline{\mathcal{W}} - \mathcal{V})\, d\mu = 0 \tag{3.13}$$

Theorem 3.6 *We assume (3.4)-(3.9). Let μ be a measure such that*

$$\int [(\rho(V_{j_1}) + \overline{\rho(V_{j_1})} F] d\mu = 0 \quad \forall\, j_1 \in J_1 \tag{3.14}$$

then

$$\int (\Delta + \overline{\mathcal{V}} + \, c\, L_{J_1,J_2}) F d\mu = 0$$

$$\textit{with} \quad L_{J_1,J_2} = \sum_{j_1 \in J_1, j_2 \in J_2} (\iota(V_{j_1}) + \overline{\iota(V_{j_1})}\,)\ \overline{V_{j_2}} \tag{3.15}$$

On the other hand, let $\overline{\Delta}$ and $\mathcal{W}$ as in (3.11) and let μ be a measure such that

$$\int [(\rho(V_{j_2}) + \overline{\rho(V_{j_2})} F] d\mu = 0 \quad \forall\, j_2 \in J_2 \tag{3.16}$$

then

$$\int (\overline{\Delta} + \overline{\mathcal{W}} + \, c\, M_{J_1,J_2}) F d\mu = 0$$

$$\textit{with} \quad M_{J_1,J_2} = \sum_{j_1 \in J_1, j_2 \in J_2} (\iota(V_{j_2}) + \overline{\iota(V_{j_2})}\,)\ \overline{V_{j_1}} \tag{3.17}$$

Proof.

$$\sum_{j_1 \in J_1, j_2 \in J_2} \int [(\rho(V_{j_2}) + \overline{\rho(V_{j_2})}) \overline{V_{j_1}} F] d\mu = 0$$

implies (3.17). We prove (3.15) in a similar way. •

Definition 3.7 *We call the operators (3.15)-(3.17) OU-operators.*

3.3 OU-operators on $\mathcal{D}$ and their invariant measures

Let μ be a real measure on $\mathcal{D}$ and F be a differentiable function with compact support in $\mathcal{D}$. We consider B_1, B_2 and Δ^K as in (2.3). We have (1.28)-(1.29). We shall apply subsection 3.2 to this particular case.

The relation (2.8) extends as

$$\int \rho(V)F\, d\mu \;=\; -\int F(z) \times [(1-c)\iota(V)(z) + V(\log k)(z)]\, d\mu \tag{3.18}$$

where $\iota(V)$ are given by (2.9) and c is a constant. In the following, we assume that

$$d\mu = \phi(\frac{z_1 - \overline{z_1}}{2i} - z_2\overline{z_2})dz_1 d\overline{z_1} dz_2 d\overline{z_2} \tag{3.19}$$

Since $\iota(V) = 0$ for $V = L_{-1}$, $V = V_\gamma$, $V = H_\beta$, Lemma 2.4 stays true with

$$\int(\rho(V) + \rho(\overline{V}))F\, d\mu = 0 \qquad for \quad V = L_{-1},\ V = V_\gamma,\ V = H_\beta \tag{3.20}$$

Corollary 3.8 *Let $d\mu$ be given by (3.19), then*

$$\int(B_2 + 4z_1\frac{\partial}{\partial z_1})\, F\, d\mu = 0 \tag{3.21}$$

Proof. We deduce (3.21) from (2.16) and (2.12-(3.20). •

From (3.18), Lemma 2.5 becomes

Lemma 3.9 *Assume that V_0 is one of the vector fields L_1, W_δ, H_α, then*

$$\int(\rho(V_0) + \overline{\rho(V_0)})F\, d\mu = 0 \quad \forall\, F \qquad \textit{if and only if} \;\; \phi(u) = u^{3(c-1)} \tag{3.22}$$

Moreover, in that case, we have $\int(\rho(V) + \overline{\rho(V)})F\, d\mu = 0$ for any $V \in \mathcal{G}$.

Proof. To calculate (3.22) when $V_0 = W_\gamma$ and u is given by (0.1), we use

$$V_\gamma(u) = z_2\overline{\gamma} - \overline{z_2}\gamma \quad \text{and} \quad (W_\gamma + \overline{W_\gamma})(u) = 2\, i\, u\, V_\gamma(u) \tag{3.23}$$

Corollary 3.10 *Assume that $d\mu = u^{3(c-1)}dz_1 d\overline{z_1} dz_2 d\overline{z_2}$, then*

$$\int(\overline{B_2} + \mathcal{V})F d\mu = 0$$

$$\textit{where} \quad \mathcal{V} = 12c(2iz_2\overline{z_2}\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}) \;-\; 4(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}) \tag{3.24}$$

and

$$\int[B_1 - 4\overline{z_1}\frac{\partial}{\partial \overline{z_1}} \;+\; 6c(2z_1\frac{\partial}{\partial \overline{z_1}} - \overline{z_2}\frac{\partial}{\partial \overline{z_2}})]\, F\, d\mu = 0 \tag{3.25}$$

Proof. (1.29)-(ii) implies $W_iV_1 + W_1V_{-i} + 4H_\beta^2 = -4(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2})$. Then we use (3.22). This gives (3.24). Since

$$C = 4[\rho(L_1) + \overline{\rho(L_1)}]\overline{L_{-1}} - [\rho(H_\alpha) + \overline{\rho(H_\alpha)}]\overline{H_\alpha} - [\rho(H_\beta) + \overline{\rho(H_\beta)}]\overline{H_\beta}$$

satisfies $\int CFd\mu = 0$, we obtain (3.25). •

From (2.3), $4\Delta^K = B + \overline{B}$ where $B = B_1 + \overline{B_2}$ is in Theorem 2.6.

Theorem 3.11 *Let c be a real constant. Let $d\mu = u^{3(c-1)}dz_1d\overline{z_1}dz_2d\overline{z_2}$, then for a differentiable function F with compact support in $\mathcal{D}$,*

$$\int \Delta^K F + 3c\mathcal{V}F\,d\mu = 0 \quad \text{with} \quad \mathcal{V} = (x_1\frac{\partial}{\partial x_1} - y_1\frac{\partial}{\partial y_1}) - \frac{1}{2}(x_2\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_2}) \tag{3.26}$$

This differs from (2.2).

Proof. By adding (3.25) to the conjugate of (3.21), we find that $d\mu$ is an invariant measure for the operator

$$B + c \times (12z_1\frac{\partial}{\partial \overline{z_1}} - 6\overline{z_2}\frac{\partial}{\partial \overline{z_2}}) \tag{3.27}$$

Remark 3.12 *Integrating by parts, we find with (0.8)*

$$\int (\Delta^K + 6ic\,u\frac{\partial}{\partial \overline{z_1}})F\,u^{3c}\,dv = 0 \tag{3.28}$$

This is equivalent to (2.2).

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