On Finite Groups with Some Non-nilpotent Subgroups Being TI-Subgroups

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Abstract

Let $G$ be a finite non-nilpotent group. We prove that if there exists a non-nilpotent subgroup $H$ of $G$ such that for every non-nilpotent subgroup $K$ of $G$ satisfying $K \cap H \neq 1$ we always have that $K$ is a TI-subgroup, then all non-nilpotent subgroups of $G$ are TI-subgroups.

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1 Introduction

Let $G$ be a finite group and $H$ a subgroup of $G$. If $H^g \cap H = 1$ or $H$ for every $g \in G$, then $H$ is said to be a TI-subgroup of $G$. 

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In [4] G. Walls described finite groups with all subgroups being TI-subgroups. As generalizations, S. Li [1] determined non-nilpotent finite groups with all second maximal subgroups being TI-subgroups. In [3] the second author of this paper and C. Zhang investigated non-nilpotent finite groups with all non-nilpotent subgroups being TI-subgroups.

**Theorem 1.1** [3] Let $G$ be a non-nilpotent group. If all non-nilpotent subgroups of $G$ are TI-subgroups, then $G$ is solvable, and all non-nilpotent subgroups of $G$ are normal.

By Theorem 1.1, we can easily get the following fact for a non-nilpotent group, here we omit its proof.

**Proposition 1.2** Let $G$ be a non-nilpotent group having exactly $n$ non-nilpotent subgroups. Suppose that $G$ has exactly $\delta(G) > 1$ non-normal non-nilpotent TI-subgroups. Then $\delta(G) \leq n - 2$.

The following example can show that there exists a finite group $G$ satisfying $\delta(G) = n - 2$:

Let $G = D_{24}$ be a dihedral group of order 24. One has $n = 4$ and $\delta(G) = 2$. Then $\delta(G) = n - 2$.

In this paper, our main goal is to investigate non-nilpotent finite groups with some particular non-nilpotent subgroups being TI-subgroups.

We have the following result, the proof of which is given in Section 3.

**Theorem 1.3** Let $G$ be a non-nilpotent group. If there exists a non-nilpotent subgroup $H$ of $G$ such that for every non-nilpotent subgroup $K$ of $G$ satisfying $K \cap H \neq 1$ we always have that $K$ is a TI-subgroup, then all non-nilpotent subgroups of $G$ are TI-subgroups.

For proof of Theorem 1.3, the following lemma is essential, the proof of which is given in Section 2.

**Lemma 1.4** Let $G$ be a non-nilpotent group. If there exists a non-nilpotent subgroup $H$ of $G$ such that for every non-nilpotent subgroup $K$ of $G$ satisfying $K \cap H \neq 1$ we always have that $K$ is a TI-subgroup, then $H \unlhd G$.

The following properties of the Frobenius group are needed to be introduced.

**Lemma 1.5** [2] Let $G$ be a Frobenius group with the kernel $N$ and complement $K$. Then

1. $N$ is nilpotent;
2. If $R \unlhd G$, then either $R \leq N$ or $N < R$. 

2 Proof of Lemma 1.4

**Lemma 2.1** Let $G$ be a non-nilpotent group and $H$ a subnormal non-nilpotent subgroup of $G$. If for every non-nilpotent subgroup $K$ of $G$ satisfying $H \leq K$ we always have that $K$ is a TI-subgroup, then $H \leq G$.

**Proof.** Assume that $H$ is not normal in $G$. Then $H^g \cap H = 1$ for every $g \in G \setminus N_G(H)$. Since $H$ is subnormal in $G$. Let $H = G_r \leq G_{r-1} \leq \ldots \leq G_1 \leq G_0 = G$ be a composition series from $H$ to $G$. For every subgroup $M$ of $G$ satisfying $H \leq M$, by the hypothesis, one has that $M$ is a TI-subgroup of $G$. By induction on $r$, we have $G_{r-1} \leq G$. Then $H^g \leq G_{r-1}^g = G_{r-1}$.

(1) Suppose that $G_{r-1}/H$ is a non-abelian simple group. Since $HH^g/H \leq G_{r-1}/H$ and $HH^g/H \cong H^g \neq 1$, one has $HH^g/H = G_{r-1}/H$. Then $G_{r-1} = HH^g$. It follows that $H \cong H^g \cong G_{r-1}/H$ is a non-abelian simple group. Let $K/H$ be a maximal subgroup of $G_{r-1}/H$. Then $K$ is a maximal subgroup of $G_{r-1}$. Since $K$ is a TI-subgroup of $G$ and $K \nleq G_{r-1}$, one has $Kx \cap K = 1$ for every $x \in G_{r-1} \setminus K$. Then $G_{r-1}$ is a Frobenius group with complement $K$. Let $N$ be the Frobenius kernel of $G_{r-1}$. By Lemma 1.5 (2), one has either $H \leq N$ or $N < H$.

(i) Suppose $H \leq N$. By Lemma 1.5 (1), one has that $N$ is nilpotent. It follows that $H$ is nilpotent, a contradiction.

(ii) Suppose $N < H$. Since $H$ is a non-abelian simple group, it follows that $N = 1$, a contradiction.

(2) Suppose that $G_{r-1}/H$ is an abelian simple group. Let $|G_{r-1}/H| = p$, where $p$ is a prime. It follows that $H$ is a maximal subgroup of $G_{r-1}$, then $G_{r-1} = HH^g$. Since $H^g \cap H = 1$, we can get that $H$ is a cyclic group of order $p$, a contradiction.

By (1) and (2), our assumption is not true and so $H \leq G$. □

Next we give the proof of Lemma 1.4.

**Proof of Lemma 1.4** Suppose $H \nleq G$. Consider the series: $H = H_1 \leq N_G(H_1) = H_2 \leq N_G(H_2) = H_3 \leq \ldots \leq N_G(H_{r-1}) = H_r \leq N_G(H_r) \leq G$. If there exists a positive integer $r$ such that $N_G(H_r) = G$, then $H$ is subnormal in $G$. By Lemma 2.1, one has $H \leq G$, a contradiction. Then $N_G(H_r) < G$ for every positive integer $r$. It follows that there must exist a positive integer $r$ such that $H_r = N_G(H_r)$. By the hypothesis, one has $H_r^g \cap H_r = 1$ for every $g \in G \setminus N_G(H_r) = G \setminus H_r$. Then $G$ is a Frobenius group with complement $H_r$.

Let $N$ be the Frobenius kernel of $G$. It is easy to see that $N \rtimes H$ is still a Frobenius group. For every maximal subgroup $L$ of $H$, $NL$ is non-nilpotent and $NL \cap H = L(N \cap H) = L \neq 1$. Thus, by the hypothesis, $NL$ is a TI-subgroup. However, since $(NL)^g \cap NL = NL^g \cap NL \geq N \neq 1$ for every $g \in G$, we have $NL \leq G$. It follows that $L = L(N \cap H) = NL \cap H \leq H$. By the
choice of $L$, one has that $H$ is nilpotent, a contradiction. It implies that our assumption is not true and so $H \leq G$. \hfill \Box

\section{Proof of Theorem 1.3}

\textbf{Proof.} Suppose that the theorem is not true. Let $M$ be a non-nilpotent subgroup of $G$ and $M$ not a TI-subgroup. By the hypothesis, one has $M \cap H = 1$. Note that $H \leq G$ by Lemma 1.4. Consider the group $H \times M$. For every maximal subgroup $L$ of $M$, one has $L \neq 1$. By the hypothesis, one has that $H \times L$ is a TI-subgroup of $G$. Since $(HL)^g \cap HL = HL^g \cap HL \geq H \neq 1$ for every $g \in G$, one has $HL \leq G$. Then $L = L(H \cap M) = HL \cap M \leq M$. By the choice of $L$, we have that $M$ is nilpotent, this contradicts that $M$ is non-nilpotent. So all non-nilpotent subgroups of $G$ are TI-subgroups. \hfill \Box

\section{References}


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